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Drift-diffusion System in the Critical Case

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1 Introduction

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We consider two dimensional drift-diffusion system:

$$\begin{cases} \partial_t n - \Delta n + \nabla\cdot(n\nabla\psi) = 0, & t>0, x\in\mathbb{R}^2, \\ \partial_t p - \Delta p - \nabla\cdot(p\nabla\psi) = 0, & t>0, x\in\mathbb{R}^2, \\ -\Delta\psi = \kappa(p-n), \quad x\in\mathbb{R}^2, \\ \quad n(0,x) = n_0(x), \qquad p(0,x) = p_0(x). \end{cases} \tag{1.1}$$

The existence and well-posedness of the solution of the drift-diffusion system (1.1) is obtained by the integral formula via the semigroup representation with a contraction mapping argument. See for example, Kurokiba-Ogawa [33]. This method is also valid for the two dimensional simplified Keller-Segel system:

$$\begin{cases} \partial_t u - \Delta u + \nabla\cdot(u\nabla\psi) = 0, & t>0, x\in\mathbb{R}^2, \\ -\Delta\psi = \kappa u, \quad x\in\mathbb{R}^2, \\ \quad u(0,x) = u_0(x). \end{cases} \tag{1.2}$$

The basic function space for x variable is $L^p(\mathbb{R}^n)$, where $\frac{n}{2} < p < n$, $n = 2,3$ and this framework is analogous to the result for the Navier-Stokes system. While the energy method works also well and we may derive the local well-posedness for two dimensional case critical case $p = 2$ (see for this case [32]). Note that we need to introduce a weighted $L^2(\mathbb{R}^n)$ class since we need to control the solution of the Poisson equation in two dimensions.

On the other hand, if we consider the other critical case $p = 1$ by the method of the integral equation, we should emphasize that the system (1.1) or simpler model (1.2) has the similar scaling structure to two dimensional vortex equation of the Navier-Stokes equation:

$$\begin{cases} \partial_t\omega - \Delta\omega + u\nabla\omega = 0, & t>0, x\in\mathbb{R}^2, \\ -\Delta u = \nabla^\perp\omega = (\partial_1\omega_2, -\partial_2\omega_1), \quad x\in\mathbb{R}^2, \\ \quad \omega(0,x) = \nabla^\perp u_0(x). \end{cases} \tag{1.3}$$

When we consider the two dimensional vortex equation (1.3), we choose the basic function space as $L^1(\mathbb{R}^2)$ and we may derive the existence and uniqueness of the solution of the integral equation

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according to the results Giga-Miyakawa-Osada [21], Giga-Kambe [22]. The space $L^1(\mathbb{R}^2)$ is the invariant space under the scaling scaling

$$\begin{cases} n_\lambda(t,x) = \lambda^2 n(\lambda^2 t, \lambda x), \\ p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x), \\ \psi_\lambda(t,x) = \psi(\lambda^2 t, \lambda x), \end{cases} \tag{1.4}$$

that keep the equation invariant. It is important to solve the equation in such an invariant class because we may employ so called the *Fujita-Kato Principle* for the semilinear equation.

We first recall how to obtain the solution in the basic class $L^1(\mathbb{R}^2)$ following the results [21] and [22]. The similar argument is possible to apply for the system scaling (1.1) and (1.2) and for simplicity, we only treat the case (1.2).

We first introduce the corresponding integral equation for the Keller-Segel system (1.2);

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\nabla(u\nabla\psi)ds.$$

Let $\|u\| \equiv \sup_{t\in I} t^{1-1/p}\|u(t)\|_p$, where $I = [0,T)$ and $4/3 \le p < 2$. Then we have

$$\begin{aligned} \|u(t)\|_p \le & \|e^{t\Delta}u_0\|_p + C\int_0^t |t-s|^{-(1/r-1/p)-1/2}\|u\nabla\psi\|_r ds \\ \le & \|e^{t\Delta}u_0\|_p + C\int_0^t |t-s|^{-(1/r-1/p)-1/2}\|u(s)\|_p\|\nabla\psi\|_s ds \\ \le & \|e^{t\Delta}u_0\|_p + C\int_0^t |t-s|^{-1/p}\|u(s)\|_p^2 ds \\ \le & \|e^{t\Delta}u_0\|_p + C\int_0^t |t-s|^{-1/p}s^{-2(1-1/p)}ds\big(\sup_{t\in I} t^{(1-1/p)}\|u(t)\|_p\big)^2, \end{aligned} \tag{1.5}$$

