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Heat-flow monotonicity underlying some sharp inequalities in geometric and harmonic analysis

By

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Abstract

The intention of this article is to provide a summary of recent collaborative work of the author on heat-flow monotonicity underlying certain fundamental inequalities in euclidean geometric and harmonic analysis. The paradigm is to allow the input function (or functions) to evolve according to certain nonlinear heat-flow and ask whether the induced quantity is monotone for all positive time.

§ 1. Introduction

For $d \in \mathbb{N}$ and $t > 0$ let H_t denote the heat kernel on $\mathbb{R}^d$ given by

$$H_t(x) := \frac{1}{t^{d/2}} e^{-\pi|x|^2/t}.$$

Certain inequalities which are intrinsically geometric are known to be underpinned by an associated monotone quantity which arises by allowing the (nonnegative) input functions to evolve under nonlinear heat-flow of the form

$$f \longmapsto (H_t * f^p)^{1/p} \tag{1.1}$$

for some $p > 0$. We illustrate this with the celebrated geometric Brascamp–Lieb inequality of Ball [3] and Barthe [5]; a powerful inequality which counts the multilinear Hölder and Loomis–Whitney inequalities as special cases. For $j = 1, \ldots, m$ let $d_j \in \mathbb{N}$, $p_j \geq 1$ and let $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ be a linear mapping such that $B_j^* B_j$ is a projection and

$$\sum_{j=1}^{m} \frac{1}{p_j} B_j^* B_j = I_d,$$

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where I_d is the identity mapping on $\mathbb{R}^d$. Then the geometric Brascamp–Lieb inequality,

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j(B_j x)\, dx \leq \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^{d_j})} \tag{1.2}$$

for nonnegative functions $f_j \in L^{p_j}(\mathbb{R}^{d_j})$, is a consequence of the nondecreasingness of $Q : (0, \infty) \to (0, \infty)$ given by

$$Q(t) = \int_{\mathbb{R}^d} \prod_{j=1}^m u_j(t, B_j x)^{1/p_j}\, dx.$$

Here, $u_j : (0, \infty) \times \mathbb{R}^{d_j} \to (0, \infty)$ is given by

$$u_j(t, \cdot) = H_t * f_j^{p_j} \tag{1.3}$$

and thus solves the heat equation $\partial_t u_j = \frac{1}{4\pi} \Delta u_j$ on $\mathbb{R}^{d_j}$ with initial data $f_j^{p_j}$. In particular, if each function f_j is sufficiently well-behaved (such as bounded with compact support) then

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j(B_j x)\, dx = \lim_{t \to 0} Q(t) \leq \lim_{t \to \infty} Q(t) = \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^{d_j})}.$$

This type of heat-flow proof of the geometric Brascamp–Lieb inequality is due to Carlen, Lieb and Loss [19] in the case of rank one mappings and Bennett, Carbery, Christ and Tao [16] in the general rank case. We remark that a closely related argument shows that the geometric Brascamp–Lieb inequality is recoverable from a monotone quantity in which the input functions evolve according to the "Mehler-flow" $\widetilde{u}_j : (0, \infty) \times \mathbb{R}^{d_j} \to (0, \infty)$ which satisfies

$$\partial_t \widetilde{u}_j = \tfrac{1}{2\pi} \Delta \widetilde{u}_j + \langle x, \widetilde{u}_j \rangle + d_j \widetilde{u}_j.$$

See [8] for a proof of this observation in the rank one case; the general rank case follows by a straightforward modification of the argument.

A somewhat different use of heat-flow as a tool for proving (1.2) can be found in [10]. The approach in [10], inspired by work of Borell, simultaneously provides a proof of the reverse geometric Brascamp–Lieb inequality due to Barthe [4], [5]. The survey article [7] contains further discussion of this alternative type of heat-flow approach to geometric inequalities.

Another significant instance of the fruitfulness of the heat-flow monotonicity approach outlined here is the proof of the multilinear Kakeya maximal inequalities in [17]. The technique has also proved successful outside the euclidean realm. In [19] and [20] respectively, Carlen, Lieb and Loss adapted the technique to prove certain multilinear inequalities in the spirit of (1.2) via heat-flows on the sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ and the permutation group S_d on d letters (see also [9] for an extension in the spherical case).

