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The Periodic Box-ball System and Tropical Curves

By

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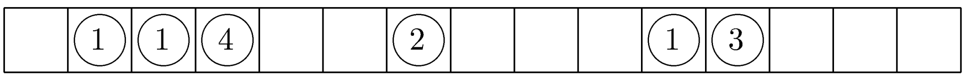
Abstract

In this article, we study the box-ball system with finitely many kinds of balls. The box-ball system is obtained from the hungry discrete Toda equation (hpd Toda eq.) by ultradiscretization. We study the applications of the algebraic geometry and the tropical geometry to the ultradiscrete integrable system(s).

§ 1. Introduction

The periodic box-ball system (pBBS) is a discrete dynamical system which consists of an array of finitely many boxes in a line and a finite number of balls (see the figure below). We regard that the rightmost box is connected to the leftmost one. In this article, we consider pBBSs with finitely many kinds of balls. We distinguish these kinds of balls by indices of positive integers written on them.



The *time evolution rule of pBBS* is defined as follows.

i) For each filled box, create a copy of the ball with index 1.

ii) Move each copy respectively to the nearest empty box on the right of it.

iii) Delete the original balls with index 1.

iv) Do the same procedure i)–iii) for the balls with index 2.

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v) Repeat these procedures for the balls with $3, 4, \ldots$ until all the balls have moved.