where $1/r = 1/p + 1/s$ and $1/s = 1/p - 1/2$. Namely we have $1/r - 1/p + 1/2 = 1/p$. Then we need

$$\frac{1}{r} = \frac{2}{p} - \frac{1}{2} \le 1$$

and

$$\frac{2}{p} \le \frac{3}{2}$$

implies $4/3 \le p$. Hence under the condition $4/3 \le p < 2$, the integral

$$\int_0^t |t-s|^{-1/p}s^{-2(1-1/p)}ds = t^{(1-1/p)}B$$

converges, where $B > 0$ is a constant determined by the beta function. Then we have

$$\sup_I t^{1-1/p}\|u(t)\|_p \le \sup_I t^{1-1/p}\|e^{t\Delta}u_0\|_p + B\big(\sup_{t\in I} t^{(1-1/p)}\|u(t)\|_p\big)^2.$$

Now we see by $t \to 0$ the first term $\sup_I t^{1-1/p}\|e^{t\Delta}u_0\|_p$ can be small (this follows from the fact C_0^∞ is dense L^p and the initial data u_0 may be approximated by a C_0^∞ function that as $t \to 0$ we have the a priori bound for the solution. Analogous method may implies the estimate for the difference of the solutions. To show that the solution is belonging to $L^1(\mathbb{R}^2)$, we treat the integral equation in L^1 and use the L^1-L^1 boundedness for the heat flow to have

$$
\begin{aligned}
\|u(t)\|_1 \le& \|e^{t\Delta}u_0\|_1 + C\int_0^t |t-s|^{-1/2}\|u\nabla\psi\|_1 ds \\
\le& \|u_0\|_1 + C\int_0^t |t-s|^{-1/2}\|u(s)\|_{4/3}\|\nabla\psi\|_4 ds \\
\le& \|u_0\|_1 + C\int_0^t |t-s|^{-1/2}\|u(s)\|_{4/3}\|u(s)\|_{4/3} ds \\
\le& \|u_0\|_1 + C\int_0^t |t-s|^{-1/2}s^{-1/2}ds\big(\sup_{t\in I} t^{1/4}\|u(t)\|_{4/3}\big)^2,
\end{aligned}
\tag{1.6}
$$

where

$$
\|\nabla(-\Delta)^{-1}u(s)\|_4 \le C\|u(s)\|_{4/3}, \qquad \frac{1}{4} = \frac{3}{4} - \frac{1}{2}
$$

is the Hardy-Littlewood-Sobolev inequality in $n = 2$. The last integration is finite and it follows the uniform boundedness of $\sup_{t\in I} t^{1/4}\|u(t)\|_{4/3}$ and hence the solution belongs to L^1.

Giga-Miyakawa-Osada [21] applied the above method to the vortex equation (1.3) to show the existence and uniqueness of the solution for the initial data in $L^1(\mathbb{R}^2)$.

Proposition 1.1 (Giga-Miyakawa-Osada) *Let $4/3 \le p$. For $\omega_0 \in L^1(\mathbb{R}^2)$ the two dimensional vortex equation of the Navier-Stokes system* (1.3) *has a unique local solution $\omega(t) \in C\big([0,T); L^1(\mathbb{R}^2)\big) \cap C\big((0,T); L^p(\mathbb{R}^2)\big)$.*

We may apply the similar method to the 2 dimensional simplified Keller-Segel system and we may have the following:

Theorem 1.2 *For $u_0 \in L^1(\mathbb{R}^2)$, there exists a unique time local solution u for the two dimensional Keller-Segel equation* (1.2) *and it satisfies $u(t) \in C\big([0,T); L^1(\mathbb{R}^2)\big) \cap C\big((0,T); L^p(\mathbb{R}^2)\big)$.*

With this regards, the result in Kurokiba-Ogawa [32] can be extended into the case $1 \le p \le 2$ when we consider $n = 2$.