Heuristically, the flow $u_j(t,x)$ in (1.3) asymptotically behaves like $\|f_j\|_{L^{p_j}}^{p_j} H_t(x)$ for large time t and consequently it is well-suited to inequalities which are sharp when evaluated on centred gaussians. Classical examples of which are "interior cases" of the Young convolution inequality on $\mathbb{R}^d$; that is, (by duality) the inequality

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_1(x)f_2(y)f_3(x-y)\,dxdy \leq C\prod_{j=1}^{3}\|f_j\|_{L^{p_j}(\mathbb{R}^d)} \tag{1.4}$$

for nonnegative functions $f_j \in L^{p_j}(\mathbb{R}^d)$ and $p_j \in (1,\infty)$ such that $\sum_{j=1}^{3}\frac{1}{p_j} = 2$. In this case, $(\frac{1}{p_1},\frac{1}{p_2},\frac{1}{p_3'})$ lies on interior of the triangle, T, whose vertices lie at $(0,1,0)$, $(1,0,0)$ and $(1,1,1)$ and it is known that the sharp constant in (1.4) may be expressed as

$$C = \|H_{\sigma_1}^{1/p_1} * H_{\sigma_2}^{1/p_2}\|_{L^{p_3'}(\mathbb{R}^d)}$$

(due to Beckner [11], [12] and Brascamp and Lieb [18]). Here, $\sigma_1,\sigma_2 > 0$ satisfy $\frac{1}{p_1}(1-\frac{1}{p_1})\sigma_2 = \frac{1}{p_2}(1-\frac{1}{p_2})\sigma_1$ and p_3' is the conjugate exponent to p_3. For such exponents, via a change of variables the inequality in (1.4) sits under the umbrella of the geometric Brascamp–Lieb inequality and is therefore recoverable from a monotone quantity which arises from a modification of the heat-flow in (1.3). See [19] and [16] for further details.

In Section 2 of this article we give a unified and direct heat-flow monotonicity treatment of the Young convolution inequality on $\mathbb{R}^d$ and its reverse form. With nonsharp constant the reverse form was first noticed by Leindler [25] and the sharp form was proved by Brascamp and Lieb [18]. For sufficiently well-behaved functions $f_j \in L^{p_j}(\mathbb{R}^d)$ we show that the norm

$$\|f_1 * f_2\|_{L^p(\mathbb{R}^d)}$$

exhibits monotonicity as each f_j evolves according to heat-flow of form (1.1). In particular, provided $\frac{1}{p_1}+\frac{1}{p_2} = 1+\frac{1}{p}$ the induced quantity is nondecreasing for $p_1,p_2 \geq 1$ and nonincreasing for $p_1,p_2 \leq 1$.

In Section 3 we pursue the applicability of the paradigm in the context of the Strichartz space-time estimates for the homogeneous Schrödinger equation. When the Lebesgue space exponents for which such estimates hold conspire to allow us to "multiply out" the Strichartz norm we observe a rather dramatic monotonicity under the flow (1.1) with $p = 2$.

Finally, in Section 4 we consider the classical Hausdorff–Young inequality on $\mathbb{R}^d$. Whenever the conjugate exponent p' is an even integer the $L^{p'}(\mathbb{R}^d)$ norm of the Fourier transform of f is nondecreasing as the function f evolves under the flow in (1.1). This follows from [16] since the multiplied out expression for the norm coincides with a geometric Brascamp–Lieb inequality via a change of variables. However, we produce explicit counterexamples to show that this monotonicity property fails substantially whenever

$p' > 2$ is not an even integer. We remark that such considerations are reasonable given Beckner's famous theorem on the gaussian extremisability of the Hausdorff–Young inequality [11], [12].

§ 2. Convolution inequalities

Let $d \in \mathbb{N}$. Suppose $0 < p_1, p_2, p < \infty$ satisfy the scaling condition

$$\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p} \tag{2.1}$$

and $\sigma_1, \sigma_2 \geq 0$ satisfy the relation

$$\frac{1}{p_1}\left(1 - \frac{1}{p_1}\right)\sigma_2 = \frac{1}{p_2}\left(1 - \frac{1}{p_2}\right)\sigma_1. \tag{2.2}$$

Let $Q : (0, \infty) \to (0, \infty)$ be given by

$$Q(t) = \|u_1(t, \cdot)^{1/p_1} * u_2(t, \cdot)^{1/p_2}\|_{L^p(\mathbb{R}^d)},$$

where $u_j : (0, \infty) \times \mathbb{R}^d \to (0, \infty)$ is given by

$$u_j(t, \cdot) = H_{\sigma_j t} * f_j^{p_j} \tag{2.3}$$

for some nonnegative $f_j \in L^{p_j}(\mathbb{R}^d)$. Thus, u_j satisfies the heat equation

$$\partial_t u_j = \frac{\sigma_j}{4\pi}\Delta u_j.$$

Theorem 2.1 (Bennett, B. [13]). *If $p_1, p_2 \geq 1$ then $Q(t)$ is nondecreasing for each $t > 0$ and if $p_1, p_2 \leq 1$ then $Q(t)$ is nonincreasing for each $t > 0$.*

Notice that Theorem 2.1 contains certain "boundary cases". In particular, we are including the exponents corresponding to the boundary of the triangle T, defined in the Introduction, with the exception of the vertices $(0, 1, 0)$, $(1, 0, 0)$ and their connecting edge.

