RIMS Kôkyûroku Bessatsu
B14 (2009), 35–51

On an extension of the Voronoï-Hardy identity and multiple Fourier series

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In this lecture we state the Voronoï-Hardy identity and a related convergence problem of multiple Fourier series, particularly the Fourier series of $f_a(x)$, which is the periodization of the characteristic function $\chi_a(x)$ of a d-dimensional ball with centor 0 and the radius $a > 0$. In the cases of $d = 1, 2$, the Fourier series of $f_a(x)$ converges to $\bar{f}_a(x)$ (=the normalization of $f_a(x)$) for all x and reveals the Gibbs-Wilbraham phenomenon at every discontinuous point. On the other hand, the cases of $d = 3, 4$ are the same as the cases of $d = 1, 2$ except for revealing the Pinsky phenomenon at $x = 0$. In the cases of $d \geq 5$, the Pinsky phenomenon is revealed at $x = 0$, and the Gibbs-Wilbraham phenomenon at every discontinuous points, moreover it diverges at every rational point and converges at almost all x.

The method of the proof was done by an identity modeled on Hardy's identity (§ 4), by the estimations of lattice points problem by B. Novák (§ 3) and by analysis of the Fourier transform of χ_a (§ 2.).

A part of this researchs was based on the joint work with Prof. Eiichi Nakai and Kazuya Ootsubo.

Contents

§ 1. The Voronoï-Hardy identity and its extension

Let $\mathbf{R}^d$, $\mathbf{Z}^d$ and $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d := (-1/2, 1/2]^d$ be the d-dimensional Euclidean space, integer lattice and torus, respectively. The Fourier coefficients of an integrable function f on $\mathbf{T}^d$ and its spherical partial sum are defined by

$$\hat{f}(n) := \int_{\mathbf{T}^d} f(x)e^{-2\pi inx}\,dx, \quad n = (n_1, \cdots, n_d) \in \mathbf{Z}^d,$$

$$S_\lambda(f)(x) := \sum_{|n|<\lambda} \hat{f}(n)e^{2\pi inx}, \quad |n| := \sqrt{\sum_{k=1}^{d} n_k{}^2}, \quad x \in \mathbf{T}^d,$$

respectively, where nx is the inner product $\sum_{k=1}^{d} n_k x_k$.

Also, the Fourier transform of an integrable function F on $\mathbf{R}^d$ and its spherical partial sum are defined by

$$\hat{F}(\xi) := \int_{\mathbf{R}^d} F(x)e^{-2\pi i\xi x}\,dx, \quad \xi = (\xi_1, \cdots, \xi_d) \in \mathbf{R}^d,$$

$$\sigma_\lambda(F)(x) := \int_{|\xi|<\lambda} \hat{F}(\xi)e^{2\pi ix\xi}\,d\xi, \quad |\xi| := \sqrt{\sum_{k=1}^{d} \xi_k{}^2}, \quad x \in \mathbf{R}^d,$$

respectively. For $a > 0$, let

$$\chi_a(x) := \begin{cases} 1, & |x| \le a, \\ 0, & |x| > a, \end{cases} \qquad \bar{\chi}_a(x) := \begin{cases} 1 & |x| < a, \\ \frac{1}{2} & |x| = a, \\ 0 & |x| > a, \end{cases} \qquad x \in \mathbf{R}^d,$$

and let their periodization be $f_a(x)$, $\bar{f}_a(x)$, i.e.,

$$f_a(x) := \sum_{n\in\mathbf{Z}^d} \chi_a(x+n), \quad \bar{f}_a(x) := \sum_{n\in\mathbf{Z}^d} \bar{\chi}_a(x+n), \quad x \in \mathbf{T}^d.$$

Then, using the Poisson summation formula, we have that

$$\hat{f}_a(n) = \hat{\chi}_a(n) = \begin{cases} \dfrac{\pi^{\frac{d}{2}}a^d}{\Gamma(d/2+1)}, & n = 0, \\ \left(\dfrac{a}{|n|}\right)^{\frac{d}{2}} J_{\frac{d}{2}}(2\pi a|n|), & n \neq 0, \end{cases}$$

and

$$S_\lambda(f_a)(x) = \frac{\pi^{\frac{d}{2}}a^d}{\Gamma(d/2+1)} + a^{\frac{d}{2}} \sum_{0<|n|<\lambda} \frac{J_{\frac{d}{2}}(2\pi a|n|)}{|n|^{\frac{d}{2}}} e^{2\pi inx},$$

where $J_\nu(s)$ is the Bessel function.

In connection with lattice points problem, it is known as Hardy's identity that

$$\sum_{|n|<a} 1 + \frac{1}{2} \sum_{|n|=a} 1 = \pi a^2 + a \sum_{n \neq 0} \frac{J_1(2\pi a|n|)}{|n|}.$$

But this identity was first stated by Voronoï. Hardy-Landau[3] (1924) mentioned the history of this identity.

"This identity was first stated by Voronoï[17](1905), who expressly disclaimed possesing an accurate proof, and only two proofs have been published, each of which presents very serious difficulties of its own, particularly when x is an integer. The first, by Hardy[2] (1915), depends on the theory of analytic functions, Cauchy's theorem, and the general theory of Dirichlet's series of type $\sum a_n e^{-s\sqrt{n}}$: and in particular on a very difficult theorem of Marcel Riesz. The second, by Landau[9] (1920), involves real analysis only, and is in principle simpler; but, like all proofs based upon the so-called 'Pfeiffer's method,' it involves complicated distinctions between different geometrical figures, and is intricate and difficult in detail."

Hardy's identity shows that

$$\bar{f}_a(0) = \lim_{\lambda \to \infty} S_\lambda(f_a)(0) \quad \text{for } d = 2.$$

As an extension of Hardy's identity, we can show that, for $d = 2$,

$$\bar{f}_a(x) = \lim_{\lambda \to \infty} S_\lambda(f_a)(x) = \lim_{\lambda \to \infty} \left(\pi a^2 + a \sum_{0<|n|<\lambda} \frac{J_1(2\pi a|n|)}{|n|} e^{2\pi i n x} \right)$$

for all $x \in \mathbf{T}^2$ (Kuratsubo[4](1996), Brandolini-Colzani[1](1999)).