For the vortex equation (1.3), the solution $\omega(t)$ satisfies the maximum principle and the uniform a priori bound follows. This shows the solution globally exists. On the other hand, for the Keller-Segel system (1.2) and the drift-diffusion system (1.1), the solutions of those system do not satisfy the maximum principle and hence we need to employ the entropy functional to establish the existence of the global solution. Indeed, Nagai-Senba-Yoshida [45], Biler [2], Nagai-Senba-Suzuki [44] employ the entropy functional to show the existence of the time global solution of (1.2) for a bounded domain Ω:

$$
\begin{aligned}
\int_{\Omega|} u\log u dx - \frac{1}{2}\int_\Omega u\psi dx + \int_0^t\int_\Omega u\big|\nabla(\log u - \psi)\big| dxdt \\
\le \int_\Omega u_0 \log u_0 dx - \frac{1}{2}\int_\Omega u_0(-\Delta)^{-1}u_0 dx.
\end{aligned}
\tag{1.7}
$$

We may derive the Lyapunov function from this entropy functional and the justification of this functional is very important. We see by a formal computation that

$$
\int_\Omega u\log u dx - \frac{1}{2}\int_\Omega u\psi dx = \int_\Omega u\log u dx - \frac{1}{2}\|\nabla\psi\|_2^2,
$$

however the second term of the right hand side does not have a sense as far as we consider the positive solution. Namely since $|x| \to \infty$

$$\psi \simeq \log |x|^{-1}$$

it follows that $\nabla\psi \notin L^2(\mathbb{R}^2)$.

This difficulty can be recovered by introducing "zero mean solutions". If we consider the case when the difference of the two carriers densities is equal to zero, we may consider the zero mean solution for the difference of the solution of the drift-diffusion system and one may justify the above integration by parts for this type of solutions. With this regards, we introduce the Hardy class $\mathcal{H}^1(\mathbb{R}^2)$ and then the entropy functional can have its meaning for the solution in this class. For the Navier-Stokes system, Miyakawa [37], [38] introduced the solution in the Hardy space. For the solution of the Navier-Stokes system, it is natural to introduce the Hardy space if the solution has sufficiently fast decay at $|x| \to \infty$ because of the divergence free condition.

For our case, we need to arrange the system in order to adjust the Hardy space with zero mean value. Introducing $v = n + p$ and $w = n - p$, the equivalent form of the system (1.1) is as follows:

$$\begin{cases} \partial_t v - \Delta v + \nabla \cdot (w\nabla\psi) = 0, \qquad t > 0, x \in \mathbb{R}^2, \\ \partial_t w - \Delta w + \nabla \cdot (v\nabla\psi) = 0, \qquad t > 0, x \in \mathbb{R}^2, \\ -\Delta\psi = -\kappa w, \quad x \in \mathbb{R}^2, \\ \qquad v(0,x) = n_0(x) + p_0(x), \qquad w(0,x) = n_0(x) - p_0(x). \end{cases} \tag{1.8}$$

In this system we may naturally assume that w has the "zero mean value". For v, we consider the deviation from the average $v - \bar{v}$, where

$$\bar{v} = \int_{\mathbb{R}^2} v(t,x)dx = \int_{\mathbb{R}^2} v_0(x)dx$$

and we obtain the system:

$$\begin{cases} \partial_t v - \Delta v + \nabla \cdot (w\nabla\psi) = 0, \qquad t > 0, x \in \mathbb{R}^2, \\ \partial_t w - \Delta w + \kappa\bar{v}w + \nabla \cdot (v\nabla\psi) = 0, \qquad t > 0, x \in \mathbb{R}^2, \\ -\Delta\psi = -\kappa w, \quad x \in \mathbb{R}^2, \\ \qquad v(t,x) \to 0, \quad w(t,x) \to 0, \quad |x| \to \infty, \\ \qquad v(0,x) = v_0 \equiv n_0(x) + p_0(x) - \overline{n_0 + p_0}, \\ \qquad w(0,x) = w_0 \equiv n_0(x) - p_0(x). \end{cases} \tag{1.9}$$

We choose the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ as the basic function class and shows the time local existence and well-posedness.

Definition. For $\lambda > 0$ and $\phi \in \mathcal{S}(\mathbb{R}^2)$, we let $\phi_\lambda = \lambda^{-2}\phi(\lambda^{-1}x)$ then for $0 < p < \infty$, we define the Hardy space $\mathcal{H}^p$ by

$$\mathcal{H}^p = \mathcal{H}^p(\mathbb{R}^2) = \left\{ f \in L^p_{loc}(\mathbb{R}^2); \quad \|f\|_{\mathcal{H}^p} \equiv \left\| \sup_{\lambda>0} |\phi_\lambda * f| \right\|_p < \infty \right\}.$$

In particular, it is well known that for $p = 1$, the dual space of the Hardy space $\mathcal{H}^1$ coincides with a class of the *bounded mean oscillation*)

$$BMO = \{f \in L^1_{loc}(\mathbb{R}^2); \|f\|_{BMO} < \infty\}.$$

Our main result is the following:

Theorem 1.3 (Ogawa-Shimizu[47]) *Let $\kappa = \pm 1$ and the initial data $(v_0, w_0) \in \mathcal{H}^1(\mathbb{R}^2) \times \mathcal{H}^1(\mathbb{R}^2)$ then there exists $T > 0$ and a unique solution (v, w) of (1.9) and satisfying $v,\, w \in C\big([0,T); \mathcal{H}^1\big) \cap L^2(0,T; \dot{\mathcal{H}}^{1,1}) \cap C\big((0,T); \dot{\mathcal{H}}^{2,1}\big) \cap C^1\big((0,T); \mathcal{H}^1\big)$. Moreover the solution flow map $(v_0, w_0) \to (v, w)$ is the Lipschitz continuous $\mathcal{H}^1(\mathbb{R}^2)^2 \to C\big([0,T); \mathcal{H}^1\big)^2$.*

The equation (1.9) has the invariant scale in the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ and according to the Fujita-Kato principle, the existence of the time global solution in the same class for the small initial data immediately follows if the initial data is sufficiently small and $\kappa \bar{v} > 0$. However such a result is not quite important for this kind of system since the repulsive drift-diffusion system should have a global solution in large data and the attractive system (that has the opposite sign of the nonlinear coupling) has a finite time blowing up solution and it is required for identifying the threshold value for the global existence of the solution.

When $f \in \mathcal{H}^1(\mathbb{R}^2)$, it is known that the following estimate holds: For $f \in \mathcal{H}^1(\mathbb{R}^2)$, the Fourier transform $\hat{f}$ is subject to

$$\int_{\mathbb{R}^2} \frac{|\hat{f}(\xi)|^2}{|\xi|^2} d\xi \le C \|f\|_{\mathcal{H}^1}^2. \tag{1.10}$$

In this case, if $-\Delta\psi = w$ and $w \in \mathcal{H}^1(\mathbb{R}^2)$, then we have

$$\|\nabla\psi\|_2^2 = \int_{\mathbb{R}^2} |\xi\widehat{\psi}|^2 d\xi = \int_{\mathbb{R}^2} \frac{|\widehat{w}|^2}{|\xi|^2} d\xi \le C\|w\|_{\mathcal{H}^1}^2$$

and this shows the entropy functional remains finite for the solution in the Hardy space.

2 L^1-type Energy Inequality

The proof of Theorem 1.3 essentially rely on the endpoint type maximal regularity. The detailed proof can be found in [47]. We summarize the crucial part. For the solution of the initial value problem of the heat equation, the key estimate is considered as a type of the energy estimate: It is well known that the solution of heat equation

$$\begin{cases} \partial_t u - \Delta u = 0, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = \phi(x). \end{cases} \tag{2.1}$$

satisfies the energy inequality:

$$\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds \le \|u_0\|_2^2$$

In particular, for the smooth solution we may derive the energy equality for the solution. The L^p version of the estimate with $1 < p < \infty$ is known as the parabolic estimate. We establish the corresponding estimate for $p = 1$ in the parabolic estimate when we exchange L^1 space into $\mathcal{H}^1$.

Theorem 2.1 *Let $e^{t\Delta}$ be the heat semi group and $\phi \in \mathcal{H}^1$. Then we have*

$$\left(\int_0^T \|\nabla e^{t\Delta}\phi\|_{\mathcal{H}^1}^2 dt\right)^{1/2} \le C\|\phi\|_{\mathcal{H}^1}, \tag{2.2}$$

where C is a positive constant independent of $T > 0$.

Theorem 2.2 *Let $1 < \theta < \infty$ and u be a solution of the inhomogeneous heat equation:*

$$\begin{cases} \partial_t u - \Delta u = f, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = \phi(x). \end{cases} \tag{2.3}$$

Then we have

$$\|\nabla u\|_{L^2(I;\dot{\mathcal{B}}^0_{1,2})} \le C\left(\|\phi\|_{\mathcal{H}^1} + \|f\|_{L^2(I;\mathcal{H}^{-1,1})}\right),$$

where $I = \mathbb{R}_+$.

This estimate is essentially equivalent to the endpoint type maximal regularity for the solution of the heat equation in the Besov space (cf. Ogawa-Shimizu [48]). The detailed proof requires the interpolation argument in the real analytic method.

The proof of Theorem 2.1 needs various estimates in the real and harmonic analysis and the meaning of the estimate is not directly understandable. To explain a heuristic reason that the estimate in Theorem 2.1 holds, we show the following proposition.