One can show that

$$\lim_{t \to 0} Q(t) = \|f_1 * f_2\|_{L^p(\mathbb{R}^d)} \quad \text{and} \quad \lim_{t \to \infty} Q(t) = C\|f_1\|_{L^{p_1}(\mathbb{R}^d)}\|f_2\|_{L^{p_2}(\mathbb{R}^d)},$$

at least for bounded and compactly supported f_j, and the constant C is given by

$$C = \|H_{\sigma_1}^{1/p_1} * H_{\sigma_2}^{1/p_2}\|_{L^p(\mathbb{R}^d)}.$$

Consequently, from Theorem 2.1 we recover the sharp Young convolution inequality and its reverse form which state that, for nonnegative functions $f_j \in L^{p_j}(\mathbb{R}^d)$, the difference

$$C\|f_1\|_{L^{p_1}(\mathbb{R}^d)}\|f_2\|_{L^{p_2}(\mathbb{R}^d)} - \|f_1 * f_2\|_{L^p(\mathbb{R}^d)} \tag{2.4}$$

is nonnegative if $p_1, p_2 \geq 1$ and nonpositive if $p_1, p_2 \leq 1$.

In the admitted boundary cases where exactly one of p_1 and p_2 is equal to 1, say p_j, (2.2) implies that σ_j vanishes and thus the flow u_j is constant in time. Formally substituting the Dirac delta distribution supported at the origin for the heat kernel H_t at time zero, we see that $C = 1$ in this case. Moreover, for such exponents the monotonicity in Theorem 2.1 is strict (this agrees with the known fact that extremisers do not exist in the corresponding Young convolution inequality) and directly follows from the following explicit formula, a by-product of our proof of Theorem 2.1.

$$\begin{aligned} Q'(t) =& \frac{\varepsilon}{8\pi Q(t)^{p-1}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u_1^{1/p_1} * u_2^{1/p_2})(x)^{p-2} u_1(x-y)^{1/p_1} u_2(y)^{1/p_2} \times \\ & u_1(x-z)^{1/p_1} u_2(z)^{1/p_2} \left| \left(\tfrac{\sigma_1}{p_1}|\tfrac{1}{p_1}-1|\right)^{1/2} \tfrac{\nabla u_1}{u_1}(x-y) + \left(\tfrac{\sigma_2}{p_2}|\tfrac{1}{p_2}-1|\right)^{1/2} \tfrac{\nabla u_2}{u_2}(y) \right. \\ & \left. - \left(\tfrac{\sigma_1}{p_1}|\tfrac{1}{p_1}-1|\right)^{1/2} \tfrac{\nabla u_1}{u_1}(x-z) - \left(\tfrac{\sigma_2}{p_2}|\tfrac{1}{p_2}-1|\right)^{1/2} \tfrac{\nabla u_2}{u_2}(z) \right|^2 \, dxdydz \end{aligned}$$

for each $t > 0$. Here ε is defined to be 1 if $p_1, p_2 \geq 1$ and -1 if $p_1, p_2 \leq 1$, and we have suppressed the t-variable in the integrand. Furthermore, the known characterisation of extremals for which the quantity in (2.4) is zero is fully recoverable from the above expression for $Q'(t)$.

Proof of Theorem 2.1. For $j = 1, 2$ let v_j denote the time dependent vector field on $\mathbb{R}^d$ given by $v_j = \nabla \log u_j$. Let $u : (0, \infty) \times \mathbb{R}^d \to (0, \infty)$ be given by

$$u^{1/p} = u_1^{1/p_1} * u_2^{1/p_2} \tag{2.5}$$

and let σ be given by

$$\sigma p = \sigma_1 p_1 + \sigma_2 p_2. \tag{2.6}$$

We claim that $\partial_t u - \frac{\sigma}{4\pi}\Delta u$ is nonnegative when $p_1, p_2 \geq 1$ and nonpositive when $p_1, p_2 \leq 1$. Since $Q(t)^p = \int u(t, \cdot)$ the claimed monotonicity in Theorem 2.1 follows by differentiating through the integral and an application of the divergence theorem.

To see the claim, first observe that

$$\begin{aligned} & \frac{1}{u^{(p-2)/p}} \left[4\pi \partial_t u - \tfrac{1}{p}(\sigma_1 p_1 + \sigma_2 p_2) \Delta u \right] \\ &= \tfrac{p\sigma_1}{p_1}(1 - \tfrac{1}{p_1}) u^{1/p} (u_1^{1/p_1} |v_1|^2 * u_2^{1/p_2}) + \tfrac{p\sigma_2}{p_2}(1 - \tfrac{1}{p_2}) u^{1/p} (u_1^{1/p_1} * u_2^{1/p_2} |v_2|^2) + \\ & \tfrac{1}{p_1 p_2}(\sigma_1(p - p_1) + \sigma_2(p - p_2)) u^{1/p} (u_1^{1/p_1} v_1 * u_2^{1/p_2} v_2) - (p-1)(\sigma_1 p_1 + \sigma_2 p_2) |\nabla(u^{1/p})|^2, \end{aligned}$$

