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Global DIV-CURL Lemma in 3D bounded domains

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1 Introduction.

Let Ω be an open set in $\mathbb{R}^3$. It is well-known that if $u_j \rightharpoonup u$, $v_j \rightharpoonup v$ weakly in $L^2(\Omega)$ and if $\{\mathrm{div}\ u_j\}_{j=1}^{\infty}$ and $\{\mathrm{rot}\ v_j\}_{j=1}^{\infty}$ are bounded in $L^2(\Omega)$, then it holds that $u_j \cdot v_j \rightharpoonup u \cdot v$ in the sense of distributions in Ω. This is the original Div-Curl lemma. For instance, we refer to Tartar [5]. The purpose of this article is to deal with a similar lemma to bounded domains where the convergence $u_j \cdot v_j \to u \cdot v$ holds in the sense that

$$\int_\Omega u_j \cdot v_j dx \to \int_\Omega u \cdot v dx \quad \text{as } j \to \infty. \tag{1.1}$$

Our result may be regarded as a *global* version of the Div-Curl lemma, which includes the previous one. To obtain such a global version, we need to pay an attention to the behaviour of $\{u_j\}_{j=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$ on the boundary $\partial\Omega$ of Ω. Indeed, an additional bound of $\{u_j \cdot \nu|_{\partial\Omega}\}_{j=1}^{\infty}$, or that of $\{v_j \times \nu|_{\partial\Omega}\}_{j=1}^{\infty}$ in $H^{\frac{1}{2}}(\partial\Omega)$ on the boundary $\partial\Omega$ plays an essential role for our convergence, where ν denotes the unit outward normal to $\partial\Omega$.

In what follows, we impose the following assumption on the domain Ω:

Assumption. Ω is a bounded domain in $\mathbb{R}^3$ with C^∞-boundary $\partial\Omega$.

Before stating our result, we first recall the generalized trace theorem for $u \cdot \nu$ and $u \times \nu$ on $\partial\Omega$ defined on the Banach spaces $E^q_{div}(\Omega)$ and $E^q_{rot}(\Omega)$ for $1 < q < \infty$, where

$$\begin{aligned} E^q_{div}(\Omega) &\equiv \{u \in L^q(\Omega); \mathrm{div}\ u \in L^q(\Omega)\} \text{ with the norm } \|u\|_{E^q_{div}} = \|u\|_q + \|\mathrm{div}\ u\|_q, \\ E^q_{rot}(\Omega) &\equiv \{u \in L^q(\Omega); \mathrm{rot}\ u \in L^q(\Omega)\} \text{ with the norm } \|u\|_{E^q_{rot}} = \|u\|_q + \|\mathrm{rot}\ u\|_q. \end{aligned}$$

Here and in what follows, $\|\cdot\|_q$ denotes the usual L^q-norm over Ω. It is known that there are bounded operators γ_ν and τ_ν on $E^q_{div}(\Omega)$ and $E^q_{rot}(\Omega)$ with properties that

$$\begin{aligned} &\gamma_\nu : u \in E^q_{div}(\Omega) \mapsto \gamma_\nu u \in W^{1-1/q',q'}(\partial\Omega)^*, \qquad \gamma_\nu u = u \cdot \nu|_{\partial\Omega} \text{ if } u \in C^1(\bar{\Omega}) , \\ &\tau_\nu : u \in E^q_{rot}(\Omega) \mapsto \tau_\nu u \in W^{1-1/q',q'}(\partial\Omega)^*, \qquad \tau_\nu u = u \times \nu|_{\partial\Omega} \text{ if } u \in C^1(\bar{\Omega}) , \end{aligned}$$

respectively, where $1/q + 1/q' = 1$. The range $W^{1-1/q',q'}(\partial\Omega)^*$ of γ_ν and τ_ν is the dual space of $W^{1-1/q',q'}(\partial\Omega)$ which is the image of the trace on $\partial\Omega$ of functions in $W^{1,q'}(\Omega)$. Indeed, the following generalized Stokes formula holds

$$(u, \nabla p) + (\text{div } u, p) = \langle \gamma_\nu u, \gamma_0 p\rangle_{\partial\Omega} \quad \text{for all } u \in E^q_{div}(\Omega) \text{ and all } p \in W^{1,q'}(\Omega), \tag{1.2}$$