Proposition 2.3 *Let $e^{t\Delta}$ be the heat semi group and let $\phi \in L^1(\mathbb{R}^2)$. Then the following estimate*

$$\left(\int_0^T \|\nabla e^{t\Delta}\phi\|_1^2 dt\right)^{1/2} \le C\|\phi\|_1 \tag{2.4}$$

generally fails.

Proof of Proposition 2.3. Noting the L^1-L^∞ estimate for the Fourier transform:

$$\sup_\xi |\hat{f}(\xi)| \le \frac{1}{2\pi}\|f\|_1$$

we set $f = \nabla e^{t\Delta}\phi$ to see that

$$\begin{aligned}
\int_0^\infty \|\nabla e^{t\Delta}\phi\|_1^2 dt \ge& 4\pi^2 \int_0^\infty \left(\sup_\xi |\xi e^{-t|\xi|^2}\hat{\phi}(\xi)|\right)^2 dt \\
=& 4\pi^2 \int_0^\infty \frac{1}{t}\left(\sup_\xi |\sqrt{t}\xi e^{-t|\xi|^2}\hat{\phi}(\xi)|\right)^2 dt \\
& \text{(where we choose some } \eta \text{ instead of taking supremum over } \xi) \\
\ge& 4\pi^2 \int_0^\infty \frac{1}{t}\left(|\eta e^{-|\eta|^2}|\right)^2 |\hat{\phi}(\sqrt{t}^{-1}\eta)|^2 dt \\
\ge& C\int_0^\infty \frac{|\hat{\phi}(\sqrt{t}^{-1}\eta)|^2}{t} dt \\
& \text{letting} \frac{|\eta|}{\sqrt{t}} = r, \text{ we see from} \qquad dt = -2\frac{|\eta|^2}{r^3}dr \\
=& C\int_0^\infty \frac{|\hat{\phi}(r\omega)|^2}{r} dr.
\end{aligned}$$

Here $\omega = \eta/|\eta|$ is a unit vector. Integrating in $\omega \in \mathbb{S}^1$ and taking the average, we obtain that

$$\int_0^\infty \|\nabla e^{t\Delta}\phi\|_1^2 dt \geq C(\eta) \int_{\mathbb{R}^2} \frac{|\hat{\phi}(\xi)|^2}{|\xi|^2} d\xi. \tag{2.5}$$

By taking appropriate ϕ, the right hand side diverges in generally (for instance, choose $\hat{\phi}(0) \neq 0$). Even if the integral average of ϕ is 0, we may not obtain the finiteness of the right hand side only assuming that $\phi \in L^1(\mathbb{R}^2)$. ❑

We should note that Proposition 2.3 itself gives an another proof of the estimate (1.10). Note that Proposition 2.3 can be derived without using (1.10).

To establish the solvability of the system (1.1) in the Hardy space we need to show the bilinear estimate for the nonlinear term. The following estimate is a generalization of the well known estimate for the product of the divergence free vector and rotation free vector due to Coifman-Lions-Mayer-Semmes [12].

Proposition 2.4 *Let $\nabla w \in \mathcal{H}^1(\mathbb{R}^2)$ and $\nabla\psi \in \dot{H}^1 \cap L^\infty$. Then we have the following estimate:*

$$\|\nabla \cdot (w\nabla\psi)\|_{\mathcal{H}^1} \leq C\big(\|w\|_2\|\Delta\psi\|_2 + \|\nabla w\|_{\mathcal{H}^1}\|\nabla\psi\|_\infty\big).$$

Analogous estimate of the above proposition is known in the Triebel-Lizorkin space, however it does not include the endpoint case: For $f,\ g \in \dot{F}^s_{p,\sigma} \cap L^r$ with $1 < p \leq \infty$, $1 \leq \sigma \leq \infty$ $1/p = 1/r + 1/q$ and $s > 0$, it holds that

$$\|fg\|_{\dot{F}^s_{p,\sigma}} \leq C\big(\|f\|_r\|g\|_{\dot{F}^s_{q,\sigma}} + \|f\|_{\dot{F}^s_{q,\sigma}}\|g\|_r\big)$$