On the other hand, it is known that $S_\lambda(f_a)(x)$ converges to $\bar{f}_a(x)$ a.e. $x \in \mathbf{T}^d$ as $\lambda \to \infty$ for all dimensions d. However, Pinsky, Stanton and Trapa[15] (1993) (See [16], p. 245) proved that, if $d \geq 3$, then $S_\lambda(f_a)(0)$ diverges as $\lambda \to \infty$. This is called the Pinsky phenomenon. Calculating numerical data by computer, Kuratsubo-Nakai-Ootsubo[8] (2006) gave the graphs of $S_\lambda(f_a)(x)$ with $a = 1/4$ and $1 \leq d \leq 6$. We can see the Gibbs-Wilbraham phenomenon and the Pinsky phenomenon on the graphs. Moreover, if $d = 5$ and $d = 6$, then we can see the third phenomenon, that is, $S_\lambda(f_a)(x)$ diverges at many points. Recently, Kuratsubo[6] proved that, if $d \geq 5$, then $S_\lambda(f_a)(x)$ diverges for all rational points x.

In this paper, we deal with analysis of the Fourier inversion formula of the indicator function $\chi_a(x)$ of a d-dimensional ball and with an extension of the Voronoï-Hardy identity where the identity means

$$S_\lambda(f_a)(x) = \bar{f}_a(x) + \mathcal{G}_{a,\lambda}(x) + \mathcal{P}_{a,\lambda}(x) + \mathcal{K}_{a,\lambda}(x),$$

where $\mathcal{G}_{a,\lambda}(x)$ contributes to the Gibbs-Wilbraham phenomenon, $\mathcal{P}_{a,\lambda}(x)$ is the essence of the Pinsky phenomenon and $\mathcal{K}_{a,\lambda}(x)$ results from lattice points problem, and in the particular case of $d=2$, $\mathcal{P}_{a,\lambda}(x)=0$ and $\mathcal{K}_{a,\lambda}(x)=o(1)$.

§ 2. Analysis of the Fourier inversion formula of χ_a

From the formulae for Bessel function :

$$\frac{d}{ds}\left(s^{\nu}J_{\nu}(s)\right)=s^{\nu}J_{\nu-1}(s),\ \frac{d}{ds}\left(\frac{J_{\nu}(s)}{s^{\nu}}\right)=-\frac{J_{\nu+1}(s)}{s^{\nu}},$$

$$\int_0^{\lambda}J_{\nu+1}(ts)J_{\nu+2}(s)\,ds=\int_0^{\lambda}\left(s^{\nu+1}J_{\nu+1}(ts)\right)\left(-\frac{J_{\nu+1}(s)}{s^{\nu+1}}\right)'ds$$

$$=-\left(\lambda^{\nu+1}J_{\nu+1}(t\lambda)\right)\left(\frac{J_{\nu+1}(\lambda)}{\lambda^{\nu+1}}\right)+\int_0^{\lambda}\left(s^{\nu+1}J_{\nu+1}(ts)\right)'\left(\frac{J_{\nu+1}(s)}{s^{\nu+1}}\right)ds.$$

Therefore we have the following Lemma.

Lemma (2.1) For $t>0,\ \lambda\geq 0$,

$$\int_0^{\lambda}J_{\nu+1}(ts)J_{\nu+2}(s)\,ds=t\int_0^{\lambda}J_{\nu}(ts)J_{\nu+1}(s)\,ds-J_{\nu+1}(t\lambda)J_{\nu+1}(\lambda).$$

For the partial sum of the Fourier transform of the indicator function of a d-dimensional ball, we define the one variable function ${\chi_{a,\lambda}}^{[d]}$ as ${\chi_{a,\lambda}}^{[d]}(r):=\sigma_{\lambda}(\chi_a)(x)$ for $|x|=r$. Then we have

$$\begin{aligned}{\chi_{a,\lambda}}^{[d]}(r)&=(\frac{a}{r})^{\frac{d}{2}-1}\int_0^{2\pi\lambda a}J_{\frac{d}{2}-1}(\frac{r}{a}s)J_{\frac{d}{2}}(s)\,ds\\&=(\frac{a}{r})^{\frac{d}{2}-2}\int_0^{2\pi\lambda a}J_{\frac{d}{2}-2}(\frac{r}{a}s)J_{\frac{d}{2}-1}(s)\,ds-(\frac{a}{r})^{\frac{d}{2}-1}J_{\frac{d}{2}-1}(2\pi\lambda r)J_{\frac{d}{2}-1}(2\pi\lambda a)\\&={\chi_{a,\lambda}}^{[d-2]}(r)-(\frac{a}{r})^{\frac{d}{2}-1}J_{\frac{d}{2}-1}(2\pi\lambda r)J_{\frac{d}{2}-1}(2\pi\lambda a).\end{aligned}$$

Therefore, iterating this calculus m_d times, we have

$${\chi_{a,\lambda}}^{[d]}(r)={\chi_{a,\lambda}}^{[d-2m_d]}(r)-\sum_{k=1}^{m_d}\left(\frac{a}{r}\right)^{\frac{d}{2}-k}J_{\frac{d}{2}-k}(2\pi\lambda r)J_{\frac{d}{2}-k}(2\pi\lambda a),$$

where $m_d=$ the integral part of $(d-1)/2$ ($d-2m_d=2$ if d is even and $d-2m_d=1$ if d is odd).

In the case of $d=1$ the behavior of $\sigma_{\lambda}(\chi_a)(x)$ is well known: $\sigma_{\lambda}(\chi_a)(x)$ $=\chi^{[1]}_{a,\lambda}(|x|)=\bar{\chi}_a(x)+\mathrm{sign}(|x|-a)\frac{1}{\pi}\int_{2\pi\lambda||x|-a|}^{\infty}\frac{\sin s}{s}ds-\frac{1}{\pi}\int_{2\pi\lambda(|x|+a)}^{\infty}\frac{\sin s}{s}ds$

$$= \bar{\chi}_a(x) + \mathrm{sign}(|x| - a)\frac{1}{\pi}\int_{2\pi\lambda||x|-a|}^{\infty} \frac{\sin s}{s} ds + O(\frac{1}{\lambda}) \quad \text{uniformly in } \mathbf{R}^1.$$

From this equality we see that $\sigma_\lambda(\chi_a)(x)$ converges to $\bar{\chi}_a(x)$ uniformly in the exterior of any neighborhood of $|x| = a$ and reveals the Gibbs-Wilbraham phenomenon at $|x| = a$.

In the case of $d = 2$ the result is the same as the case of $d = 1$ but the calculus is not trivial, which was proved essentially by Landau[10].

Lemma (2.2) If $d = 2$, then we have the following: $\sigma_\lambda(\chi_a)(x) = \chi_{a,\lambda}^{[2]}(|x|)$ $= \bar{\chi}_a(x) + \mathrm{sign}(|x| - a)\frac{1}{\pi}\int_{2\pi\lambda||x|-a|}^{\infty} \frac{\sin s}{s} ds + O(\frac{1}{\sqrt{\lambda}})$ unif. in $\mathbf{R}^2$. Particularly we see that $\sigma_\lambda(\chi_a)(x)$ converges to $\bar{\chi}_a(x)$ uniformly in the exterior of any neighborhood of $|x| = a$ and reveals the Gibbs-Wilbraham phenomenon at $|x| = a$.