$$(u, \text{rot } \phi) = (\text{rot } u, \phi) + \langle \tau_\nu u, \gamma_0 \phi\rangle_{\partial\Omega} \quad \text{for all } u \in E^q_{rot}(\Omega) \text{ and all } \phi \in W^{1,q'}(\Omega), \tag{1.3}$$

where γ_0 denotes the usual trace operator from $W^{1,q'}(\Omega)$ onto $W^{1-1/q',q'}(\partial\Omega)$, and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ is the duality paring between $W^{1-1/q',q'}(\partial\Omega)^*$ and $W^{1-1/q',q'}(\partial\Omega)$. Here and in what follows, $(\cdot,\cdot)$ denotes the duality paring between $L^q(\Omega)$ and $L^{q'}(\Omega)$. For a detail of (1.2) and (1.3), we refer to Borchers-Sohr [1], [2], Simader-Sohr [3] and Temam [6].

Our result now reads:

Theorem 1 *Let Ω be as in the Assumption. Let $1 < r < \infty$ with $1/r + 1/r' = 1$. Suppose that $\{u_j\}_{j=1}^\infty \subset L^r(\Omega)$ and $\{v_j\}_{j=1}^\infty \subset L^{r'}(\Omega)$ satisfy*

$$u_j \rightharpoonup u \quad \text{weakly in } L^r(\Omega), \quad v_j \rightharpoonup v \quad \text{weakly in } L^{r'}(\Omega) \tag{1.4}$$

for some $u \in L^r(\Omega)$ and $v \in L^{r'}(\Omega)$, respectively. Assume also that

$$\{\text{div } u_j\}_{j=1}^\infty \text{ is bounded in } L^q(\Omega) \text{ for some } q > \max\{1, 3r/(3+r)\} \tag{1.5}$$

and that

$$\{\text{rot } v_j\}_{j=1}^\infty \text{ is bounded in } L^s(\Omega) \text{ for some } s > \max\{1, 3r'/(3+r')\}, \tag{1.6}$$

respectively. If either

(i) *$\{\gamma_\nu u_j\}_{j=1}^\infty$ is bounded in $W^{1-1/q,q}(\partial\Omega)$,*

or

(ii) *$\{\tau_\nu v_j\}_{j=1}^\infty$ is bounded in $W^{1-1/s,s}(\partial\Omega)$,*

then it holds that

$$\int_\Omega u_j \cdot v_j dx \to \int_\Omega u \cdot v dx \quad \text{as } j \to \infty. \tag{1.7}$$

In particular, if either $\gamma_\nu u_j = 0$, or $\tau_\nu v_j = 0$ for all $j = 1, 2, \cdots$ is satisfied, then we have also (1.7).

As an immediate consequence of our theorem, we have the following Div-Curl lemma in an arbitrary open set in $\mathbb{R}^3$.

Corollary 1.1 (Tartar [5]) *Let D be an arbitrary open set in $\mathbb{R}^3$. Let $1 < r < \infty$. Suppose that $\{u_j\}_{j=1}^\infty \subset L^r(D)$ and $\{v_j\}_{j=1}^\infty \subset L^{r'}(D)$ satisfy*

$$u_j \rightharpoonup u \quad \text{weakly in } L^r(D), \quad v_j \rightharpoonup v \quad \text{weakly in } L^{r'}(D) \tag{1.8}$$

for some $u \in L^r(D)$ and $v \in L^{r'}(D)$, respectively. Assume also that

$$\{\text{div } u_j\}_{j=1}^\infty \text{ and } \{\text{rot } v_j\}_{j=1}^\infty \text{ are bounded in } L^r(D) \text{ and } L^{r'}(D), \tag{1.9}$$

respectively. Then it holds that

$$u_j \cdot v_j \rightharpoonup u \cdot v \quad \text{in the sense of distributions in } D. \tag{1.10}$$

Remarks. (i) Since Ω is a bounded domain, we may assume that $3r/(3+r) < q \leqq r$ and $3r'/(3+r') < s \leqq r'$, and hence it holds that $\{u_j\}_{j=1}^{\infty} \subset E^q_{div}(\Omega)$ and that $\{v_j\}_{j=1}^{\infty} \subset E^s_{rot}(\Omega)$. Then we have that $\{\gamma_\nu u_j\}_{j=1}^{\infty} \subset W^{1-1/q',q'}(\partial\Omega)^*$ and $\{\tau_\nu v_j\}_{j=1}^{\infty} \subset W^{1-1/s',s'}(\partial\Omega)^*$.