The proof of the above estimate requires the L^p boundedness of the maximal function and the limiting case $p = 1$ is eliminated. Our estimate is corresponding to the case $p = 1$ by observing

$$\|\nabla(fg)\|_{\dot{F}^0_{1,2}} \simeq \|\nabla(fg)\|_{\mathcal{H}^1}.$$

3 End-point Maximal Regularity

As an application of Theorem 1.1, we study the well posedness issue for the Cauchy problem of a semi-linear elliptic parabolic system. Let $u(t,x)$ and $\psi(t,x) : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (u\nabla\psi) = 0, & t > 0,\ x \in \mathbb{R}^2, \\ -\Delta\psi = \kappa u, & t > 0,\ x \in \mathbb{R}^2, \\ \quad u(0,x) = u_0(x), & x \in \mathbb{R}^2 \end{cases} \tag{3.1}$$

with $\kappa = \pm 1$. When $\kappa = 1$, the system (3.1) describes a model for the chemotaxis called (simplified) Keller-Segel system [30], Jäger-Luckhaus system or Nagai model ([26], [41], see also [1]-[5], [24], [27], [30], [32], [33], [39], [40]-[42], [52]). When $\kappa = -1$, the system (3.1) is called as a mono-polar drift-diffusion system for the semi-conductor simulation.

It would be worth to compare our result in the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ with the case for other critical spaces. One may possibly consider the homogeneous Besov space $\dot{B}^0_{1,2}$.

Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Paley-Littlewood dyadic decomposition of unity satisfying that

$$\sum_{j\in\mathbb{Z}} \hat{\phi}(2^{-j}\xi) = 1$$

for all $\xi \neq 0$. For $s \in \mathbb{R}$ and $1 \le p, \sigma \le \infty$, we define the homogeneous Besov space $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ by

$$\dot{B}^s_{p,\sigma} = \{f \in \mathcal{S}^*/\mathcal{P}; \|f\|_{\dot{B}^s_{p,\sigma}} < \infty\}$$

with the norm

$$\|f\|_{\dot{B}^s_{p,\sigma}} \equiv \begin{cases} \Big(\sum_{j\in\mathbb{Z}} 2^{js\sigma}\|\phi_j * f\|_p^\sigma\Big)^{1/\sigma}, 1 \le \sigma < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\phi_j * f\|_p, \ \sigma = \infty \end{cases} \tag{3.2}$$

and $\mathcal{P}$ denotes all polynomials.

Let $C^\infty_{0,0}(\mathbb{R}^n) = \{f \in C^\infty_0(\mathbb{R}^n); \hat{f}(0) = 0\}$. When $p = \infty$, since $C^\infty_{0,0}(\mathbb{R}^n)$ is not dense in $\dot{B}^s_{p,\sigma}$, we denote the completion of $C^\infty_{0,0}(\mathbb{R}^n)$ by the homogeneous Besov norm $\dot{B}^s_{\infty,\sigma}$ as $\dot{\mathcal{B}}^s_{\infty,\sigma}$;

$$\dot{\mathcal{B}}^s_{\infty,\sigma} = \overline{C^\infty_{0,0}(\mathbb{R}^n)}^{\dot{B}^s_{\infty,\sigma}}.$$

For $1 < \sigma$, $\sigma' < \infty$ with $1/\sigma + 1/\sigma' = 1$, since

$$(\dot{\mathcal{B}}^0_{\infty,\sigma'})^* = \dot{B}^0_{1,\sigma}, \qquad (\dot{B}^0_{1,\sigma})^* = \dot{B}^0_{\infty,\sigma'},$$

and $\dot{\mathcal{B}}^0_{\infty,\sigma'} \subsetneq \dot{B}^0_{\infty,\sigma'}$, $\dot{B}^0_{1,\sigma}$ is not reflexive.

Theorem 3.1 (Endpoint maximal regularity) (Ogawa-Shimizu[48]) *Let $1 < \rho, \sigma \le \infty$ and $I \subset \mathbb{R}_+$ be an open interval. For $f \in L^\rho(I; \dot{B}^0_{1,\rho}(\mathbb{R}^n))$ and $u_0 \in \dot{B}^{2(1-1/\rho)}_{1,\rho}(\mathbb{R}^n)$, let u be a solution of the heat equation*

$$\begin{cases} \partial_t u - \Delta u = f, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.3}$$

Then we have

$$\|\partial_t u\|_{L^\rho(I;\dot{B}^0_{1,\rho})} + \|D^2 u\|_{L^\rho(I;\dot{B}^0_{1,\rho})} \le C(\|u_0\|_{\dot{B}^{2(1-1/\rho)}_{1,\rho}} + \|f\|_{L^\rho(I;\dot{B}^0_{1,\rho})}). \tag{3.4}$$

Remark. A similar but slightly different estimate for the inhomogeneous heat equation is given by [51]. The estimate appears in terms of the different combination of the space-time norm. Indeed, their result coincides the above estimate when the norm is restricted such a way that the second exponent of the Besov space is the same as the time exponent.