Proof. Let $0 \le r \le 1 - \delta$. For $0 \le t \le r$ and $s > 0$ we have (Landau[11], Satz 518, (722))

$$\int_0^s J_0(ru)J_1(u)du = 1 - J_0(s) - sJ_0(s)\int_0^r \frac{J_1(ts)}{t^2 - 1}dt + sJ_1(s)\int_0^r \frac{tJ_0(ts)}{t^2 - 1}dt.$$

By the second mean value theorem we have

$$\int_0^r \frac{J_1(ts)}{1 - t^2}dt = \frac{1}{1 - r^2}\int_{r_1}^r J_1(ts)dt = \frac{1}{1 - r^2}\frac{J_0(r_1 s) - J_0(rs)}{s} \quad (0 \le \exists r_1 \le r) \text{ and}$$
$$\int_0^r \frac{tJ_0(ts)}{1 - t^2}dt = \frac{1}{1 - r^2}\int_{r_2}^r tJ_0(ts)dt = \frac{1}{1 - r^2}\frac{rJ_1(rs) - r_2J_1(r_2 s)}{s} \quad (0 \le \exists r_2 \le r).$$

$$\begin{aligned}
&\int_0^s J_0(ru)J_1(u)du \\
&\quad = 1 - J_0(s) + J_0(s)\left(\frac{J_0(r_1 s) - J_0(rs)}{1 - r^2}\right) - J_1(s)\left(\frac{rJ_1(rs) - r_2J_1(r_2 s)}{1 - r^2}\right) \\
&\quad = 1 + O(\frac{1}{\sqrt{s}}) \quad \text{uniformly in } r \in [0, 1 - \delta].
\end{aligned}$$

Therefore $\displaystyle\int_0^{2\pi\lambda a} J_0(\frac{|x|}{a}s)J_1(s)ds = 1 + O(\frac{1}{\sqrt{\lambda}})$ uniformly in $\{x \,|\, \frac{|x|}{a} \le 1 - \delta\}$.

Next by the same method with Satz 520 (Landau[10], p. 218), we have the following inequality.

$$\left|\int_{2\pi a\lambda}^{\infty} J_0(\frac{|x|}{a}s)J_1(s)ds + \left(\frac{a}{|x|}\right)^{1/2} \mathrm{sign}(|x| - a)\frac{2}{\pi}\int_{2\pi||x|-a|\lambda}^{\infty} \frac{\sin s}{s}ds\right|$$
$$\le \frac{C}{a\lambda}\frac{1}{\sqrt{\delta}}\left(1 + \frac{1}{\delta}\right) + \frac{1}{a\pi\lambda} \le \frac{C}{a\lambda}\left(\frac{1}{\sqrt{\delta}}\left(1 + \frac{1}{\delta}\right) + 1\right) \quad \text{if } \frac{|x|}{a} \ge \delta.$$

Therefore $\int_0^{2\pi a\lambda} J_0(\frac{|x|}{a}s)J_1(s)ds = \int_0^{\infty} J_0(\frac{|x|}{a}s)J_1(s)ds - \int_{2\pi a\lambda}^{\infty} J_0(\frac{|x|}{a}s)J_1(s)ds$

$$= \bar{\chi}_a(x) + \left(\frac{a}{|x|}\right)^{1/2} \operatorname{sign}(|x|-a)\frac{2}{\pi}\int_{2\pi||x|-a|\lambda}^{\infty}\frac{\sin s}{s}ds + O(\frac{1}{\lambda}) \text{ (unif. in } \{x \mid \frac{|x|}{a} \geq \delta\})$$

$$= \bar{\chi}_a(x) + \operatorname{sign}(|x|-a)\frac{2}{\pi}\int_{2\pi||x|-a|\lambda}^{\infty}\frac{\sin s}{s}ds$$

$$+ \left(\left(\frac{a}{|x|}\right)^{\frac{1}{2}} - 1\right) \operatorname{sign}(|x|-a)\frac{2}{\pi}\int_{2\pi||x|-a|\lambda}^{\infty}\frac{\sin s}{s}ds + O(\frac{1}{\lambda}),$$

where $\left(\left(\frac{a}{|x|}\right)^{1/2} - 1\right) \operatorname{sign}(|x|-a)\frac{2}{\pi}\int_{2\pi||x|-a|\lambda}^{\infty}\frac{\sin s}{s}ds$

$$= \frac{(\frac{a}{|x|}) - 1}{(\frac{a}{|x|})^{1/2} + 1}\operatorname{sign}(|x|-a)\frac{2}{\pi}\int_{2\pi||x|-a|\lambda}^{\infty}\frac{\sin s}{s}ds \ll ||x|-a| \left|\int_{2\pi||x|-a|\lambda}^{\infty}\frac{\sin s}{s}ds\right|$$

$$\ll \sup_{t>0}\left(t\left|\int_t^{\infty}\frac{\sin s}{s}ds\right|\right)/\lambda \ll 1/\lambda \quad \text{uniformly in } \{x \mid \frac{|x|}{a} \geq \delta\}.$$

Theorem (2.3) Suppose $d \geq 1$, $a > 0$ and m_d is the integral part of $(d-1)/2$, we have the following.

$$(1)\ \sigma_\lambda(\chi_a)(x) = \chi_{a,\lambda}{}^{[d-2m_d]}(|x|) - \sum_{k=1}^{m_d}\left(\frac{a}{|x|}\right)^{\frac{d}{2}-k} J_{\frac{d}{2}-k}(2\pi\lambda|x|)J_{\frac{d}{2}-k}(2\pi\lambda a)$$

$$= \chi_{a,\lambda}{}^{[d-2m_d]}(|x|) - \sum_{k=1}^{m_d}\left(\Gamma(\frac{d}{2}-k+1)\frac{J_{\frac{d}{2}-k}(2\pi\lambda|x|)}{(\pi\lambda|x|)^{\frac{d}{2}-k}}\right)$$

$$\times\left(\frac{\pi^{\frac{d}{2}-k-1}(\lambda a)^{\frac{d}{2}-k-\frac{1}{2}}}{\Gamma(\frac{d}{2}-k+1)}\cos(2\pi\lambda a - \frac{d-2k+1}{2}\pi) + O(\lambda^{\frac{d}{2}-k-\frac{3}{2}})\right).$$

(2) (i) If $d = 1, 2, 3$, $\sigma_\lambda(\chi_a)(x)$ is uniformly bounded in x, and if $d \geq 4$, $\sigma_\lambda(\chi_a)(x)$ is uniformly bounded in the exterior of any neighborhood of $x = 0$.