(ii) In Theorem 1, it is *un*necessary to assume both bounds of $\{\gamma_\nu u_j\}_{j=1}^{\infty}$ in $W^{1-1/r,r}(\partial\Omega)$ and $\{\tau_\nu v_j\}_{j=1}^{\infty}$ in $W^{1-1/r',r'}(\partial\Omega)$. Indeed, what we need is only one of these bounds.

2 L^r-Helmholtz-Weyl decomposition.

In this section, we recall the Helmholtz-Weyl decomposition for vector fields in $L^r(\Omega)$. For a detail, we refer [2]. According to the two types $u \cdot \nu = 0$ and $u \times \nu = 0$ of boundary conditions on $\partial\Omega$, we first define harmonic vector spaces $X_{har}(\Omega)$ and $V_{har}(\Omega)$ as

$$\begin{aligned} X_{har}(\Omega) &= \{h \in C^\infty(\bar{\Omega}); \text{div } h = 0, \text{rot } h = 0 \text{ in } \Omega \text{ with } h \cdot \nu = 0 \text{ on } \partial\Omega\}, \\ V_{har}(\Omega) &= \{h \in C^\infty(\bar{\Omega}); \text{div } h = 0, \text{rot } h = 0 \text{ in } \Omega \text{ with } h \times \nu = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

Moreover, for $1 < r < \infty$ let us define divergence-free vector fields $X^r_\sigma(\Omega)$ and $V^r_\sigma(\Omega)$ by

$$\begin{aligned} X^r_\sigma(\Omega) &\equiv \{u \in W^{1,r}(\Omega); \text{div } u = 0, \gamma_\nu u = 0\}, \\ V^r_\sigma(\Omega) &\equiv \{u \in W^{1,r}(\Omega); \text{div } u = 0, \tau_\nu u = 0\}. \end{aligned}$$

Then we have the following decomposition theorem. For a detail, we refer Kozono-Yanagisawa [2]

Proposition 2.1 ([2]) *Let Ω be as in the Assumption. Let $1 < r < \infty$.*

(1) Both $X_{har}(\Omega)$ and $V_{har}(\Omega)$ are finite dimensional vector spaces.

(2) For every $u \in L^r(\Omega)$, there are $p \in W^{1,r}(\Omega)$, $w \in V^r_\sigma(\Omega)$ and $h \in X_{har}(\Omega)$ such that u can be represented as

$$u = h + \text{rot } w + \nabla p. \tag{2.1}$$

Such a triplet $\{p, w, h\}$ is subordinate to the estimate

$$\|p\|_{W^{1,r}} + \|w\|_{W^{1,r}} + \|h\|_r \leqq C\|u\|_r \tag{2.2}$$

with the constant $C = C(\Omega, r)$ independent of u. The above decomposition (2.1) is unique. In fact, if u has another expression

$$u = \tilde{h} + \text{rot } \tilde{w} + \nabla\tilde{p}$$

for $\tilde{p} \in W^{1,r}(\Omega)$, $\tilde{w} \in V^r_\sigma(\Omega)$ and $\tilde{h} \in X_{har}(\Omega)$, then we have

$$h = \tilde{h}, \quad \text{rot } w = \text{rot } \tilde{w}, \quad \nabla p = \nabla\tilde{p}. \tag{2.3}$$

(3) For every $u \in L^r(\Omega)$, there are $p \in W^{1,r}_0(\Omega)$, $w \in X^r_\sigma(\Omega)$ and $h \in V_{har}(\Omega)$ such that u can be represented as

$$u = h + \text{rot } w + \nabla p. \tag{2.4}$$

Such a triplet $\{p, w, h\}$ is subordinate to the estimate

$$\|p\|_{W^{1,r}} + \|w\|_{W^{1,r}} + \|h\|_r \leqq C\|u\|_r \tag{2.5}$$

with the constant $C = C(\Omega, r)$ independent of u. The above decomposition (2.4) is unique. In fact, if u has another expression

$$u = \tilde{h} + \mathrm{rot}\ \tilde{w} + \nabla \tilde{p}$$

for $\tilde{p} \in W_0^{1,r}(\Omega)$, $\tilde{w} \in X_\sigma^r(\Omega)$ and $\tilde{h} \in V_{har}(\Omega)$, then we have

$$h = \tilde{h}, \quad \mathrm{rot}\ w = \mathrm{rot}\ \tilde{w}, \quad p = \tilde{p}. \tag{2.6}$$