We show an example of the time evolution of pBBS below. The letter "m" means a ball with index m, and the sign "." means an empty box. (t : time).

```
t=0    ..1135...24...............
  1    ......1135.24.............
  2    ..........135124..........
  3    .............35.1124......
  4    ...............35...1124..
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We call a sequence of nondecreasing positive integers (for example "1135", "24") *soliton*. The number of solitons does not change under the time evolution [8]. An N-*soliton system* is a pBBS with N solitons. (The pBBS in the figure above is a 2-soliton system.)

Let $M \in \mathbb{Z}_{>0}$. Let us consider N-soliton systems with M kinds of balls. Choose one soliton arbitrarily and regard it as the "leftmost" soliton formally. (Of course actual leftmost soliton cannot exist by periodicity). At time $t \in \mathbb{Z}$, let $Q_n^{t+\frac{m-1}{M}}$ be the number of balls with index m in the n-th soliton from the left, and W_n^t be the number of empty boxes between the n-th soliton and the $(n+1)$-th soliton. Note that $Q_{n+N}^t \equiv Q_n^t$ and $W_{n+N}^t \equiv W_n^t$. For $n \in \mathbb{Z}$ and $t \in (1/M) \cdot \mathbb{Z}$, the time evolution rule of pBBS is expressed as follows [8]:

$$Q_n^{t+1} = \min\left[W_n^t,\ Q_n^t + X_n^t\right], \quad W_n^{t+\frac{1}{M}} = Q_{n+1}^t + W_n^t - Q_n^{t+1}, \tag{1.1}$$
$$(X_n^n := \max_{k=0}^{N-1}\left[\textstyle\sum_{l=1}^{k}\left(Q_{n-l}^t - W_{n-l}^t\right)\right].)$$
$$\textstyle\sum_{n=1}^{N} Q_n^t < \sum_{n=1}^{N} W_n^t. \tag{1.2}$$

The inequality (1.2) reflects the fact that the number of empty boxes must be larger than the number of moving balls. The integer $W_n^{t+\frac{k}{M}}$ ($k = 1, 2, \ldots, M-1$) is the number of empty boxes between two solitons when the balls with index k having finished moving.

Let $\{Q_n^0, Q_n^{1/M}, \ldots, Q_n^{(M-1)/M}, W_n^0\}_n$ be the *initial condition* of pBBS. Let us consider the general solution $\{Q_n^t, W_n^t\}_{n,t}$. Although it seems to be quite difficult to solve the system (1.1)–(1.2) directly, the following proposition gives us another method to obtain the solution.

Proposition 1.1. ([5, 8]) *Let $\varepsilon > 0$ be a positive parameter. Assume that there exists a set of functions $\{I_n^t(\varepsilon), V_n^t(\varepsilon)\}_{n,t}$ such that*

$$I_n^{t+1} = I_n^t + V_n^t - V_{n-1}^{t+\frac{1}{M}}, \quad V_n^{t+\frac{1}{M}} = \frac{I_{n+1}^t V_n^t}{I_n^{t+1}}, \quad (I_{n+N}^t \equiv I_n^t,\ V_{n+N}^t \equiv V_n^t) \tag{1.3}$$
$$\textstyle\prod_{n=1}^{N} I_n^t < \prod_{n=1}^{N} V_n^t. \tag{1.4}$$

If the limits

$$Q_n^t := -\lim_{\varepsilon\to 0^+}(\varepsilon \log I_n^t), \quad W_n^t := -\lim_{\varepsilon\to 0^+}(\varepsilon \log V_n^t)$$

exist, then the set $\{Q_n^t, W_n^t\}_{n,t}$ *satisfies* (1.1) *and* (1.2).

The discrete system (1.3)–(1.4) is called *hungry periodic d-Toda equation* (hpd Toda eq.), and the operation '$-\lim_{\varepsilon\to 0^+}\varepsilon\log\cdot$' is usually called *ultradiscretization.* According to the proposition above, the general solution of pBBS is the ultradiscretization of the solution of hpd Toda equation.

In this article, we research the method to calculate the general solutions of pBBSs. This article is organized as follows: In section 2, we introduce the method to solve the hpd Toda equation according to [3]. In section 3, we review the integration theory over tropical curves according to [4]. The results introduced in these two sections are applied to pBBS in section 4. We study the *relative cycle* of pBBS by using the theory of tropical curves.

§ 2. Solutions of the hpd Toda equation

We briefly introduce the theory of hpd Toda equation in this section. See [3], for details.

§ 2.1. Linearization

Let y be a complex parameter. Define the matrices $L_t(y)$ and $R_t(y)$ as follows.

$$L_t(y) = \begin{pmatrix} 1 & & & V_N^t\cdot 1/y \\ V_1^t & 1 & & \\ & \ddots & \ddots & \vdots \\ & & V_{N-1}^t & 1 \end{pmatrix}, \quad R_t(y) = \begin{pmatrix} I_1^t & 1 & & \\ & I_2^t & \ddots & \\ & & \ddots & 1 \\ y & & & I_N^t \end{pmatrix}.$$

The hpd Toda equation (1.3)–(1.4) is equivalent to the following equation:

$$L_{t+\frac{1}{M}}(y)R_{t+1}(y) = R_t(y)L_t(y), \qquad t\in(1/M)\cdot\mathbb{Z}. \tag{2.1}$$

Define the new matrix $X_t(y) := L_t(y)R_{t+\frac{M-1}{M}}(y)\cdots R_{t+\frac{1}{M}}(y)R_t(y)$. Then equation (2.1) becomes

$$X_{t+\frac{1}{M}}(y)R_t(y) = R_t(y)X_t(y), \qquad t\in(1/M)\cdot\mathbb{Z}. \tag{2.2}$$

By equation (2.2), the characteristic polynomial of $X_t(y)$ does not depend on time t. Therefore the algebraic curve defined by $\det(X_t(y) - xE) = 0$ does not depend on t

either. (E is the unit matrix). Let

$$\begin{aligned}\Phi(x,y) &:= \det(X_t(y) - xE)\\ &= f_0(x)\,y^M + f_1(x)\,y^{M-1} + \cdots + f_M(x) + h\,y^{-1},\end{aligned}$$

where $f_i(x)$ $(i = 0, 1, \ldots, M)$ is a polynomial in x of degree $\leq iN/M$, and h is a constant [3, 7]. The plane complex curve defined by $\Phi = 0$ is called *spectral curve*. The spectral curve is determined by the initial data $\{I_n^0, I_n^{\frac{1}{M}}, \ldots, I_n^{\frac{M-1}{M}}, V_n^0\}_n$ of hpd Toda equation. Let C be the plane curve defined by $\Phi = 0$. The *isospectral set* $\mathcal{T}_C$ is a set of matrices $X(y)$ of which the spectral curve is C.