(ii) If $d = 1, 2$, $\sigma_\lambda(\chi_a)(x)$ is uniformly convergent in the exterior of any neighborhood of spherical surface $|x| = a$, and if $d \geq 3$, $\sigma_\lambda(\chi_a)(x)$ is uniformly convergent in the exterior of any neighborhood of spherical surface $|x| = a$ and of $x = 0$.

(iii) If $d \geq 3$, we have

$$\liminf_{\lambda\to\infty}\frac{\sigma_\lambda(\chi_a)(0)-1}{\lambda^{\frac{d-3}{2}}} = -\frac{\pi^{\frac{d-4}{2}}a^{\frac{d-3}{2}}}{\Gamma(\frac{d}{2})}, \quad \limsup_{\lambda\to\infty}\frac{\sigma_\lambda(\chi_a)(0)-1}{\lambda^{\frac{d-3}{2}}} = \frac{\pi^{\frac{d-4}{2}}a^{\frac{d-3}{2}}}{\Gamma(\frac{d}{2})}.$$

(iv) For all $d = 1, 2, \cdots$, the Gibbs-Wilbraham phenomenon for $\sigma_\lambda(\chi_a)(x)$ reveals at spherical surface $|x| = a$. That is

$$\lim_{\lambda\to\infty} \chi_{a,\lambda}{}^{[d]}(a - \frac{1}{2\lambda}) = \frac{1}{2} + \frac{1}{\pi}\int_0^\pi \frac{\sin s}{s}\, ds = 1 + 0.08949\cdots$$

and

$$\lim_{\lambda\to\infty} \chi_{a,\lambda}{}^{[d]}(a + \frac{1}{2\lambda}) = \frac{1}{2} - \frac{1}{\pi}\int_0^\pi \frac{\sin s}{s}\, ds = -0.08949\cdots.$$

§ 3. On the results of B. Novák for lattice points problem

Břeislav Novák (1938-2003) published many outstanding papers on lattice points problem from the 1960's to the 1980's. We introduce only a small parts of his work which is used here and is contained in [11], [12], [13] and [14].

$$\begin{aligned} P_\alpha(t:x) &:= \frac{1}{\Gamma(\alpha+1)}\sum_{|n|^2<t}(t-|n|^2)^\alpha e^{2\pi i n x} - \frac{\pi^{\frac{d}{2}} t^{\frac{d}{2}+\alpha}}{\Gamma(\frac{d}{2}+\alpha+1)}\delta(x), \\ M_\alpha(t,x) &:= \int_0^t |P_\alpha(s:x)|^2 ds \quad \text{and} \\ T_\alpha(t,x) &:= \Big(\frac{1}{t}M_\alpha(t:x)\Big)^{\frac{1}{2}} = \Big(\frac{1}{t}\int_0^t |P_\alpha(s:x)|^2 ds\Big)^{\frac{1}{2}}, \end{aligned}$$

$$\text{where } \delta(x) := \begin{cases} 1 & \text{if } x \in \mathbf{Z}^d, \\ 0 & \text{otherwise.} \end{cases}$$

I. For every x we have

$$P_\alpha(t:x) = \begin{cases} O(t^{\frac{d}{2}-1}) & 0 \le \alpha < \frac{d}{2}-2, \\ O(t^{\frac{d}{2}-1}\log t) & 0 < \alpha = \frac{d}{2}-2, \\ O(t^{\frac{d}{4}+\frac{\alpha}{2}}) & \alpha > \frac{d}{2}-2, \end{cases}$$

and

$$\liminf_{t\to+\infty} \frac{1}{t^{\frac{d-1}{4}}}\Big(\frac{1}{t}\int_0^t |P_0(s:x)| ds\Big) > 0.$$

II. For every $x \in \mathbf{Q}^d$, we have

$$\begin{aligned} 0 \le \alpha < \tfrac{d}{2}-2 &\implies P_\alpha(t:x) = O(t^{\frac{d}{2}-1}) \text{ (If } x \notin \mathbf{Q}^d \text{ , then } o(t^{\frac{d}{2}-1})). \\ 0 \le \alpha < \tfrac{d-3}{2} &\implies M_\alpha(t:x) = K_\alpha(x)t^{d-1} + O(t^\beta) \ (K_\alpha(x) > 0). \end{aligned}$$

III. For almost all x we have

$$0 \leq \alpha < \frac{d}{2} - 2 \Longrightarrow P_\alpha(t:x) = O(t^{\frac{d}{4}+\frac{\alpha}{2}} \log^{3d} t)$$

and

$$M_0(t:x) = O(t^{\frac{d+1}{2}} \log^{3d+2} t).$$

IV. On $\gamma(x)$, where $\gamma(x) = \gamma(x_1, x_2, \cdots, x_d)$

$= \sup\{\beta \mid \langle x_j k\rangle \ll k^{-\beta}$ for all j have infinitely many solutions in $k \in \mathbf{N}\}$,

we have

$$0 \leq \alpha \leq \frac{d}{2} - 2 - \frac{1}{\gamma(x)} \Longrightarrow \limsup_{t\to+\infty} \frac{\log|P_\alpha(t:x)|}{\log t} = \frac{d}{2} - 1 - \frac{\frac{d}{2} - 1 - \alpha}{2(\gamma(x)+1)},$$
$$0 \leq \alpha \leq \frac{d}{2} - \frac{3}{2} - \frac{1}{2\gamma(x)} \Longrightarrow \limsup_{t\to+\infty} \frac{\log|M_\alpha(t:x)|}{\log t} = d - 1 - \frac{\frac{d}{2} - 1 - \alpha}{\gamma(x)+1} \quad \text{and}$$
$$\alpha > \frac{d}{2} - \frac{3}{2} - \frac{1}{2\gamma(x)} \ \text{and}\ \alpha \geq 0 \Longrightarrow \limsup_{t\to+\infty} \frac{\log|M_\alpha(t:x)|}{\log t} = \frac{d}{2} + \alpha + \frac{1}{2}.$$

Remark. It is well known $\frac{1}{d} \leq \gamma(x) \leq +\infty$ for all x, $\gamma(x) = +\infty$ for $x \in \mathbf{Q}^d$, and $\gamma(x) = \frac{1}{d}$ for almost all $x \in R^d$.

We can prove the following two lemmas by using these.