An immediate consequence of the above theorem is

Corollary 2.1 *Let Ω be as in the Assumption.*

(1) By the unique decompositions (2.1) and (2.4) we have two kinds of direct sums in algebraic and topological sense

$$L^r(\Omega) = X_{har}(\Omega) \oplus \mathrm{rot}\ V_\sigma^r(\Omega) \oplus \nabla\ W^{1,r}(\Omega), \tag{2.7}$$

$$L^r(\Omega) = V_{har}(\Omega) \oplus \mathrm{rot}\ X_\sigma^r(\Omega) \oplus \nabla\ W_0^{1,r}(\Omega) \tag{2.8}$$

for $1 < r < \infty$.

(2) Let S_r, R_r and Q_r be projection operators associated with both (2.1) and (2.4) from $L^r(\Omega)$ onto $X_{har}(\Omega)$, rot *$V_\sigma^r(\Omega)$ and $\nabla\ W^{1,r}(\Omega)$, and from $L^r(\Omega)$ onto $V_{har}(\Omega)$,* rot *$X_\sigma^r(\Omega)$ and $\nabla\ W_0^{1,r}(\Omega)$, respectively, i.e.,*

$$S_r u \equiv h, \quad R_r u \equiv \mathrm{rot}\ w, \quad Q_r u \equiv \nabla p. \tag{2.9}$$

Then we have

$$\|S_r u\|_r \leqq C\|u\|_r, \quad \|R_r u\|_r \leqq C\|u\|_r, \quad \|Q_r u\|_r \leqq C\|u\|_r \tag{2.10}$$

for all $u \in L^r(\Omega)$, where $C = C(r)$ is the constant depending only on $1 < r < \infty$. Moreover, there holds

$$\begin{cases} S_r^2 = S_r, & S_r^* = S_{r'}, \\ R_r^2 = R_r, & R_r^* = R_{r'} \\ Q_r^2 = Q_r, & Q_r^* = Q_{r'}, \end{cases} \tag{2.11}$$

where S_r^*, R_r^* and Q_r^* denote the adjoint operators on $L^{r'}(\Omega)$ of S_r, R_r and Q_r, respectively.

If u has an additional regularity such as div $u \in L^q(\Omega)$ and rot $u \in L^q(\Omega)$ for some $1 < q \leqq r$, then we may choose the scalar and the vector potentials p and w in (2.1) and (2.4) in the class $W^{2,q}(\Omega)$. More precisely, we have

Proposition 2.2 *Let Ω be as in the Assumption and let $1 < r < \infty$. Suppose that $u \in L^r(\Omega)$.*

(1) *Let us consider the decomposition (2.1).*

(i) *If, in addition,* rot $u \in L^q(\Omega)$ *for some $1 < q \leqq r$, then the vector potential w of u in (2.1) can be chosen as $w \in W^{2,q}(\Omega) \cap V_\sigma^r(\Omega)$ with the estimate*

$$\|w\|_{W^{2,q}} \leqq C(\|\mathrm{rot}\ u\|_q + \|u\|_r). \tag{2.12}$$

(ii) *If, in addition,* div $u \in L^q(\Omega)$ *with $\gamma_\nu u \in W^{1-1/q,q}(\partial\Omega)$ for some $1 < q \leqq r$, then the scalar potential p of u in (2.1) can be chosen as $p \in W^{2,q}(\Omega) \cap W^{1,r}(\Omega)$ with the estimate*

$$\|p\|_{W^{2,q}} \leqq C(\|\mathrm{div}\ u\|_q + \|u\|_r + \|\gamma_\nu u\|_{W^{1-1/q,q}(\partial\Omega)}). \tag{2.13}$$