Assume C is smooth hereafter. If $(x, y) \in C$ and $X(y) \in \mathcal{T}_C$, then the number x is an eigenvalue of $X(y)$. Let $\boldsymbol{v}(x, y)$ be an eigenvector of $X(y)$ which belongs to x. Naturally $\boldsymbol{v}(x, y)$ is regarded as an element of $\mathbb{P}^{N-1}$. Moreover, when x is a simple root of $\Phi(x, y)$ (for fixed y), the element $\boldsymbol{v}(x, y) \in \mathbb{P}^{N-1}$ is determined uniquely. As C is smooth, the map $(x, y) \mapsto \boldsymbol{v}(x, y)$ can be extended to the proper map $\psi_{X(y)} : C \to \mathbb{P}^{N-1}$ uniquely. It is known that the image $\psi_{X(y)}(C) \subset \mathbb{P}^{N-1}$ is a projective curve of degree $d := g + N - 1$ ([3], §2.2). (g is the genus of C).

Let us consider the twisting sheaf $\mathcal{O}_{\mathbb{P}^{N-1}}(1)$ over $\mathbb{P}^{N-1}$ and its pullback

$$\psi^*_{X(y)}\mathcal{O}_{\mathbb{P}^{N-1}}(1).$$

The sheaf $\psi^*_{X(y)}\mathcal{O}_{\mathbb{P}^{N-1}}(1)$ can be identified to the element of $\mathrm{Pic}^d C$ which is defined by the pullback of $C \cap H \subset \mathbb{P}^{N-1}$. ($H$ is a hyperplane). The map $\varphi_C : \mathcal{T}_C \ni X(y) \mapsto \psi^*_{X(y)}\mathcal{O}_{\mathbb{P}^{N-1}}(1) \in \mathrm{Pic}^d C$ is called *eigenvector map*.

Let $\boldsymbol{v}(x, y) = (g_1, g_2, \ldots, g_{N-1}, 1)^T$, where $g_1, \ldots, g_{N-1}$ are appropriate meromorphic functions of x and y. Then the element $\varphi_C(X(y)) \in \mathrm{Pic}^d C$ is the positive divisor $\mathcal{E}$ of minimal degree such that $(g_k) + \mathcal{E} \geq 0 \quad (\forall k)$. In fact, $\mathcal{E}$ is the pullback of the hyperplane $\{X_N = 0\}$, where $(X_1 : X_2 : \cdots : X_N)$ is a set of homogeneous coordinates for $\mathbb{P}^{N-1}$.

Remark 2.1. *By concrete calculations, we can obtain the relation :* $(g_1)_\infty = \mathcal{E}$ *([3], Remark 2.4), ([7]). This relation is quite convenient to calculate the image of given matrix $X(y)$ by the eigenvector map.*

Let $P : (x, y) = (\infty, \infty)$ and $Q : (x, y) = (\infty, 0)$ be two points on C. It follows that:

•1 There exists a general effective divisor $\mathcal{D}$ of degree g s.t. $\varphi_C(X(y)) = (g_1)_\infty = \mathcal{D} + (N - 1)Q$.

•2 $\varphi_C : \mathcal{T}_C \to \mathrm{Pic}^d C$ is injective.

· See [7]).

By the statement •2, to analyze $X(y) \in \mathcal{T}_C$ and to analyze $\varphi_C(X(y)) \in \mathrm{Pic}^d C$ is equivalent.

Remark 2.2. *There uniquely exist two points $P(\infty,\infty)$, $Q(\infty,0)$ on C. This is a special property of the spectral curves of hpd Toda equation. In [7], they researched the spectral curves on which $\{\text{g.c.d.}(N,M)\}$ points $P_1, P_2, \ldots . P_G : (x,y) = (\infty,\infty)$ exist. $(G = \{\text{g.c.d.}(N,M)\})$.*

Let us proceed to the hpd Toda equation. Let $X_t(y) \in \mathcal{T}_C$ be the matrix defined by the equation below (2.1). Let σ (resp. τ) be the action on $\mathcal{T}_C$ induced by the shift of the indices of Q_n^t and W_n^t as $n \mapsto n+1$ (resp. $t \mapsto t+1$). These actions naturally induce the action on Image(φ_C).

It is possible to calculate these actions concretely. Define the complex numbers $y_0, y_1, \ldots, y_{M-1}$ by $y_k := (-1)^N \cdot \prod_{n=1}^N I_n^{k/M}$. Hence it can be checked easily that there exist M points $A_0(0,y_0), A_1(0,y_1), \ldots, A_{M-1}(0,y_{M-1})$ on C.

Theorem 2.1.

1. $\varphi_C(\sigma(X(y))) = \varphi_C(X(y)) + P - Q$,
2. $\varphi_C(\tau(X(y))) = \varphi_C(X(y)) + A_0 + A_1 + \cdots + A_{M-1} - M \cdot P$.

Proof. [3], Prop 2.6 and Prop 2.16. □

The points $P, Q, A_0, \ldots, A_{M-1}$ do not depend on t nor n. And the actions of σ and τ are commutative. Therefore, the arbitrary composite of σ and τ can be expressed as:

Corollary 2.2. *Let $\mathcal{D}_n := P - Q$, $\mathcal{D}_t := A_0 + A_1 + \cdots + A_{M-1} - M \cdot P$. Then*

$$\varphi_C(\sigma^r \circ \tau^s(X(y))) = \varphi_C(X(y)) + r\mathcal{D}_n + s\mathcal{D}_t.$$

□

In a word, the actions of σ and τ is *linearized* on $\mathrm{Pic}^d C$.

§ 3. The lattice integral over tropical curve

We introduce the theory of tropical curves in this section. The *lattice integral* over tropical curve is an analogy of the holomorphic integral over complex curve [1, 6]. We study the relation between these two "integrals" , according to [4].

§ 3.1. The tropical curve

A *Tropical curve* is an algebraic curve defined over the tropical semifield $\mathbb{T} := \mathbb{R} \cup \{+\infty\}$ equipped with the min-plus operation: "$x + y$"$=\min\{x, y\}$ and "xy"$=x + y$.

Here we consider *tropical plane curves* only.

Definition 3.1. *Let $\phi_1, \phi_2, \dots, \phi_n$ be polynomials of the form: $\phi_i(X,Y) = a_iX + b_iY + c_i$ $(a_i, b_i \in \mathbb{Z}, c_i \in \mathbb{R})$. A tropical plane curve $\mathfrak{C}$ is a subset of $\mathbb{R}^2$ defined by*

$$(3.1) \qquad \mathfrak{C} := \{(X,Y) \in \mathbb{R}^2 \mid \text{the function } \min_{i=1}^n [\phi_i(X,Y)] \text{ is not smooth.}\}$$

Hereafter we call these curves "tropical curves" simply. A tropical curve consists of finitely many segments and half-lines, that are called *edges* of the tropical curve. A *vertex* of a tropical curve is a point which is contained by more than two segments or half-lines.