where we have used a freedom afforded by the fact that the derivative of a convolution of two (suitable) functions may equally well land on either. The proof completes by noticing that the right-hand side of the above expression evaluated at $(t,x)\in(0,\infty)\times\mathbb{R}^d$ coincides with

$$\frac{\varepsilon}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u_1(t,x-y)^{1/p_1}u_2(t,y)^{1/p_2}u_1(t,x-z)^{1/p_1}u_2(t,z)^{1/p_2}\times$$
$$|\Lambda_1^{1/2}v_1(t,x-y)+\Lambda_2^{1/2}v_2(t,y)-\Lambda_1^{1/2}v_1(t,x-z)-\Lambda_2^{1/2}v_2(t,z)|^2\,dydz,$$

where $(\Lambda_1,\Lambda_2):=(\frac{p\sigma_1}{p_1}|1-\frac{1}{p_1}|,\frac{p\sigma_2}{p_2}|1-\frac{1}{p_2}|)$ and ε is defined to be 1 if $p_1,p_2\geq 1$ and -1 if $p_1,p_2\leq 1$. This follows by expanding the square in the integrand and the hypotheses (2.1) and (2.2). This completes the proof of Theorem 2.1. $\square$

At the heart of our proof of Theorem 2.1 is a closure property for solutions of heat *inequalities*. Essentially, we have shown that if, for $j=1,2$, we have $p_j\geq 1$ and $u_j:(0,\infty)\times\mathbb{R}^d\to(0,\infty)$ satisfies

$$\partial_t u_j\geq\frac{\sigma_j}{4\pi}\Delta u_j$$

then

$$\partial_t u\geq\frac{\sigma}{4\pi}\Delta u,$$

where u and σ are given by (2.5) and (2.6), respectively. Similarly, if $p_j\leq 1$ and

$$\partial_t u_j\leq\frac{\sigma_j}{4\pi}\Delta u_j$$

for $j=1,2$ then

$$\partial_t u\leq\frac{\sigma}{4\pi}\Delta u.$$

Here we have ignored some technical details which relate to the finiteness of various Lebesgue integrals arising in the argument. In [13] we identify a natural list of further (technical) ingredients which guarantee the existence of such integrals and are also closed under the operation $(p_1,p_2,\sigma_1,\sigma_2,u_1,u_2)\mapsto(p,\sigma,u)$.

We note that the heat-flow monotonicity associated to the geometric Brascamp–Lieb inequality discussed in the Introduction here also rests on a similar closure property of heat inequalities under the operation $(u_1,\ldots,u_m)\mapsto u$ where u is the "geometric mean" given by

$$u(t,\cdot)=\prod_{j=1}^m u_j(t,B_j\,\cdot)^{1/p_j}.$$

This observation is implicit in the work [19] and [16]. In [13] we also note that a similar closure property holds for harmonic addition; that is $(u_1,u_2)\mapsto u$ where

$$\frac{1}{u}=\frac{1}{u_1}+\frac{1}{u_2}.$$

This closure property is easily seen to imply a "harmonic triangle inequality".

An advantage of the perspective of closure is that one may iterate and this allows us to deduce the following generalisation of Theorem 2.1 in a rather cheap way. Suppose $0 < p_1, \ldots, p_n, p < \infty$ satisfy

$$\sum_{j=1}^{n} \frac{1}{p_j} = n - 1 + \frac{1}{p} \tag{2.7}$$

and let $0 \leq \sigma_1, \ldots, \sigma_n < \infty$ satisfy

$$\frac{1}{p_j}\left(1 - \frac{1}{p_j}\right)\sigma_k = \frac{1}{p_k}\left(1 - \frac{1}{p_k}\right)\sigma_j$$

for each $j, k = 1, \ldots, n$. Let $Q : (0,\infty) \to (0,\infty)$ be given by

$$Q(t) = \|u_1(t,\cdot)^{1/p_1} * \cdots * u_n(t,\cdot)^{1/p_n}\|_{L^p(\mathbb{R}^d)}$$

where $u_j : (0,\infty) \times \mathbb{R}^d \to (0,\infty)$ is given by (2.3) for some nonnegative $f_j \in L^{p_j}(\mathbb{R}^d)$.

Theorem 2.2 (Bennett, B. [13]). *If $p_1, \ldots, p_n \geq 1$ then $Q(t)$ is nondecreasing for each $t > 0$ and if $p_1, \ldots, p_n \leq 1$ then $Q(t)$ is nonincreasing for each $t > 0$.*

As one may expect, from Theorem 2.2 (and its proof) we recover the sharp n-fold Young convolution inequality, its reverse form and a complete characterisation of extremals.