Lemma (3.1)
If $d \geq 5$ and $x \in \mathbf{Q}^d$ then we have $\limsup_{\lambda\to\infty} \left(\lambda^{\frac{5-d}{2}} |P_0(\lambda^2 : x)\Lambda_0(a:\lambda^2)|\right) > 0.$

Lemma (3.2)
If $d \geq 5$ and $x \in \mathbf{R}^d$, then we have $\limsup_{\lambda\to\infty} \left(\lambda^{\frac{5-d}{2}} |P_l(\lambda^2 : x)\Lambda_l(a:\lambda^2)|\right) = 0 \ (l \geq 1).$

§ 4. On the result of Kuratsubo-Nakai-Ootsubo

The following theorem was proved completely in our paper (Kuratsubo, Nakai and Ootsubo[8]) and the method depends on analysis of the Fourier inversion formula of χ_a, the method of Kuratsubo[4], [5] and the results of B. Novák.

Theorem (4.1) (An extension of the Voronoï-Hardy identity)
Let f_a and $\bar{f}_a$ be as in § 1. Then

$$S_\lambda(f_a)(x) = \bar{f}_a(x) + \mathcal{G}_{a,\lambda}(x) + \mathcal{P}_{a,\lambda}(x) + \mathcal{K}_{a,\lambda}(x),$$

where

$$\begin{aligned}
\mathcal{G}_{a,\lambda}(x) &= \operatorname{sign}(|x|-a)\frac{1}{\pi}\int_{2\pi\lambda||x|-a|}^{\infty}\frac{\sin s}{s}\,ds \\
&+\sum_{n\neq 0}\left(\frac{a}{|x-n|}\right)^{\frac{d+1}{2}+k_d}\operatorname{sign}(|x-n|-a)\frac{1}{\pi}\int_{2\pi\lambda||x-n|-a|}^{\infty}\frac{\sin s}{s}\,ds+O(1/\lambda), \\
\mathcal{P}_{a,\lambda}(x) &= -\sum_{k=1}^{m_d}\left(\frac{a}{|x|}\right)^{\frac{d}{2}-k} J_{\frac{d}{2}-k}(2\pi\lambda|x|)J_{\frac{d}{2}-k}(2\pi\lambda a), \\
\mathcal{K}_{a,\lambda}(x) &=
\end{aligned}$$

$$a^{\frac{d}{2}}\sum_{l=0}^{k_d}\left\{\frac{(\pi a)^l}{\Gamma(\alpha+1)}\left(\sum_{|n|<\lambda}(\lambda^2-|n|^2)^l e^{2\pi i n x}-\int_{|\xi|<\lambda}(\lambda^2-|\xi|^2)^l e^{2\pi i\xi x}\,d\xi\right)\frac{J_{\frac{d}{2}+l}(2\pi a\lambda)}{\lambda^{\frac{d}{2}+l}}\right\},$$

where $k_d = \left(\text{the smallest integer } k \text{ which satisfies } k > \frac{d-1}{2}\right) = m_d+1$. The terms have the following properties:

(1) (a) For all dimensions d, $\mathcal{G}_{a,\lambda}(x) \to 0$ as $\lambda \to \infty$ for all $x \in \mathbf{T}^d$. The convergence is uniform in the exterior of any neighborhood of $\bigcup_{n\in\mathbf{Z}^d}\{x \in \mathbf{T}^d : |x-n| = a\}$.

(b) For all dimensions d, we can choose $x_\lambda \in \mathbf{T}^d$ so that $|x_\lambda - n| = a \mp \frac{1}{2\lambda}$ for all $n \in G_x = \{n \in \mathbf{Z}^d : |x-n| = a\}$ and $\mathcal{G}_{a,\lambda}(x_\lambda) \to \pm c_g j_a(x)$ as $\lambda \to \infty$, where

$$c_g = \frac{1}{\pi}\int_{\pi}^{\infty}\frac{\sin s}{s}\,ds\ (= -0.08949\cdots),\quad j_a(x) = \lim_{\delta\to\infty}\left(\sup_{|x-y|<\delta} f_a(y) - \inf_{|x-y|<\delta} f_a(y)\right).$$

(2) (a) If $1 \le d \le 2$, then $\mathcal{P}_{a,\lambda}(x) = 0$.

(b) If $d \ge 3$, then

$$\liminf_{\lambda\to\infty}\frac{\mathcal{P}_{a,\lambda}(0)}{\lambda^{\frac{d-3}{2}}} = -\frac{\pi^{\frac{d-4}{2}}a^{\frac{d-3}{2}}}{\Gamma(\frac{d}{2})},\quad \limsup_{\lambda\to\infty}\frac{\mathcal{P}_{a,\lambda}(0)}{\lambda^{\frac{d-3}{2}}} = \frac{\pi^{\frac{d-4}{2}}a^{\frac{d-3}{2}}}{\Gamma(\frac{d}{2})},$$

and $\mathcal{P}_{a,\lambda(x)} \to 0$ as $\lambda \to \infty$ for $x \neq 0$. The convergence is uniform in the exterior of any neighborhood of $x = 0$.

(3) (a) If $1 \le d \le 4$, then $\mathcal{K}_{a,\lambda}(x) \to 0$ as $\lambda \to \infty$ uniformly for all $x \in \mathbf{T}^d$.

(b) If $d = 5$, then $\mathcal{K}_{a,\lambda}(x) \to 0$ as $\lambda \to \infty$ for all non-rational points x and $0 < \limsup_{\lambda\to\infty}|\mathcal{K}_{a,\lambda}(x)| < \infty$ for all rational points x.

(c) If $d \ge 6$, then $\mathcal{K}_{a,\lambda}(x) \to 0$ as $\lambda \to \infty$ for a. e. x and $0 < \limsup_{\lambda\to\infty}\frac{|\mathcal{K}_{a,\lambda}(x)|}{\lambda^{\frac{d-5}{2}}} < \infty$ for all rational points x.

§ 5. On the graphs for $d = 6$

When $d = 6$, $a = 9/8$ and $x \in \mathbf{T}^6$, $|x+n| \le 9/8 \Longrightarrow |n| \le 9/8 + |x| \le 9/8 + \sqrt{6}/2 < 3$.