(2) *Let us consider the decomposition (2.4).*

(i) *If, in addition,* div $u \in L^q(\Omega)$ *for some* $1 < q \leqq r$*, then the scalar potential* p *of* u *in (2.4) can be chosen as* $p \in W^{2,q}(\Omega) \cap W_0^{1,r}(\Omega)$ *with the estimate*

$$\|p\|_{W^{2,q}} \leqq C\|\text{div }u\|_q. \tag{2.14}$$

(ii) *If, in addition,* rot $u \in L^q(\Omega)$ *with* $\tau_\nu u \in W^{1-1/q,q}(\partial\Omega)$ *for some* $1 < q \leqq r$*, then the vector potential* w *of* u *in (2.4) can be chosen as* $w \in W^{2,q}(\Omega) \cap X_\sigma^r(\Omega)$ *with the estimate.*

$$\|w\|_{W^{2,q}} \leqq C(\|\text{rot }u\|_q + \|u\|_r + \|\tau_\nu u\|_{W^{1-1/q,q}(\partial\Omega)}). \tag{2.15}$$

Here $C = C(\Omega, r, q)$ *is the constant depending only on* Ω*,* r *and* q*.*

3 Proof of Theorem 1.

(i) Let us first consider the case when $\{\gamma_\nu u_j\}_{j=1}^\infty$ is bounded in $W^{1-1/q,q}(\partial\Omega)$. In such a case, we make use of the decomposition (2.1). Let S_r, R_r and Q_r be the projection operators from $L^r(\Omega)$ onto $X_{har}(\Omega)$, rot $V_\sigma^r(\Omega)$ and $\nabla W^{1,r}(\Omega)$ defined by (2.9), respectively. Notice that the identity

$$(u, v) = (S_r u, S_{r'} v) + (R_r u, R_{r'} v) + (Q_r u, Q_{r'} v) \tag{3.1}$$

holds for all $u \in L^r(\Omega)$ and all $v \in L^{r'}(\Omega)$. Indeed, by the generalized Stokes formula (1.2) and (1.3), we have

$$(\nabla p, h) = -(p, \text{div }h) + \langle \gamma_\nu h, \gamma_0 p\rangle_{\partial\Omega} = 0,$$
$$(\text{rot }w, h) = (w, \text{rot }h) + \langle \tau_\nu w, \gamma_0 h\rangle_{\partial\Omega} = 0$$

for all $p \in W^{1,r}(\Omega)$, $w \in V_\sigma^r(\Omega)$ and $h \in X_{har}(\Omega)$, Similarly, we have

$$(\text{rot }w, \nabla p) = \langle \gamma_\nu(\text{rot }w), \gamma_0 p\rangle_{\partial\Omega} = 0 \quad \text{for all } w \in V_\sigma^r(\Omega),\ p \in W^{1,r'}(\Omega).$$

Thus we obtain (3.1).

Now, by (3.1), we see that the convergence (1.7) can be reduced to

$$(S_r u_j, S_{r'} v_j) \ \to\ (S_r u, S_{r'} v), \tag{3.2}$$
$$(R_r u_j, R_{r'} v_j) \ \to\ (R_r u, R_{r'} v), \tag{3.3}$$
$$(Q_r u_j, Q_{r'} v_j) \ \to\ (Q_r u, Q_{r'} v). \tag{3.4}$$

By Proposition 2.1 (1), the ranges of S_r and $S_{r'}$ are of finite dimension, which means that both S_r and $S_{r'}$ are finite rank operators, therefore compact. Hence, we have by (1.4) that

$$S_r u_j \to S_r u \quad \text{strongly in } L^r(\Omega), \qquad S_{r'} v_j \to S_{r'} v \quad \text{strongly in } L^{r'}(\Omega),$$

from which it follows (3.2).

Next, we apply Proposition 2.2 (1) to (3.3) and (3.4). Since Ω is bounded, we may assume that

$$\max\left\{1, \frac{3r}{3+r}\right\} < q \leqq r, \quad \max\left\{1, \frac{3r'}{3+r'}\right\} < s \leqq r'.$$