We conclude this section by describing an extension of our results when the scaling condition (2.7) is relaxed. Let $1 \leq p_1, \ldots, p_n, p < \infty$ be such that

$$\sum_{j=1}^{n} \frac{1}{p_j} \geq n - 1 + \frac{1}{p} \tag{2.8}$$

and suppose that $0 \leq \alpha_1, \ldots, \alpha_n \leq 1$ satisfy

$$\sum_{j=1}^{n} \frac{\alpha_j}{p_j} = n - 1 + \frac{1}{p}.$$

Let $0 \leq \sigma_1, \ldots, \sigma_n < \infty$ satisfy

$$\frac{1}{p_j}\left(1 - \frac{\alpha_j}{p_j}\right)\sigma_k = \frac{1}{p_k}\left(1 - \frac{\alpha_k}{p_k}\right)\sigma_j$$

for each $j, k = 1, \ldots, n$. Finally, let $Q : (0,\infty) \to (0,\infty)$ be given by

$$Q(t) = t^{d(\sum_{j=1}^{n} 1/p_j - (n-1) - 1/p)/2}\|u_1(t,\cdot)^{1/p_1} * \cdots * u_n(t,\cdot)^{1/p_n}\|_{L^p(\mathbb{R}^d)}$$

where $u_j : (0,\infty) \times \mathbb{R}^d \to (0,\infty)$ is given by (2.3) for some nonnegative $f_j \in L^{p_j}(\mathbb{R}^d)$.

Theorem 2.3 (Bennett, B. [13]). *For each $t > 0$, $Q(t)$ is nondecreasing.*

The idea behind this extension originates in [16]. As was the case with Theorems 2.1 and 2.2, one can view Theorem 2.3 as a corollary to a closure property associated to heat inequalities. The additional ingredient under the relaxed scaling condition (2.8) is that the differential inequalities

$$\sigma_j \mathrm{div}(\nabla \log u_j)(t,\cdot) \geq -\frac{2d\pi}{t} \tag{2.9}$$

for $u_j : (0,\infty) \times \mathbb{R}^d \to (0,\infty)$ and $j = 1,\dots,n$ imply

$$\sigma \mathrm{div}(\nabla \log u)(t,\cdot) \geq -\frac{2d\pi}{t},$$

where u is given by

$$u^{1/p} = u_1^{1/p_1} * \cdots * u_n^{1/p_n}$$

and σ is given by

$$\sigma p = \sum_{j=1}^{n} \sigma_j p_j.$$

We remark that if u_j satisfies the heat equation $\partial_t u_j = \frac{\sigma_j}{4\pi}\Delta u_j$ with nonnegative initial data then (2.9) follows from a certain log-convexity property of solutions to heat equations; see Corollary 8.7 of [16].

§ 3. Strichartz estimates for the homogeneous Schrödinger equation

For $d \in \mathbb{N}$ let the Fourier transform $\widehat{f} : \mathbb{R}^d \to \mathbb{C}$ of a Lebesgue integrable function f on $\mathbb{R}^d$ be given by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx. \tag{3.1}$$

For each $s \in \mathbb{R}$ let $e^{is\Delta}$ denote the Fourier multiplier operator given by

$$\widehat{e^{is\Delta} f}(\xi) = e^{-is|\xi|^2} \widehat{f}(\xi),$$

for all f belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$. Thus for each $f \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$e^{is\Delta} f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i(x\cdot\xi - s|\xi|^2)} \widehat{f}(\xi)\, d\xi$$

and we have that $e^{is\Delta} f$ solves the homogeneous Schrödinger equation

$$i\frac{\partial u}{\partial s} = -\Delta u$$

on $\mathbb{R}^d$ with initial data f. It is now well-known that the solution operator $e^{is\Delta}$ satisfies the Strichartz space-time estimate

$$\|e^{is\Delta}f\|_{L^p(\mathbb{R},L^q(\mathbb{R}^d))} \leq C\|f\|_{L^2(\mathbb{R}^d)} \tag{3.2}$$

for some finite constant C if and only if (p,q,d) is Schrödinger admissible; i.e. $p,q \geq 2$, $(p,q,d) \neq (2,\infty,2)$ and $\frac{2}{p}+\frac{d}{q}=\frac{d}{2}$. See [24] for the endpoint case $(2,2d/(d-2),d)$ and references therein for earlier contributions.

From the orientation of euclidean harmonic analysis, we recall the familiar fact that $e^{is\Delta}\widehat{f}$ coincides with the adjoint restriction operator (the extension operator) associated to a paraboloid applied to f. The estimate in (3.2) for $p=q=2+4/d$ is classical and is due to Strichartz [26] who followed arguments of Stein and Tomas [27].