Therefore

$$f_a(x) = \sum_{n \in \mathbf{Z}^d} \chi_a(x) = \sum_{n_1=-2}^{2} \sum_{n_2=-2}^{2} \cdots \sum_{n_6=-2}^{2} \chi_a(x+n).$$

The main term of $\mathcal{K}_{a,\lambda}(x)$ is the first term ($l = 0$), where $\mathcal{K}_{a,\lambda}(x) =$

$$a^3 \sum_{l=0}^{3} \left\{ \frac{(\pi a)^l}{\Gamma(\alpha+1)} \left(\sum_{|n|<\lambda} (\lambda^2 - |n|^2)^l e^{2\pi i n x} - \int_{|\xi|<\lambda} (\lambda^2 - |\xi|^2)^l e^{2\pi i \xi x}\, d\xi \right) \frac{J_{3+l}(2\pi a \lambda)}{\lambda^{3+l}} \right\}.$$

Particularly we remark the sign of $J_3(2\pi a\lambda)$ and it is determined by $\cos(2\pi(9/8)\lambda - 7\pi/4)$.

$$\cos(2\pi(9/8)\lambda - 7\pi/4) = \begin{cases} \cos(\pi/4) = 1/\sqrt{2} & \text{if } \lambda = 800, \\ \cos(\pi/2) = 0 & \text{if } \lambda = 801, \\ \cos(3/4\pi) = -1/\sqrt{2} & \text{if } \lambda = 802, \\ \cos(\pi) = -1 & \text{if } \lambda = 803, \\ \cos(5/4\pi) = -1/\sqrt{2} & \text{if } \lambda = 804, \\ \cos(3/2\pi) = 0 & \text{if } \lambda = 805, \\ \cos(-\pi/4) = 1/\sqrt{2} & \text{if } \lambda = 806, \\ \cos(0) = 1 & \text{if } \lambda = 807. \end{cases}$$

On the other hand $\mathcal{P}_{a,\lambda}(x) = -\sum_{k=1}^{2} \left(\frac{a}{|x|}\right)^{3-k} J_{3-k}(2\pi\lambda|x|) J_{3-k}(2\pi\lambda a)$

$$= \left(\Gamma(3)\frac{J_2(2\pi\lambda|x|)}{(\pi\lambda|x|)^2}\right) \left(\frac{\pi(\lambda a)^{3/2}}{2}\cos(2\pi\lambda a - 5/4\pi) + O(\lambda^{1/2})\right)$$

$$+ \left(\Gamma(2)\frac{J_1(2\pi\lambda|x|)}{\pi\lambda|x|}\right) \left(\pi(\lambda a)^{1/2}\cos(2\pi\lambda a - 3/4\pi) + O(\lambda^{-1/2})\right).$$

When $\lambda = 803$, $\cos(2\pi\lambda a - 5/4\pi) = 0$ and $\cos(2\pi\lambda a - 3/4\pi) = -1$, and when $\lambda = 807$, $\cos(2\pi\lambda a - 5/4\pi) = 0$ and $\cos(2\pi\lambda a - 3/4\pi) = 1$.

Figure 1: The graph of $f_a(x_1, 0, 0, 0, 0, 0)$
Figure 2: The graph of $S_{a,800}(x_1, 0, 0, 0, 0, 0)$, Figure 3: The graph of $S_{a,801}(x_1, 0, 0, 0, 0, 0)$
Figure 4: The graph of $S_{a,802}(x_1, 0, 0, 0, 0, 0)$, Figure 5: The graph of $S_{a,803}(x_1, 0, 0, 0, 0, 0)$
Figure 6: The graph of $S_{a,804}(x_1, 0, 0, 0, 0, 0)$, Figure 7: The graph of $S_{a,805}(x_1, 0, 0, 0, 0, 0)$
Figure 8: The graph of $S_{a,806}(x_1, 0, 0, 0, 0, 0)$, Figure 9: The graph of $S_{a,807}(x_1, 0, 0, 0, 0, 0)$
$(a = 9/8)$

Acknowledgment. Author would like to thank K. Ootsubo for the offer of the figures from 1 to 9 who is one of collabolators([8]).