The condition appeared in (3.1) is rewritten as:

$$(3.2) \qquad \sharp\{i \mid \phi_i(X,Y) = \min_{i=1}^n [\phi_i(X,Y)]\} \geq 2.$$

Hence, for $A = (X,Y) \in \mathfrak{C}$, we define the non-negative integer $I(A)$ by

$$I(A) := \sharp\{i \mid \phi_i(X,Y) = \min_{i=1}^n [\phi_i(X,Y)]\} - 2.$$

Definition 3.2. *The tropical curve $\mathfrak{C}$ is regular if*

$$I(A) = 0 \Leftrightarrow A \text{ is a inner point of an edge}, \quad I(A) = 1 \Leftrightarrow A \text{ is a vertex.}$$

Remark 3.1. *$\mathfrak{C}$ is regular $\Rightarrow$ $\mathfrak{C}$ is* 3-valent.

For the purpose of arguing about the ultradiscretization of algebraic curves, it is convenient to consider the algebraic curves over the Puiseux field. Let $\boldsymbol{e} := e^{-1/\varepsilon}$ and $K := \bigcup_{d=1}^{\infty} \mathbb{C}((\boldsymbol{e}^{1/d}))$. The Puiseux field K has the valuation map $\mathrm{val} : K \to \mathbb{Q} \cup \{+\infty\}$ $(\mathrm{val}(\boldsymbol{e}) = 1, \mathrm{val}(0) = +\infty)$. When the ultradiscrete limit $X := -\lim_{\varepsilon \to 0} \varepsilon \log x$ $(x \in K)$ exists, it follows that $\mathrm{val}(x) = X$.

Define the subring $R := \{x \in K \mid \mathrm{val}(x) \geq 0\} \subset K$ and the multiplicative group $R^* := \{x \in K \mid \mathrm{val}(x) = 0\}$.

Let $\Phi(x,y) = \sum_{i,j} r_{i,j} x^i y^j \in K[x,y]$ be a polynomial over K. The *tropicalization* of the algebraic curve $C : \Phi(x,y) = 0$ is a set

$$\mathrm{Trop}C := \{(X,Y) \in \mathbb{R}^2 \mid \min_{i,j} [\mathrm{val}(r_{i,j}) + iX + jY] \text{ is not smooth.}\}.$$

§ 3.2. Examples

3.2.1. Example I

$$(3.3) \qquad \Phi(x,y) = y^2 + (x^2 + \boldsymbol{e}x + \boldsymbol{e}^5)y + \boldsymbol{e}^{11}$$

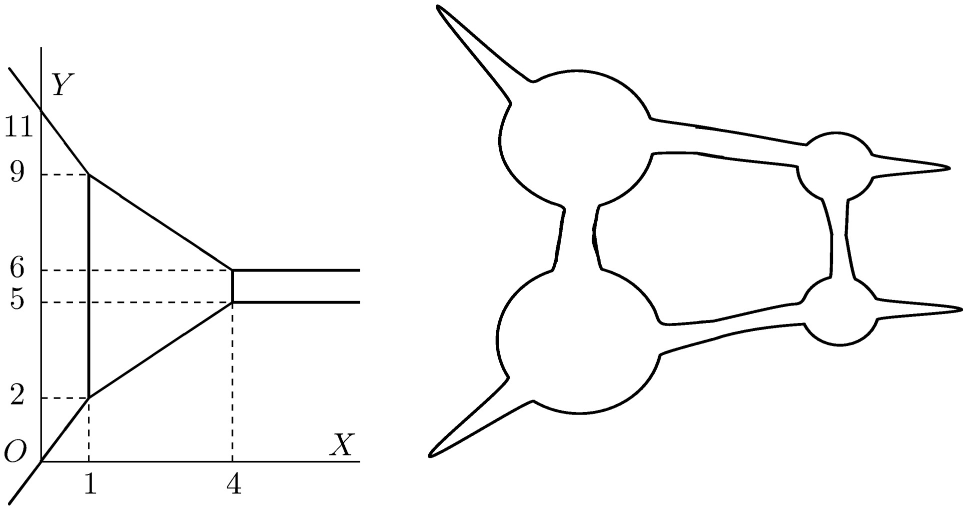


Figure 1. Trop C and C (Example I).

Let C be the algebraic curve defined by Φ. Then it follows that:

$$\operatorname{Trop} C = \{(X, Y) \mid \min[2Y, 2X + Y, X + Y + 1, Y + 5, 20] \text{ is not smooth.}\}$$

(figure 1).

Trop C has four vertices. The vertex $(X, Y) = (1, 2)$ is associated with the points in the curve C that are expressed by $(x, y) = (r\boldsymbol{e}^1, s\boldsymbol{e}^2)$ $(r, s \in R^*)$. Substituting $x = r\boldsymbol{e}^1$ and $y = s\boldsymbol{e}^2$ to $\Phi(x, y) = 0$, we obtain $0 = s^2\boldsymbol{e}^4 + (r^2\boldsymbol{e}^2 + r\boldsymbol{e}^2 + \boldsymbol{e}^5)s\boldsymbol{e}^2 + \boldsymbol{e}^{11} = (s^2 + r^2s + rs)\boldsymbol{e}^4 + s\boldsymbol{e}^7 + \boldsymbol{e}^{11}$. When we take the limit $\varepsilon \to 0^+$ $(\boldsymbol{e} \to 0)$, the factor $s^2 + r^2s + rs$ must be dominant. Now we consider the Riemann surface defined by $s^2 + r^2s + rs = 0$. First note that $r, s \in R^*$, which implies $r, s \neq 0$. The surface $\{s + r^2 + r = 0\}$ is a Riemann sphere with genus 0.

Similarly, vertices $(1, 9)$, $(4, 5)$, $(4, 6)$ are also associated with the following "small" Riemann surfaces respectively:

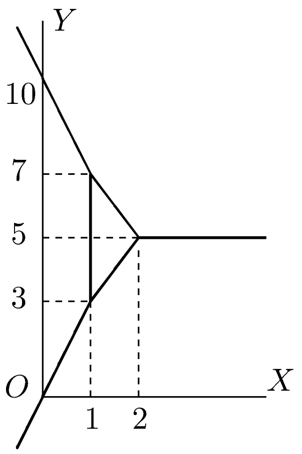
$$r^2s + rs + 1 = 0, \qquad s^2 + rs + s = 0, \qquad rs + s + 1 = 0.$$

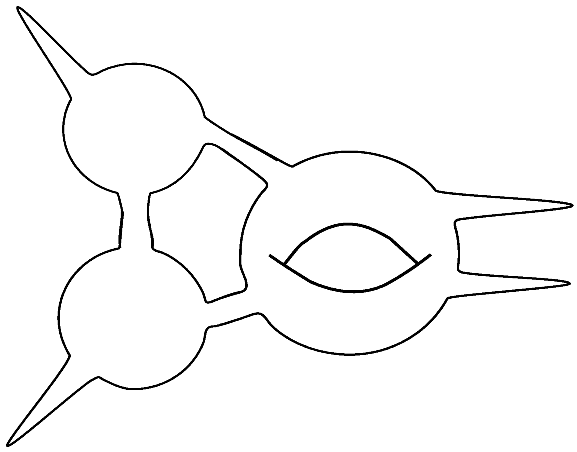
Each curve is of genus 0.

Figure 1 shows us the relation between C and Trop C schematically. Each Riemann sphere is associated with a vertex. Let us call these "small" Riemann surfaces *subsurfaces*.

3.2.2. Example II

$$C:\ \Phi(x, y) = y^2 + (x^3 + \boldsymbol{e}x^2 + \boldsymbol{e}^3x + \boldsymbol{e}^6)y + \boldsymbol{e}^{10} = 0$$





The subsurfaces associated with $(1,3)$, $(1,7)$ and $(2,5)$ are expressed by:

$$s + r^3 + r^2 = 0, \qquad r^3 s + r^2 s + 1 = 0, \qquad s^2 + r^2 s + rs + 1 = 0$$

respectively. The first two are of genus 0, and the third one is of genus 1.

The figure above shows us the relation between C and $\operatorname{Trop} C$. Although the genus of C is 2, the genus of $\operatorname{Trop} C$ is 1.

Definition 3.3. *The algebraic curve C over K is non-degenerate if its subsurfaces are irreducible and of genus* 0.

§ 3.3. Lattice integral

The *lattice integral* is a linear form over a tropical curve, which is defined by the metric over rational graph [6].

Let E be an edge of $\operatorname{Trop} C$. The point $x \in E$ satisfies $E : x = v_0 + tv, (0 \le t \le \ell)$. The vector v_0 is the starting point of E and v is the primitive tangent vector for E. Now we define the *tropical length* of E by $\ell(E) := \ell$.

Let $\beta_1, \dots, \beta_g$ be a basis of $H_1(\operatorname{Trop} C, \mathbb{Z})$. We can define a natural bi-linear form $(\cdot,\cdot)$ over the space of paths on $H_1(\operatorname{Trop} C, \mathbb{Z})$. For this, we define $(r, r) := \ell(r)$ for non-self-intersecting path, and extend it to any pairs of paths bilinearly.

This bilinear form is called *lattice integral* because $(\beta_i, \cdot)$ can be regarded as an element $H_1(\mathbb{Z})^\vee = H^0(\Omega^1)$. ($\Omega^1$ is the cotangent sheaf of $\operatorname{Trop} C$).

A *period matrix* of $\operatorname{Trop} C$ is a $g \times g$ matrix $B_{\operatorname{Trop} C} := \Big((\beta_i, \beta_j)\Big)_{i,j}$.

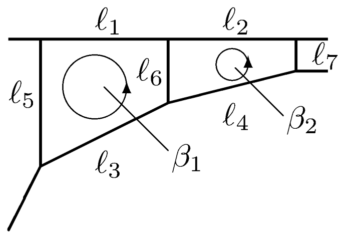