The homogeneous part of the above estimate (namely the case $f \equiv 0$) can be generalized a little further. We are particularly interested in the special case $\rho = 2$. If we reduce the regularity of the regularity of the solution, the homogeneous part of the above estimate can be written as the following:

$$\left(\int_0^T \|\nabla e^{t\Delta} u_0\|^2_{\dot{B}^0_{1,\sigma}}\, dt\right)^{1/\rho} \le C\|u_0\|_{\dot{B}^0_{1,2}}, \tag{3.5}$$

where $1 \leq \sigma \leq \infty$. This estimate is optimal in view Proposition 2.3. On the other hand, if we restrict the initial data in the Besov space $\dot{B}^0_{1,2}$ then we see

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{|\widehat{u_0}(\xi)|^2}{|\xi|^2} d\xi =&\sum_j \int_{|\xi|=2^j}^{2^{j+1}} \int_{|\omega|=1} \frac{|\widehat{u_0}(\xi)|^2}{|\xi|^2} |\xi| d|\xi| d\omega \\ =&\sum_j \int_{|\xi|=2^j}^{2^{j+1}} \int_{|\omega|=1} |\widehat{u_0}(\xi)|^2 \frac{d|\xi|}{|\xi|} d\omega \\ \leq& \sum_j \sup_\xi |\widehat{\phi_j}(\xi)\widehat{u_0}(\xi)|^2 \int_{|\xi|=2^j}^{2^{j+1}} \frac{d|\xi|}{|\xi|} \int_{|\omega|=1} d\omega \\ \leq& 2\pi \log 2 \sum_j \|\phi_j * u\|_1^2 \leq C\|u_0\|^2_{\dot{B}^0_{1,2}}.\end{aligned} \tag{3.6}$$

The nature of those system is quite similar except when it describes the aggregation of chemotaxis or the repulsive behavior of charges. A distinct difference is that the former has a blowing up solutions while the latter is a stable system and has the global solution for the large data. This difference is basically stems from the sign of the coupling term. Similar situation is observed in the corresponding simpler model of the semi-linear heat equation:

$$\begin{cases} \partial_t u - \Delta u + \alpha u^p = 0, & x \in \mathbb{R}^n, \\ \quad u(0,x) = u_0(x) > 0, & x \in \mathbb{R}^n. \end{cases} \tag{3.7}$$

It is well known that if $\alpha > 0$, there exists a global solution in large data and if $\alpha < 0$ and $p > 1+2/n$ there exists a global solution for small data while there exists a blowing up solution if $p \leq 1+2/n$ (cf. Fujita [16], Hayakawa [23]). Moreover some microscopic behavior is completely classified by Giga-Kohn [22]. By this classification, we call $\alpha > 0$ is non-attractive case (defocussing case) and $\alpha < 0$ is attractive case (focusing case). We notice that the two dimensional case is corresponding to the critical case for the quadratic nonlinearity since $n = 2$ gives $p = 2$.

By the Fujita-Kato argument ([17]), it is possible to construct the strong solution from the corresponding integral equation (see Kurokiba-Ogawa [33]). However the base function space in it is L^p for $n/2 < p < n$ and again the limiting scale L^1 is excluded. Hence it is still a room for improving the Banach scale, where the wellposedness for the system may hold in valid as well as the equation remains invariant under the scaling.

The system (3.1) conserves total mass:

$$\int_{\mathbb{R}^2} u(t,x)dx = \int_{\mathbb{R}^2} u_0(x)dx.$$

Let u be in $L^1(\mathbb{R}^2) \cap L\log L(\mathbb{R}^2)$. We put a functional as

$$V(t) \equiv \int_{\mathbb{R}^2} \{(1+|u(t)|)\log(1+|u(t)|) - |u(t)|\}dx - \frac{1}{2\kappa}\int_{\mathbb{R}^2} \frac{u(t)}{|u(t)|}|\nabla\psi(t)|^2 dx \tag{3.8}$$

Then the following inequality holds.

$$\begin{aligned}\frac{d}{dt}V(t)+&\int_{\mathbb{R}^2}(1+|u(t)|)|\nabla(\log(1+|u(t)|)) - \nabla\psi(t))|^2 dx \\ \leq& \int_{\mathbb{R}^2} \{(1+|u(t)|)\log(1+|u(t)|) - |u(t)|\}dx - \|\nabla\psi(t)\|_2^2.\end{aligned} \tag{3.9}$$