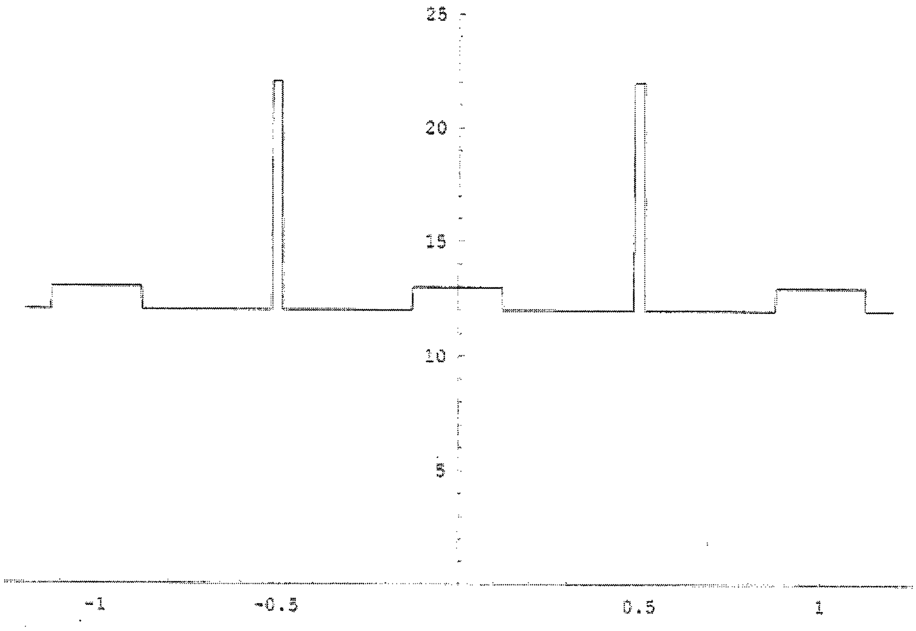


Figure 1

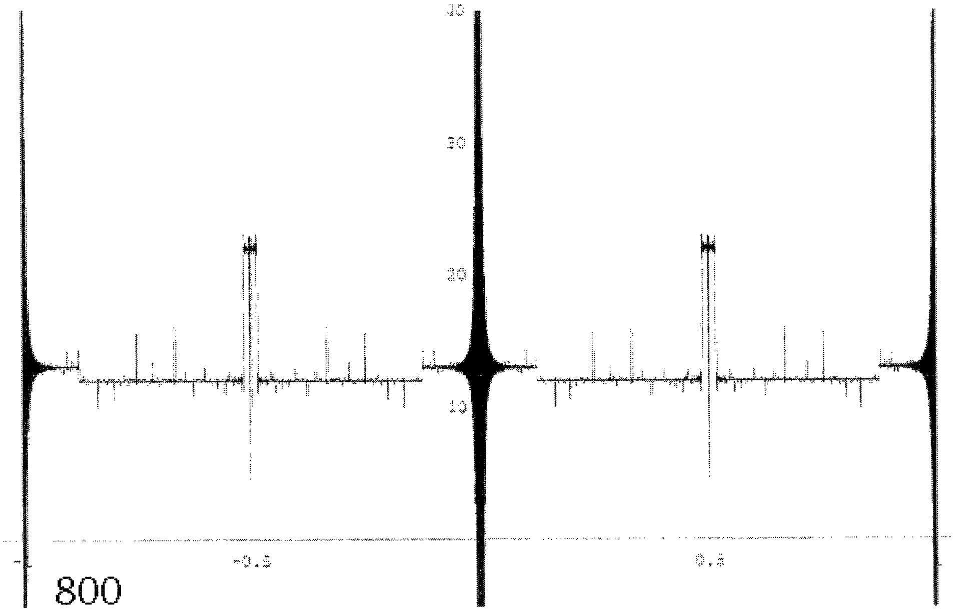


Figure 2

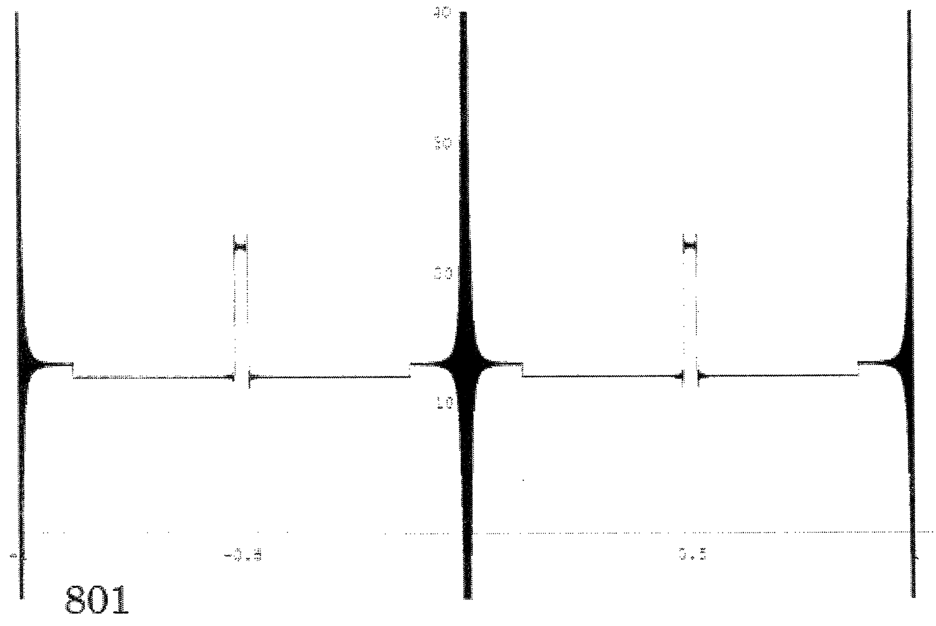
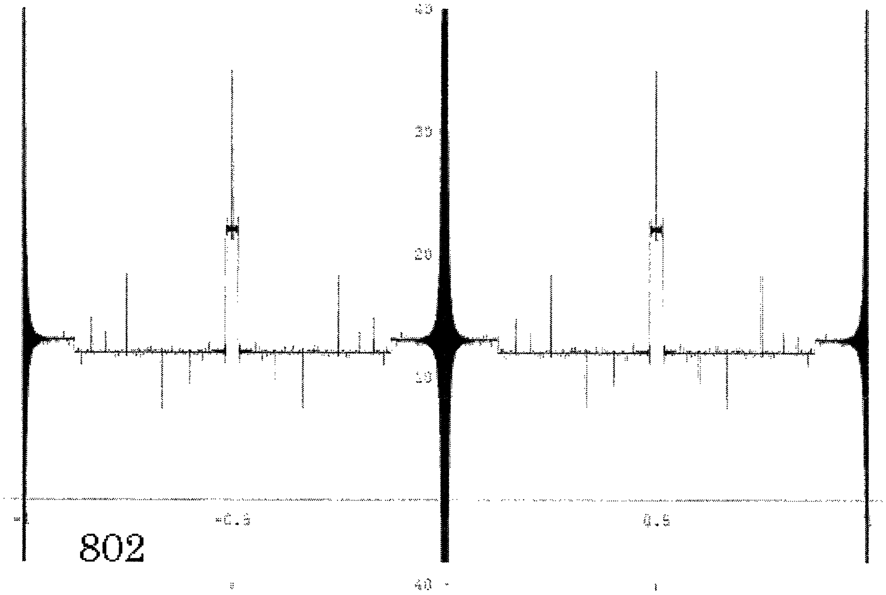


Figure 3



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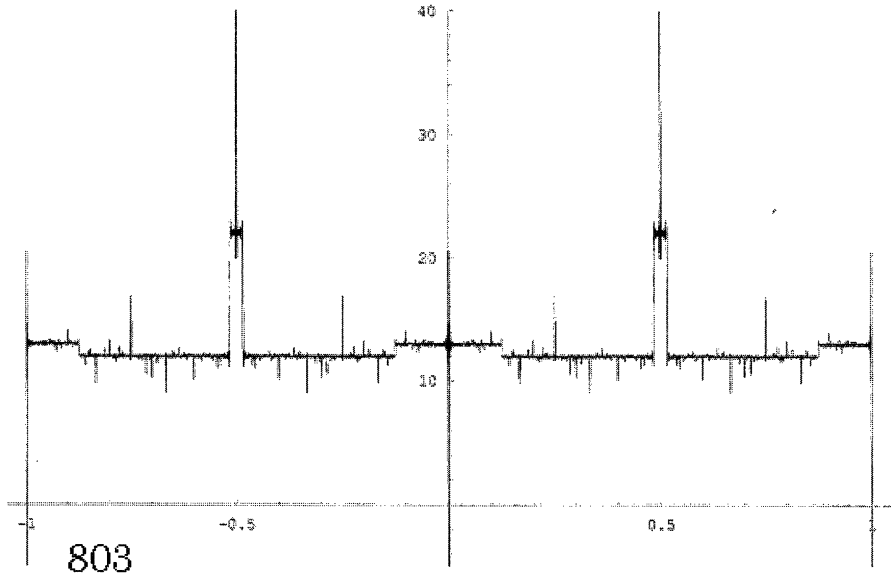
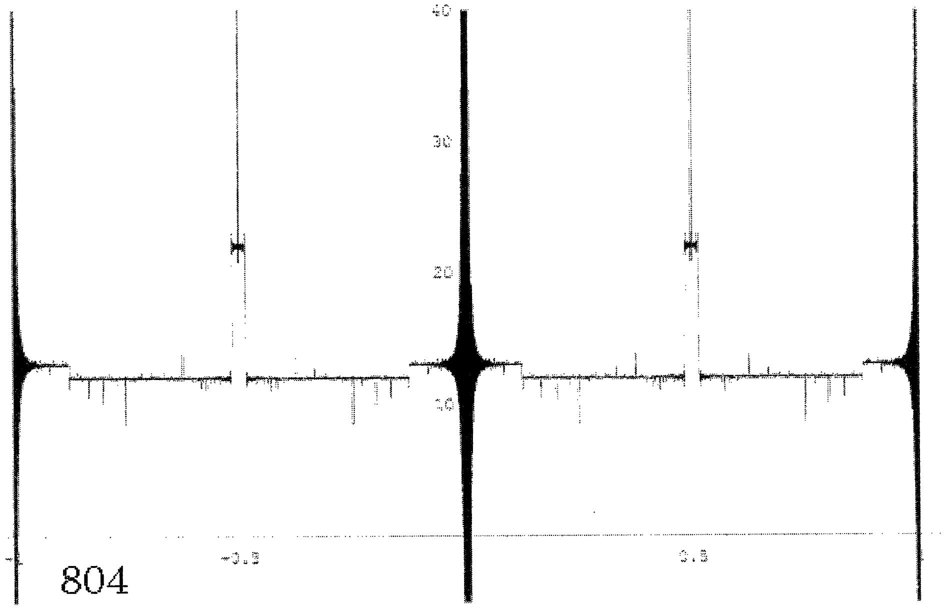