Figure 2. An example of metric graph. It follows that $(\beta_1, \beta_1) = \ell_1 + \ell_3 + \ell_5 + \ell_6$, $(\beta_2, \beta_2) = \ell_2 + \ell_4 + \ell_6 + \ell_7$, $(\beta_1, \beta_2) = (\beta_2, \beta_1) = -\ell_6$.

§ 3.4. Complex integrals

Now we study the relation between the lattice integrals and the usual complex integrals. For this, we consider the simplest case here.

Definition 3.4. *Let C be a curve given by the zeros of $\Phi \in K[x, y]$. When C satisfies the following conditions, we said that C has a good tropicalization.*
1) *C is non-degenerate,* 2) $\mathrm{Trop}\, C$ *is regular,* 3) *generic condition.*

Generic condition is an assumption for the coefficients of $\Phi \in K[x, y]$. It may happens that C becomes singular or has some strange property when these coefficients take special values. We assume the generic condition to avoid these situations. See [4],§3, for details.

Next we proceed to the complex integrals. Let C be a Riemann surface of genus g, and $\alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g \in H_1(C, \mathbb{Z})$ be a canonical basis. When choosing β_i, we make them have the same position and orientation as $\beta_1, \ldots, \beta_g \in H_1(\mathrm{Trop}, \mathbb{Z})$ on $\mathrm{Trop}\, C$ (see the figure below). Define the holomorphic differentials $\omega_1, \ldots, \omega_g$ on C s.t. $\int_{\alpha_i} \omega_j = \delta_{i,j}$, and the $g \times g$ matrix $B_C := (\int_{\beta_i} \omega_j)_{i,j}$.

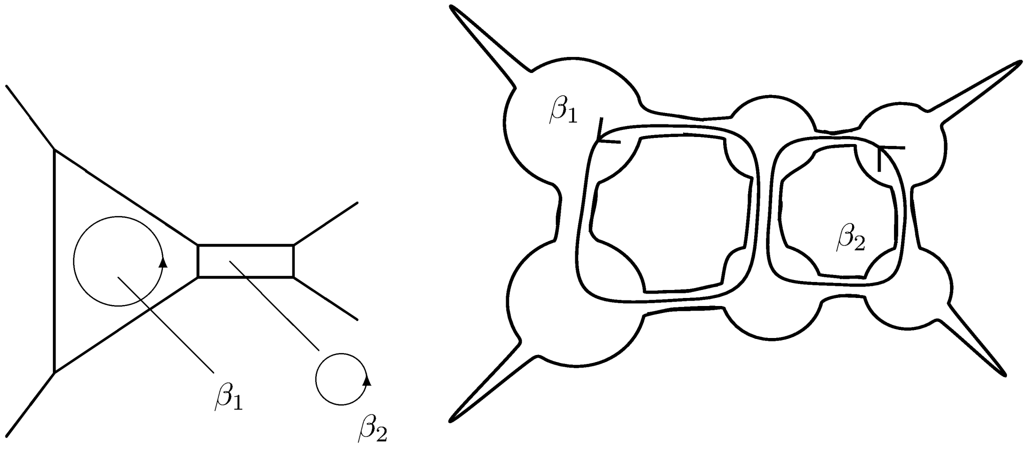
The following is the main result of [4].

Theorem 3.1. *When C has a good tropicalization, it follows that*

$$B_C \sim \frac{-1}{2\pi \mathrm{i} \varepsilon} B_{\mathrm{Trop}\, C} \qquad (\varepsilon \to 0^+).$$

Corollary 3.2. *Let γ be an oriented path on C. When we define a path γ (abuse of symbol) on* $\mathrm{Trop}\, C$ *such that it has the same position and orientation as $\gamma_{\mathrm{on}\, C}$, then* :

$$\int_\gamma \omega_i \sim \frac{-1}{2\pi \mathrm{i} \varepsilon} (\gamma, \beta_i) \qquad (\varepsilon \to 0^+).$$



§ 4. Application for pBBSs

In this section, we study the method to solve the problems about pBBSs by applying the results of theorem 2.1 and theorem 3.1.

§ 4.1. Abel-Jacobi mapping

Let $Q_n^0, Q_n^{\frac{1}{M}}, \dots, Q_n^{\frac{M-1}{M}}, W_n^0$ be an initial condition of pBBS. Using this, we define the initial condition of hpd Toda equation by $I_n^{\frac{k}{M}} := \kappa_{n,k}\, \boldsymbol{e}^{Q_n^{k/M}}$, $V_n^0 := \lambda_n\, \boldsymbol{e}^{W_n^0}$ $(k = 0, 1, \dots, M-1)$, where $\kappa_{n,k}$ and λ_n are arbitrary elements of R^*.

Let C be the spectral curve defined by the initial data. According to section 2, the eigenvector mapping φ_C sends the matrix $X(y)$ (2.2) into $\mathrm{Pic}^d C$. We can have the general effective divisor $\mathcal{D}$ such that $\varphi_C(X(y)) = \mathcal{D} + (N-1)Q$.

Fix a point $P_0 \in C$. Let $\mathrm{Div}_+^g(C)$ be the set of effective divisors of degree g on C. Define the *Abel-Jacobi mapping* $\boldsymbol{A} : \mathrm{Div}_+^g(C) \to \mathbb{C}^g/(\mathbb{Z}^g + B_C\mathbb{Z}^g)$ by

$$\boldsymbol{A}(D) := \sum_{i=1}^{g} \left(\int_{P_0}^{P_i} \omega_1, \int_{P_0}^{P_i} \omega_2, \dots, \int_{P_0}^{P_i} \omega_g \right).$$

The image of $\boldsymbol{A}$ does not depend on the choice of integral paths. By the classical result, if D is a general divisor, it is possible to determine the original divisor D uniquely from the vector $\boldsymbol{A}(D)$. Therefore, to analyze the divisor $\mathcal{D} + (N-1)Q$ and to analyze the vector $\boldsymbol{A}(\mathcal{D})$ is equivalent.

Let $\mathcal{D}(X(y))$ be the general effective divisor expressed by $\varphi_C(X(y)) - (N-1)Q$. By the statements of theorem 2.1 and corollary 2.2, the actions of σ and τ are expressed by the following equation:

$$\boldsymbol{A}\{\mathcal{D}(\sigma^r \circ \tau^s(X(y)))\} = \boldsymbol{A}\{\mathcal{D}(X(y))\} + r\boldsymbol{v}_n + s\boldsymbol{v}_t \quad \in \mathbb{C}^g/(\mathbb{Z}^g + B_C\mathbb{Z}^g), \tag{4.1}$$

where $\boldsymbol{v}_n = \left(\int_Q^P \omega_1, \dots, \int_Q^P \omega_g\right)$, $\quad \boldsymbol{v}_t = \sum_{i=0}^{M-1} \left(\int_P^{A_i} \omega_1, \dots, \int_P^{A_i} \omega_g\right)$.

§ 4.2. Fundamental and relative period of pBBS

Definition 4.1. *The fundamental period of pBBS is the number of minimal time steps until the state of pBBS returns to the same state as the initial condition.*

Although it is possible to calculate the fundamental periods of pBBSs, we consider another "period" here for simplicity.

Definition 4.2. *The relative period of pBBS is the minimal positive integer* $T > 0$ *such that* $\exists q \in \mathbb{Z} \Rightarrow Q_{n+q}^{t+T} \equiv Q_n^t$ *and* $W_{n+q}^{t+T} \equiv W_n^t$.

Remark 4.1. *The fundamental period can be calculated from the relative period by combinatorial methods [9].*

Equation (4.1) shows us

$$\varphi_C(\sigma^q \circ \tau^T(X(y))) = \varphi_C(X(y)) \quad \Leftrightarrow \quad q\boldsymbol{v}_n + T\boldsymbol{v}_t \equiv 0 \quad (\mathrm{mod}\ (\mathbb{Z}^g + B_C\mathbb{Z}^g)).$$

§ 4.3. How to calculate the relative period

In this section, we study the method to calculate the relative periods. As a concrete example, consider the state of pBBS below:

```
t=0      11...1112....111...11111......
  1      ..11.....1112...111.....11111.
  2      11..1111.....1112..111.......1
  3      ..11....11111....112..1111....
```

Then we have

$$(Q_1^0,\, Q_1^{1/2},\, Q_2^0,\, Q_2^{1/2},\, Q_3^0,\, Q_3^{1/2},\, Q_4^0,\, Q_4^{1/2}) = (2,0,3,1,3,0,5,0),$$
$$(W_1^0, W_2^0, W_3^0, W_4^0) = (3,4,3,6).$$

(Note that we can define the "leftmost" soliton arbitrarily). Define $I_1^0 = \kappa_{1,0}\,\boldsymbol{e}^2$, $I_1^{1/2} = \kappa_{1,1}\,\boldsymbol{e}^0$; $I_2^0 = \kappa_{2,0}\,\boldsymbol{e}^3, \dots$; $V_1^0 = \lambda_1\boldsymbol{e}^3, V_1^0 = \lambda_2\boldsymbol{e}^4, \dots$ *etc.* $(\kappa_{n,0}, \kappa_{n,1}, \lambda_n \in R^*)$.

Therefore, the determinant $\Phi(x,y) = \det(X(y) - xE)$ can be expanded:

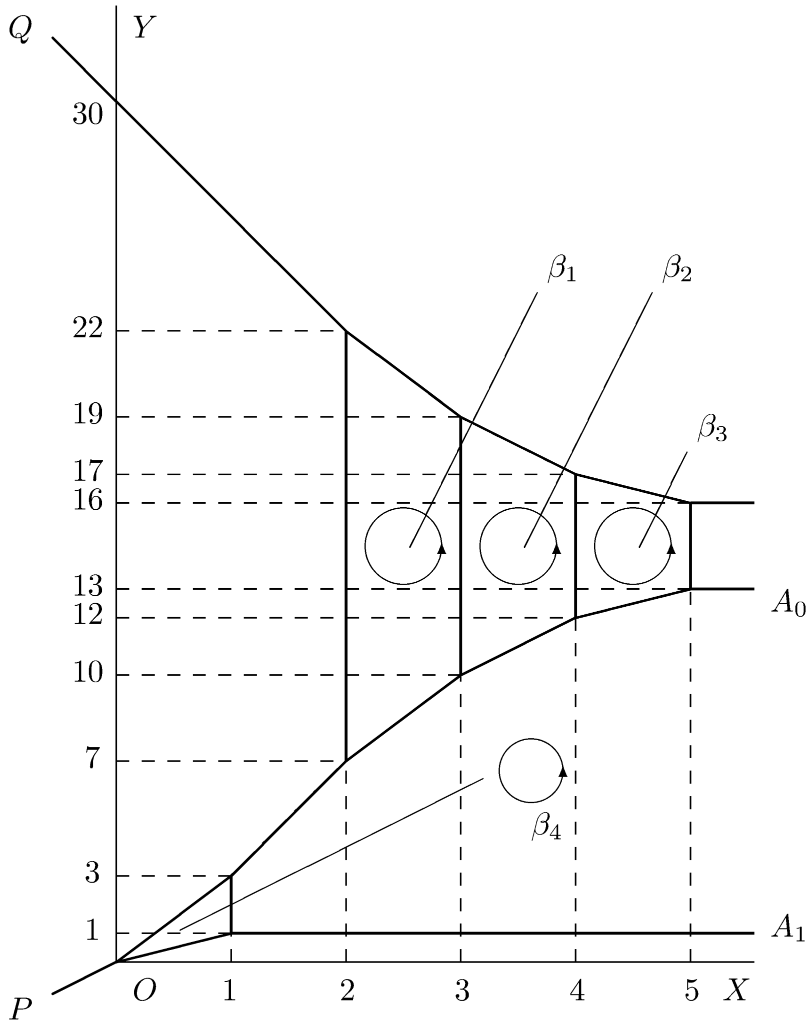
$$\Phi = y^2 - 2(x^2 + r_{2,1}\boldsymbol{e}^0 x + r_{2,0}\boldsymbol{e}^1)y + (x^4 + r_{1,3}\boldsymbol{e}^2 x^3 + r_{1,2}\boldsymbol{e}^5 x^2 + r_{1,1}\boldsymbol{e}^9 x + r_{1,0}\boldsymbol{e}^{14}) - r\boldsymbol{e}^{30} y^{-1}$$

$(r_{i,j} \in R^*,\ \forall i,j)$.

Because the field K is algebraically closed, we can rewrite Φ into:

$$\begin{aligned}\Phi &= y^2 - 2(x+u_{2,0}\boldsymbol{e}^0)(x+u_{2,1}\boldsymbol{e}^1)y \\ &\quad + (x+u_{1,0}\boldsymbol{e}^2)(x+u_{1,1}\boldsymbol{e}^3)(x+u_{1,2}\boldsymbol{e}^4)(x+u_{1,3}\boldsymbol{e}^5) - r\boldsymbol{e}^{30}y^{-1}\end{aligned} \tag{4.2}$$

$(u_{i,j} \in R^*)$. Denote the algebraic curve $\{\Phi = 0\}$ by C. Hereafter we consider the tropical curve $\mathrm{Trop}\, C$. See the figure in the previous page.



The point $A_0 : (x,y) = \left(0, \prod_{n=1}^{4} I_n^0\right) \in C$ is associated with the point $(X,Y) = (\mathrm{val}\,(0), \mathrm{val}\,(\prod_{n=1}^{4} I_n^0)) = (+\infty, 13)$ on $\mathrm{Trop}\, C$. We will abuse the symbol A_0 to mean the point on the tropical curve. Similarly, we use (abuse) the symbol A_1, P and Q as $A_1 : (X,Y) = (+\infty, 1)$, $P : (X,Y) = (-\infty,-\infty)$, $Q : (X,Y) = (-\infty,+\infty) \in \mathrm{Trop}\, C$. $(\mathrm{val}\,(\infty) = -\mathrm{val}\,(0) = -\infty)$.

Fix oriented paths $\gamma_{Q\to P}, \gamma_{P\to A_0}, \gamma_{P\to A_1}, \subset \operatorname{Trop} C$ which connect two points $Q \to P$, $P \to A_0$, $P \to A_1$ respectively.

Then we obtain the following data from the figure of $\operatorname{Trop} C$.

$$B_{\operatorname{Trop} C} = \begin{pmatrix} 26 & -9 & 0 & 0 \\ -9 & 16 & -5 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix},$$

$$\boldsymbol{v}_{n,\operatorname{Trop} C} := \big((\beta_i, \gamma_{Q\to P})\big)_i \equiv (15,0,0,1) \pmod{B_{\operatorname{Trop} C}\,\mathbb{Z}^4},$$
$$\boldsymbol{v}_{t,\operatorname{Trop} C} := \big((\beta_i, \gamma_{P\to A_0} + \gamma_{P\to A_1})\big)_i \equiv (1,1,1,0) \pmod{B_{\operatorname{Trop} C}\,\mathbb{Z}^4}.$$

The vectors

$$\boldsymbol{v}_{n,\operatorname{Trop} C}, \quad \boldsymbol{v}_{t,\operatorname{Trop} C} \pmod{B_{\operatorname{Trop} C}\,\mathbb{Z}^4}$$

do not depend on the choice of $\gamma_{Q\to P}, \gamma_{P\to A_i}$.

Here we note the relation:

$$\begin{aligned} N\,\boldsymbol{v}_{n,\operatorname{Trop} C} &= 4\,\boldsymbol{v}_{n,\operatorname{Trop} C} \equiv (60,0,0,4) \\ &= 3\,(26,-9,0,0) + 2\,(-9,16,-5,0) + (0,-5,10,0) + (0,0,0,4) \\ &\equiv (0,0,0,0) \pmod{B_{\operatorname{Trop} C}\,\mathbb{Z}^4}, \end{aligned}$$

that reflects the fact that the action $\sigma : n \mapsto n+1$ satisfies $\sigma^N = \mathrm{id}$.

Assume C has a good tropicalization. Theorem 3.1 and corollary 3.2 imply

$$B_C \sim \frac{-1}{2\pi \mathrm{i} \varepsilon} B_{\operatorname{Trop} C},$$
$$\boldsymbol{v}_n \sim \frac{-1}{2\pi \mathrm{i} \varepsilon} \boldsymbol{v}_{n,\operatorname{Trop} C} \pmod{B_{\operatorname{Trop} C}\,\mathbb{Z}^4},$$
$$\boldsymbol{v}_t \sim \frac{-1}{2\pi \mathrm{i} \varepsilon} \boldsymbol{v}_{t,\operatorname{Trop} C} \pmod{B_{\operatorname{Trop} C}\,\mathbb{Z}^4}.$$

Substituting this relation into the last equation $q\boldsymbol{v}_n + T\boldsymbol{v}_t \equiv 0 \pmod{(\mathbb{Z}^g + B_C \mathbb{Z}^g)}$, we obtain

$$\begin{aligned} & q\boldsymbol{v}_n + T\boldsymbol{v}_t \equiv 0 \pmod{(\mathbb{Z}^4 + B_C \mathbb{Z}^4)} \\ & \qquad \Leftrightarrow\ \exists \boldsymbol{r}', \boldsymbol{r} \in \mathbb{Z}^4 \quad \text{s.t.} \quad q\boldsymbol{v}_n + T\boldsymbol{v}_t = \boldsymbol{r}' + B_C \boldsymbol{r} \\ & \qquad \Rightarrow\ \exists \boldsymbol{r} \in \mathbb{Z}^4 \quad \text{s.t.} \quad \frac{-1}{2\pi \mathrm{i} \varepsilon}(q\boldsymbol{v}_{n,\operatorname{Trop} C} + T\boldsymbol{v}_{t,\operatorname{Trop} C}) = \frac{-1}{2\pi \mathrm{i} \varepsilon} B_{\operatorname{Trop} C} \boldsymbol{r} \\ & \qquad \Leftrightarrow\ q\boldsymbol{v}_{n,\operatorname{Trop} C} + T\boldsymbol{v}_{t,\operatorname{Trop} C} \equiv 0 \pmod{B_{\operatorname{Trop} C} \mathbb{Z}^4}. \end{aligned} \tag{4.3}$$

(The second arrow is '$\Rightarrow$'. See section 5.)

Equation (4.3) is equivalent to:

$$q\,(B_{\operatorname{Trop} C}^{-1} \cdot \boldsymbol{v}_{n,\operatorname{Trop} C}) + T\,(B_{\operatorname{Trop} C}^{-1} \cdot \boldsymbol{v}_{t,\operatorname{Trop} C}) \equiv 0 \pmod{\mathbb{Z}^4}.$$

Therefore,

$$q\,(3/4,\ 1/2,\ 1/4,\ 1/4) + T\,(1/10,\ 8/45,\ 17/90,\ 0) \equiv 0 \pmod{\mathbb{Z}^4}.$$

From this, we can conclude that the minimal positive integer $T > 0$ which satisfies equation (4.3) is 90. In fact, the relative period of given state of pBBS is 90 (see Appendix A).

§ 5. Concluding remark

We have studied the algebraic geometrical and tropical geometrical method to analyze the pBBSs in the previous section. Through this technique, we can obtain automatically some of important results, for example, the fundamental period formula of the pBBS with one kind of balls [9].

However, we have admitted some assumptions in this article. To justify our theory, we must deal with the following subjects.

(1) Regularity of $\operatorname{Trop} C$.
To apply theorem 3.1, $\operatorname{Trop} C$ must be regular. However, some special initial condition makes $\operatorname{Trop} C$ non-regular. Because regularity is essential for the proof of theorem 3.1, this theorem fails in this situation.

(2) Generic condition.
Theorem 3.1 also requires that the coefficients of $\Phi(x,y)$ (4.2) must satisfy some generic condition. We do not describe in detail about this condition here, but it is sufficient to prove:

Conjecture 5.1. *The coefficients of* $\Phi(x,y) = \det\,(X(y) - xE)$ *are functionally independent as the functions of* $I_n^0, \ldots, I_n^{\frac{M-1}{M}}, V_n^0$.

This conjecture should be true if we want to claim "pBBS is a completely integrable system", but the author does not know the proof.

(3) Does the torus $\mathbb{R}^g / B_{\operatorname{Trop} C}\mathbb{Z}^g$ have enough information of pBBS?
Fix a spectral curve C. Let $\mathcal{T}_C$ be the iso-spectral set (section 2.1). Denote the set of initial conditions of pBBSs which give the tropical curve $\operatorname{Trop} C$ by $\mathfrak{T}_{\operatorname{Trop} C}$.

Let us consider the composite:

$$\mathfrak{T}_{\operatorname{Trop} C} \underset{**}{\overset{*}{\leftrightarrows}} \mathcal{T}_C \overset{\varphi_C}{\hookrightarrow} \operatorname{Pic}^d C \xrightarrow{\boldsymbol{A}} \mathbb{C}^g/(\mathbb{Z}^g + B_C\mathbb{Z}^g) \overset{***}{\to} \mathbb{R}^g/B_{\operatorname{Trop} C}\mathbb{Z}^g$$

'$\overset{*}{\leftarrow}$' is ultradiscretization, '$\overset{**}{\to}$' is an operation to define an initial condition of hpd Toda eq. from an initial condition of pBBS (section 4.1), '$\overset{***}{\to}$' is an operation to pick up the coefficient of $-1/(2\pi\mathrm{i}\varepsilon)$. Note that '$\overset{**}{\to}$' depends on the choice of elements R^*.

The questions are:

Q.1. Is $\chi : \mathfrak{T}_{\mathrm{Trop}\, C} \to \mathbb{R}^g / B_{\mathrm{Trop}\, C}\mathbb{Z}^g$ well-defined?

Q.2. What is the image of χ? Is χ injective?

If these are true, all arrow in (4.3) (section 4.3) become "$\Leftrightarrow$".

Remark 5.1. *For $M = 1$, R. Inoue and T. Takenawa answered these questions in [2]. In fact, they proved that χ is an isomorphism.*

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§ Appendix A.