By (1.6) and (2.12) with q and r replaced by s and r', respectively, we see that $R_{r'}v_j \equiv \text{rot } \tilde{w}_j$ with $\tilde{w}_j \in V_\sigma^{r'}(\Omega)$ satisfies $\tilde{w}_j \in W^{2,s}(\Omega) \cap V_\sigma^{r'}(\Omega)$ with the estimate

$$\|\tilde{w}_j\|_{W^{2,s}} \leqq C(\|\text{rot } v_j\|_s + \|v_j\|_{r'}) \leqq M, \quad \text{for all } j = 1, 2, \cdots$$

with a constant M independent of j. Since $1/r' > 1/s - 1/3$, the embedding $W^{2,s}(\Omega) \subset W^{1,r'}(\Omega)$ is compact, and hence we see that $\{\tilde{w}_j\}_{j=1}^\infty$ has a strongly convergent subsequence in $W^{1,r'}(\Omega)$, and hence $\{R_{r'}v_j\}_{j=1}^\infty$ has a strongly convergent subsequence in $L^{r'}(\Omega)$. Since (1.4) yields rot $\tilde{w}_j = R_{r'}v_j \rightharpoonup R_{r'}v$ weakly in $L^{r'}(\Omega)$, it holds, in fact, that

$$R_{r'}v_j \to R_{r'}v \quad \text{strongly in } L^{r'}(\Omega). \tag{3.5}$$

Obviously by (1.4), $R_r u_j \rightharpoonup R_r u$ weakly in $L^r(\Omega)$, and hence (3.3) follows.

Since $\{\gamma_\nu u_j\}_{j=1}^\infty$ is bounded in $W^{1-1/q,q}(\partial\Omega)$, we see from (1.5) and (2.13) that $Q_r u_j = \nabla p_j$ satisfies that $p_j \in W^{2,q}(\Omega)$ with the estimate

$$\|p_j\|_{W^{2,q}} \leqq C(\|\text{div } u_j\|_q + \|u_j\|_r + \|\gamma_\nu u_j\|_{W^{1-1/q,q}(\partial\Omega)}) \leqq M \quad \text{for all } j = 1, 2, \cdots$$

with a constant M independent of j. Since $1/r > 1/q - 1/3$, again by the compact embedding $W^{2,q}(\Omega) \subset W^{1,r}(\Omega)$ and by the weak convergence $\nabla p_j = Q_r u_j \rightharpoonup Q_r u$ in $L^r(\Omega)$, implied by (1.4), it holds that

$$Q_r u_j \to Q_r u \quad \text{strongly in } L^r(\Omega). \tag{3.6}$$

Since (1.4) yields $Q_{r'}v_j \rightharpoonup Q_{r'}v$ weakly in $L^{r'}(\Omega)$, we see that (3.4) follows.

(ii) We next consider the case when $\{\tau_\nu v_j\}_{j=1}^\infty$ is bounded in $W^{1-1/s,s}(\partial\Omega)$. In this case, we make use of the decomposition (2.4). Then the argument is quite similar to the former case (i) above. So, we may omit the proof. This proves Theorem 1.

Proof of Corollary 1.1. We may prove that for every $\varphi \in C_0^\infty(D)$

$$\int_D \varphi u_j \cdot v_j dx \to \int_D \varphi u \cdot v dx.$$

Let us take a bounded domain $\Omega \subset \mathbb{R}^3$ with the smooth boundary $\partial\Omega$ so that supp $\varphi \subset \Omega \subset D$. Then it suffices to prove that

$$\int_\Omega \varphi u_j \cdot v_j dx \to \int_\Omega \varphi u \cdot v dx. \tag{3.7}$$

Obviously by (1.8), it holds that

$$\varphi u_j \rightharpoonup \varphi u \quad \text{weakly } L^r(\Omega), \quad v_j \rightharpoonup v \quad \text{weakly } L^{r'}(\Omega). \tag{3.8}$$

Since div $(\varphi u_j) = \varphi \text{div } u_j + u_j \cdot \nabla\varphi$, we see by (1.8) and (1.9) that $\{\text{div } (\varphi u_j)\}_{j=1}^\infty$ is bounded in $L^r(\Omega)$ with

$$\gamma_\nu(\varphi u_j) = 0, \quad j = 1, 2, \cdots. \tag{3.9}$$

Since (1.9) states that $\{\text{rot } v_j\}_{j=1}^\infty$ is also bounded in $L^{r'}(\Omega)$, by taking $q = r$ and $s = r'$ in (1.5) and (1.6), respectively, we see that the convergence (3.7) follows from (3.8), (3.9) and Theorem 1 (i). This proves Corollary 1.1.

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