In [15] we observe the following monotonicity property of space-time norms associated to propagator $e^{is\Delta}$ for certain special exponents. For aesthetic reasons and consistency with [15], we adopt the notation

$$e^{t\Delta}f := \widetilde{H}_t * f$$

at this point for the solution to the heat equation $\partial_t u = \Delta u$ with initial data f. Here,

$$\widetilde{H}_t(x) := \frac{1}{(4\pi t)^{d/2}}\, e^{-|x|^2/4t}.$$

Theorem 3.1 (Bennett, B., Carbery, Hundertmark [15]). *Let (p,q,d) be a Schrödinger admissible triple such that q is an even integer which divides p. If f is a nonnegative integrable function on $\mathbb{R}^d$ and $\alpha \in [1/2,1]$ then $Q_\alpha : (0,\infty) \to (0,\infty)$ given by*

$$Q_\alpha(t) = t^{d(\alpha-1/2)/2}\|e^{is\Delta}(e^{t\Delta}f)^\alpha\|_{L^p(\mathbb{R},L^q(\mathbb{R}^d))}.$$

is nondecreasing for each $t>0$.

Triples (p,q,d) satisfying the hypothesis in Theorem 3.1 are $(6,6,1),(8,4,1)$ and $(4,4,2)$. For such exponents, it follows from Theorem 3.1 when $\alpha = 1/2$ and $f=|g|^2$ for some bounded and compactly supported function g on $\mathbb{R}^d$ that

$$\|e^{is\Delta}|g|\|_{L^p(\mathbb{R},L^q(\mathbb{R}^d))} = \lim_{t\to 0} Q_{1/2}(t) \leq \lim_{t\to\infty} Q_{1/2}(t) = C\|g\|_{L^2(\mathbb{R}^d)},$$

where $C=\|e^{is\Delta}\widetilde{H}_1^{1/2}\|_{L^p(\mathbb{R},L^q(\mathbb{R}^d))}$. Moreover,

$$\|e^{is\Delta}g\|_{L^p(\mathbb{R},L^q(\mathbb{R}^d))} \leq \|e^{is\Delta}|g|\|_{L^p(\mathbb{R},L^q(\mathbb{R}^d))}$$

since q is an even integer which divides p. For the $(6,6,1)$ and $(4,4,2)$ cases, we recover the sharp Strichartz estimates for the homogeneous Schrödinger equation due to Foschi

[21] and independently Hundertmark and Zharnitsky [23]. In the $(8,4,1)$ case, the heat-flow monotonicity (and hence sharp inequality) follows cheaply from the $(4,4,2)$ case via the observation

$$\|e^{is\Delta}(e^{t\Delta}f)^\alpha\|_{L^8(\mathbb{R},L^4(\mathbb{R}))}^2 = \|e^{is\Delta}(e^{t\Delta}f\otimes f)^\alpha\|_{L^4(\mathbb{R},L^4(\mathbb{R}^2))}.$$

We shall see in the proof of Theorem 3.1 below that the nondecreasingness of Q_α follows by bringing together Strichartz-norm representation formulae of Hundertmark and Zharnitsky [23] and a monotonicity property associated to the Cauchy–Schwarz inequality contained in Lemma 3.2 below. One can view the fact that we consider the exponents which allow us to mulitply out the norm as avoiding "bad" oscillatory behaviour. We add some strength to this philosophy in the next section in the context of the Fourier transform and the Hausdorff–Young inequality.

Lemma 3.2. *Suppose $n\in\mathbb{N}$, $\alpha\in[1/2,1]$ and f_1,f_2 are nonnegative integrable functions on $\mathbb{R}^n$. Then $\widetilde{Q}_\alpha:(0,\infty)\to(0,\infty)$ given by*

$$\widetilde{Q}_\alpha(t) = t^{n(\alpha-1/2)}\int_{\mathbb{R}^n}(e^{t\Delta}f_1)^\alpha(e^{t\Delta}f_2)^\alpha\,dx$$

is nondecreasing for all $t>0$.

For a proof of the above lemma, we refer the reader to [16]. When $\alpha=1/2$, matters are reduced to a special case of the monotonicity inherent in the geometric Brascamp–Lieb inequality considered in the Introduction here. In this case, the proof is particularly straightforward and produces the explicit formula

$$\widetilde{Q}'_{1/2}(t) = \frac{1}{4}\int_{\mathbb{R}^n}|\nabla(\log e^{t\Delta}f_1)-\nabla(\log e^{t\Delta}f_2)|^2(e^{t\Delta}f_1)^{1/2}(e^{t\Delta}f_2)^{1/2}\,dx \tag{3.3}$$

for each $t>0$.

Proof of Theorem 3.1. We begin with the $(6,6,1)$ case. The $(4,4,2)$ case follows by a similar argument and we omit the details.