```
t= 0    11...1112....111...11111......
        ..11.....1112...111.....11111.
        11..1111.....1112..111.......1
        ..11....11111....112..1111....
        1...11.......1111...112...1111
        .111..1111.......111...1112...
        2...11....11111.....111....111
        .1112.11.......11111...111....
```

```
        1....12.1111........111...1111
        .1111..12...11111......111....
t=10    .....11..1112....11111....111.
        111....11....1112.....1111...1
        ...1111..111.....1112.....111.
        11.....11...11111....1112....1
        ..111....11......1111....11112
        112..1111..111.......1111.....
        ...112...11...11111......1111.
        111...112..11......11111.....1
        ...111...112.111........11111.
        111...111...12..11111........1
t=20    ...111...111..112....11111....
        1.....111...11...1112.....1111
        .11111...111..11.....1112.....
        ......111...11..11111....1112.
        1112.....111..11.....1111....1
        ....11112...11..111......1111.
        111......1112.11...1111......1
        ...1111......12.111....11111..
        111....1111....12..111......11
        ...1111....1111..112..111.....
t=30    .......1111....11...112..11111
        11111......1111..11....112....
        .....11111.....11..1111...112.
        112.......11111..11....111...1
        ...11112.......11..1111...111.
        111.....11112....11....111...1
        ...1111......11112.11.....111.
        11.....1111.......12.11111...1
        ..11111....1111.....12....111.
        11.....1111....11111..12.....1
t=40    ..111......1111.....11..11112.
        112..1111......1111...11.....1
        ...112...11111.....111..111...
        1.....112.....11111...11...111
        .11111...112.......111..111...
        1.....111...11112.....11...111
```