Figure 5



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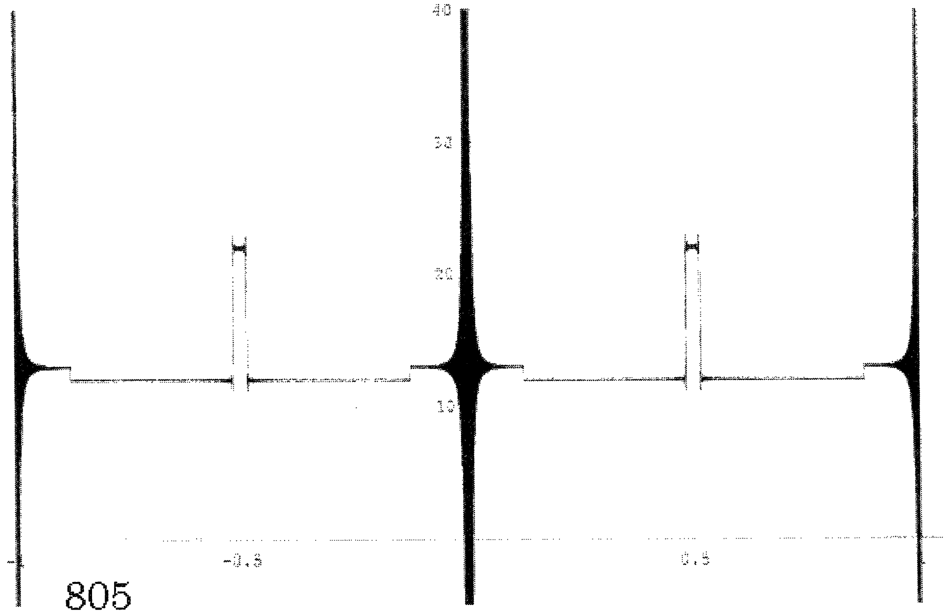

Figure 7

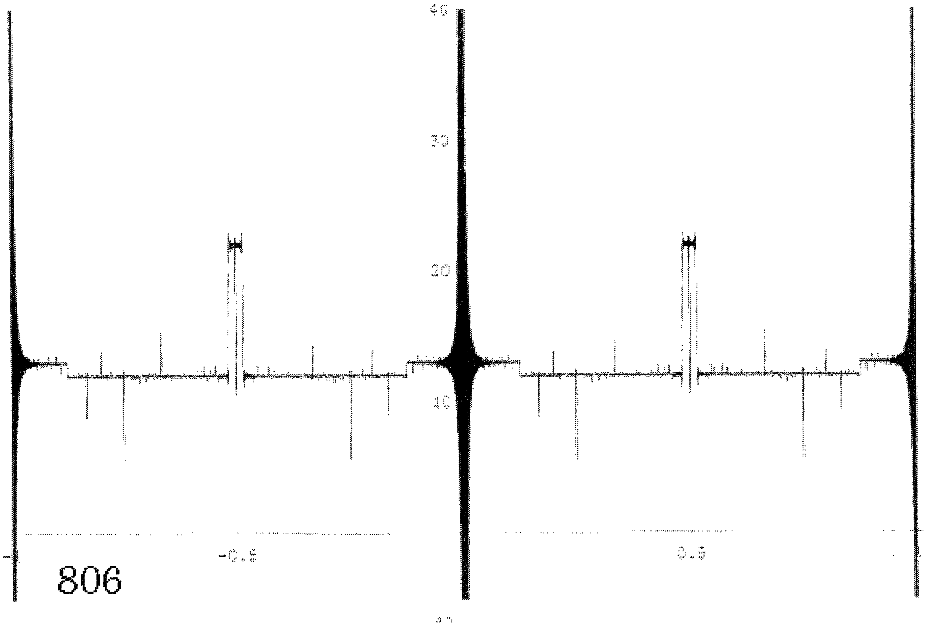

Figure 8

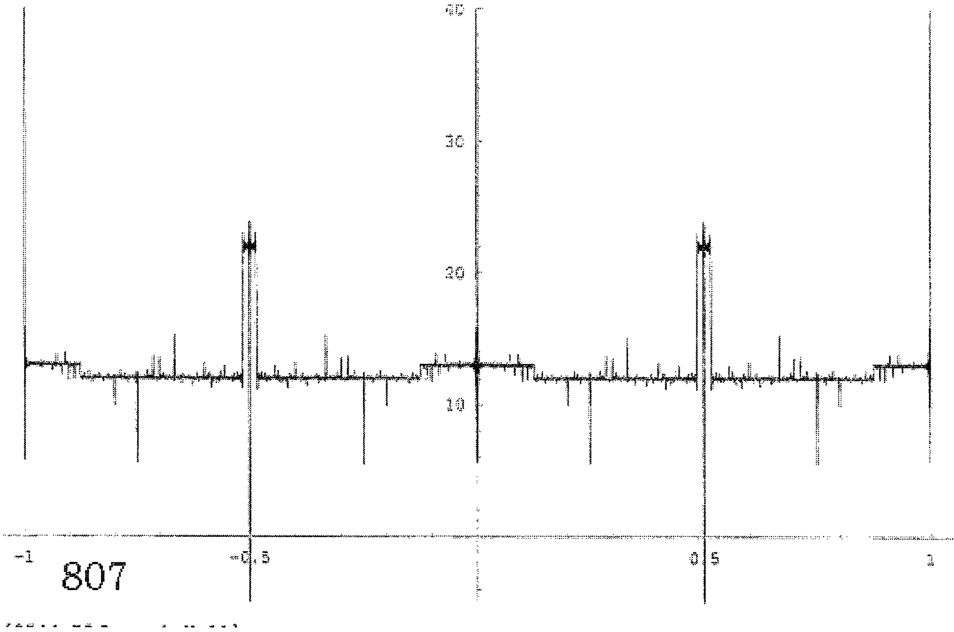

Figure 9

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