For Schwartz functions f on $\mathbb{R}$,

$$\|e^{is\Delta}f\|_{L^6(\mathbb{R}\times\mathbb{R})}^6 = \frac{1}{2\sqrt{3}}\int_{\mathbb{R}^3}F(x)PF(x)\,dx$$

where P is the projection onto functions on $\mathbb{R}^3$ which are invariant under the rotations about the direction $(1,1,1)$ and F is the three-fold tensor product of f. The identification of P as a particularly simple projection operator is due to Hundertmark and Zharnitsky [23]. For our goal of monotonicity it is important that we may write

$$PF(x) = \int_O F(\rho x)\,d\mathcal{H}(\rho)$$

where O is the group of isometries on $\mathbb{R}^3$ which coincide with the identity on the span of $(1,1,1)$ and $d\mathcal{H}$ denotes the right-invariant Haar probability measure on O.

Notice that

$$e^{t\Delta}f \otimes e^{t\Delta}f \otimes e^{t\Delta}f = e^{t\Delta}F$$

and, for each isometry ρ on $\mathbb{R}^3$,

$$(e^{t\Delta}f \otimes e^{t\Delta}f \otimes e^{t\Delta}f)(\rho\,\cdot) = e^{t\Delta}F_\rho$$

where $F_\rho := F(\rho\,\cdot)$. Therefore,

$$Q_\alpha(t)^6 = \frac{1}{2\sqrt{3}}\int_O t^{3(\alpha-1/2)}\int_{\mathbb{R}^3}(e^{t\Delta}F)^\alpha(x)(e^{t\Delta}F_\rho)^\alpha(x)\,dxd\mathcal{H}(\rho)$$

and, by Lemma 3.2 and the nonnegativity of the measure $d\mathcal{H}$, it follows that $Q_\alpha(t)$ is nondecreasing for each $t>0$. □

The proof of Theorem 3.1 combined with (3.3) produces an explicit formula the derivative of $Q_{1/2}$ at each time $t>0$ from which it is possible to recover the complete characterisation of extremals as gaussians in the corresponding Strichartz estimates. This characterisation is due to Foschi [21] and Hundertmark and Zharnitsky [23].

§ 4. The Hausdorff–Young inequality

Let f be a nonnegative integrable function on $\mathbb{R}^d$ and for $2 \le q \le p' \le \infty$ let $Q_{p,q} : (0,\infty) \to (0,\infty)$ be given by

$$Q_{p,q}(t) = t^{d(1/q-1/p')/2}\left\|\widehat{u(t,\cdot)^{1/p}}\right\|_{L^q(\mathbb{R}^d)},$$

where $u(t,\cdot) = H_t * f$ and $\widehat{\ }$ is the Fourier transform given by (3.1). By taking $q = p'$ and $f = |g|^p$ for a bounded and compactly supported function g on $\mathbb{R}^d$, if $Q_{p,q}$ were nondecreasing for each $t>0$ then

$$\|\,\widehat{|g|}\,\|_{L^{p'}(\mathbb{R}^d)} = \lim_{t\to 0} Q_{p,q}(t) \le \lim_{t\to\infty} Q_{p,q}(t) = \|\widehat{H_1^{1/p}}\|_{L^{p'}(\mathbb{R}^d)}\|g\|_{L^p(\mathbb{R}^d)}.$$

If $q = p'$ is an even integer then by Plancherel's theorem one may write $Q_{p,q}$ in terms of a $q/2$-fold convolution

$$Q_{p,q}(t) = t^{d(1/q-1/p')/2}\|u(t,\cdot)^{1/p} * \cdots * u(t,\cdot)^{1/p}\|_{L^2(\mathbb{R}^d)}^{2/q}.$$

By expanding the above $L^2(\mathbb{R}^d)$ norm it follows from [16] that $Q_{p,q}$ is nondecreasing for each $t>0$; see [14] for a verification of this fact. Alternatively, one can appeal directly to Theorem 2.3. Since p' is an even integer we have

$$\|\widehat{g}\|_{L^{p'}(\mathbb{R}^d)} \le \|\,\widehat{|g|}\,\|_{L^{p'}(\mathbb{R}^d)}$$

and thus one recovers the sharp form of the Hausdorff–Young inequality on $\mathbb{R}^d$

$$\|\widehat{g}\|_{L^p(\mathbb{R}^d)} \leq \|\widehat{H_1^{1/p}}\|_{L^{p'}(\mathbb{R}^d)}\|g\|_{L^p(\mathbb{R}^d)}$$

for p' an even integer due to Babenko [1], [2].

However, the monotonicity of $Q_{p,q}$ fails dramatically if q is not an even integer, as our next result shows.

Theorem 4.1 (Bennett, B., Carbery [14]). *Let $d \in \mathbb{N}$, $2 \leq q \leq p' \leq \infty$ and suppose q is not an even integer. Then there exists a nonnegative integrable function f on $\mathbb{R}^d$ such that $Q_{p,q}(t)$ is strictly decreasing for sufficiently small $t > 0$.*

Theorem 4.1 is of course a significant obstacle to finding a proof based on heat-flow of the sharp Hausdorff–Young inequality due to Beckner [11], [12]; i.e. for *all* $p' \in [2, \infty)$.