```
        .1111....111.....11112..11....
        .....1111...111.......112.1111
        11111....111...1111......12...
        .....1111...111....11111...12.
t=50    112......111...1111.....111..1
        ...11112....111....1111....11.
        1.......11112..111.....1111..1
        .1111........112..11111....11.
        1....11111......112....1111..1
        .1111.....11111....112.....11.
        1....1111......1111...1112...1
        .111.....1111......111....1112
        12..11111....1111.....111.....
        ..12.....1111....11111...111..
t=60    11..112......1111.....111...11
        ..11...11112.....1111....111..
        1...11......11112....1111...11
        .111..111........11112...111..
        1...11...11111........1112..11
        .111..111.....11111.......112.
        12..11...1111......11111.....1
        ..112.11.....1111.......11111.
        11...12.11111....1111........1
        ..111..12....1111....11111....
t=70    1....11..112.....1111.....1111
        .1111..11...1112.....1111.....
        .....11..111....11112....1111.
        111....11...111......11112...1
        ...1111..111...111........1112
        1112...11...111...11111.......
        ....1112.11....111.....11111..
        111.....12.1111...111.......11
        ...11111..12...111...1111.....
        ........11..1112..111....11111
t=80    11111.....11....112..1111.....
        .....11111..11.....112...1111.
        111.......11..11111...112....1
        ...1111.....11.....111...11112
```

```
       1112...11111..11......111.....
       ....1112....11..11111....111..
       11......1112..11.....1111...11
       ..11111.....112.111......111..
       1......11111...12..1111.....11
       .111.......111..112...11111..
t=90   1...11111......11...1112....11
```