Proof of Theorem 4.1. It suffices to handle $d = 1$, since if f is a one-dimensional counterexample to the monotonicity of $Q_{p,q}$, then $\bigotimes_{j=1}^d f$ is a d-dimensional counterexample. Given the special relationship that convolution and the Fourier transform enjoy, it is natural to consider the case $p = 1$ first of all.

Using the semigroup property of the heat kernel H_t, it is sufficient to find a counterexample in the form of a finite Borel measure μ on $\mathbb{R}$. To this end, let m and n be coprime integers to be chosen later, $r \in (0, 1/2)$ and

$$\mu = \delta_0 + r\delta_m + r\delta_n,$$

where δ_j denotes the Dirac delta measure supported at the integer j. Thus, $\widehat{\mu}(\xi) = 1 + re^{-2\pi i m\xi} + re^{-2\pi i n\xi}$ and if c_n denotes the nth Fourier coefficient of $|\widehat{\mu}|^q$ then it follows that we can express $Q_{1,q}(t)$ as the following power series in $e^{-\pi/qt}$:

$$Q_{1,q}(t)^q = q^{-1/2} \sum_{n \in \mathbb{Z}} c_n e^{-\pi n^2/qt}.$$

By differentiating the above expression for $Q_{1,q}(t)^q$ term by term it follows that the sign of $Q_{1,q}(t)$ as $t > 0$ approaches zero coincides with the sign of $c_1 + c_{-1}$. Now,

$$|\widehat{\mu}(\xi)|^q = \sum_{k=0}^{\infty} a_k r^k (e^{-2\pi i m\xi} + e^{-2\pi i n\xi})^k \sum_{k'=0}^{\infty} a_{k'} r^{k'} (e^{2\pi i m\xi} + e^{2\pi i n\xi})^{k'}$$

where a_k is the kth binomial coefficient in the expansion of $(1 + x)^{q/2}$. It is only here where we use the size restriction on the parameter r. Observe that if $k < q/2 + 1$ then

$a_k > 0$, and thereafter a_k is strictly alternating in sign. Now,

$$\begin{aligned} c_1 + c_{-1} &= 2 \sum_{k,k'=0}^{\infty} a_k a_{k'} r^{k+k'} \int_0^1 (e^{-2\pi i m\xi} + e^{-2\pi i n\xi})^k (e^{2\pi i m\xi} + e^{2\pi i n\xi})^{k'} e^{-2\pi i\xi}\, d\xi \\ &= 2 \sum_{k,k'=0}^{\infty} a_k a_{k'} r^{k+k'} \sum_{(\mathbf{j},\mathbf{j}')\in\Lambda_{k,k'}} \binom{k}{j_1}\binom{k'}{j_1'}, \end{aligned}$$

where

$$\begin{aligned} \Lambda_{k,k'} = \{(\mathbf{j},\mathbf{j}') = ((j_1,j_2),(j_1',j_2')) \in (\mathbb{N}_0^2)^2 \ : j_1 + j_2 = k,\ j_1' + j_2' = k' \text{ and} \\ m(j_1 - j_1') + n(j_2 - j_2') = 1\} \end{aligned}$$

and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

We claim that by choosing m and n appropriately (depending on q) we can ensure that $\Lambda_{k,k'}$ is empty *whenever* $a_k a_{k'} > 0$. Remarkably, it is not difficult to show that if m and n have the same parity the sets $\Lambda_{k,k'}$ are empty whenever k and k' have the same parity. Moreover, if one chooses m and n to be "sufficiently coprime" in the sense that whenever $\alpha m + \beta n = 1$ the vector (α, β) is sufficiently distant from the origin, then one can show that $\Lambda_{k,k'}$ is empty whenever one of k and k' is less than $q/2 + 1$. This leaves a contribution from summands with k and k' greater than $q/2 + 1$ and, as long as one summand is nonzero, it follows that $c_1 + c_{-1} < 0$ as required. For further details of these arguments we refer the reader to [14].

The manner in which the oscillation is exploited in the above argument via an infinite binomial expansion involving negative coefficients is in the spirit of the Hardy–Littlewood majorant counterexample in [22].

The idea behind our argument for $p > 1$ is the following. For large m and n which are sufficiently far apart and small $t > 0$, $H_t * \mu$ is a finite sum of "well-separated" gaussians and consequently $(H_t * \mu)^{1/p}$ is "very close" to $H_t^{1/p} * \widetilde{\mu}$, where

$$\widetilde{\mu} := \delta_0 + r^{1/p}\delta_m + r^{1/p}\delta_n.$$

Combined with the correcting power of t from the definition of Q, H_t raised to the power $1/p$ is essentially the same heat kernel at a rescaled time. Furthermore, it is not difficult to locate a suitable pair of large and well-separated integers (m, n) for which our argument for $p = 1$ works. Therefore, our analysis for the measure μ above is sufficient, modulo an error term, to produce a counterexample for all $p > 1$. This argument is made rigorous in [14]. □

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