If we restrict our problem for the positive solution, the mass conservation law assures the L^1 a priori estimate. However it is not sufficient to show the global existence of the solution for (3.1). The existence of a solution of the equation (3.1) in $L^1(\mathbb{R}^2)$ can be proved very much similar way to the case of the vorticity equation of the 2 dimensional Navier-Stokes system: For $\omega = \text{rot } u \equiv \nabla_1 u_2 - \nabla_2 u_1 : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$,

$$\begin{cases} \partial_t \omega - \Delta\omega + u \cdot \nabla\omega = 0, & t > 0,\ x \in \mathbb{R}^2, \\ -\Delta u = \nabla^{\perp}\omega, & t > 0,\ x \in \mathbb{R}^2, \\ \quad \omega(0,x) = \omega_0(x) & \qquad x \in \mathbb{R}^2, \end{cases} \tag{3.10}$$

where $\nabla^{\perp} = (\nabla_2, -\nabla_1)$ denotes the skew adjoint gradient in $\mathbb{R}^2$ (cf, Giga-Miyakawa-Osada [21], Giga-Kambe [22], Gallay-Wayne [18]). The vorticity in 2 dimensions is subject to the comparison principle and the pointwise a priori estimate follows from the equation. Our problem (3.1), however, does not have the comparison principle and hence the a priori estimate for the free energy $V(t)$ is required for the global existence or even global existence of the small data solution up to the threshold mass ([43], [44], [45], [49]). In this sense, it is interesting problem to consider the initial value problem in the space where the free energy functional is well defined.

In our first result [47], we consider the bi-polor type drift-diffusion system in the critical Hardy space $\mathcal{H}^1$ where the entropy functional is rigorously justified and showed that there exists a unique time local solution of (3.1) for any initial data in $\mathcal{H}^1$.

In this paper, we consider slightly larger class $\dot{B}^0_{1,2}$ than $\mathcal{H}^1$ to show the well-posedness of the system (3.1), where the natural free energy is still well-defined. The Besov space $\dot{B}^0_{1,2}$ has the scale where the equation is invariant under (1.4).

The equation (3.1) can be converted as the corresponding integral equations as follows.

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\mathcal{L}}\nabla \cdot (u(s)\nabla\psi(s))ds, \qquad t > 0. \tag{3.11}$$

Hereafter we denote that the inverse of 2 dimensional Laplacian $-\Delta^{-1}$ by the Newton potential

$$\psi = -(-\Delta)^{-1}\kappa u = -\frac{1}{2\pi}\int_{\mathbb{R}^n} \kappa u(y)\log|x-y|^{-1}dy, \quad n = 2. \tag{3.12}$$

By this notation, we can eliminate ψ from the integral equation.

Our well-posedness result for (3.1) in the Besov space $\dot{B}^0_{1,2}$ is the following:

Theorem 3.2 (local existence for large data) (Ogawa-Shimizu [48]) *Let $\kappa = \pm 1$. Then for $u_0 \in \dot{B}^0_{1,2}$, there exists $T > 0$ and a unique solution $u \in C([0,T); \dot{B}^0_{1,2}) \cap L^2((0,T); \dot{B}^1_{1,2})$ to (3.1). Besides the solution is belonging to $C([0,T); \dot{B}^0_{1,2}) \cap C((0,T); \dot{B}^2_{1,2}) \cap C^1((0,T); \dot{B}^0_{1,2})$ and the flow map $\dot{B}^0_{1,2} \ni u_0 \to u \in C([0,T); \dot{B}^0_{1,2})$ is the Lipschitz continuous.*

Remark. Recently, Biler-Wu [6] obtained the local existence and uniqueness of two dimensional generalized drift-diffusion system in the homogeneous Besov space $\dot{B}^{1-\alpha}_{2,\sigma}$ where $1 \le \sigma \le \infty$ and α indicates the order of the fractional Laplacian $(-\Delta)^{\alpha/2}$. Their result seems restricted for $\alpha < 2$.

In view of the scaling invariance by the scaling (1.4) and due to Fujita-Kato principle [17], [28], we may also derive the global existence of the solution under the smallness assumption on the initial data.

Theorem 3.3 (global existence for small data) *Let $\kappa = \pm 1$. Then for $u_0 \in \dot{B}^0_{1,2}$, there exists $\varepsilon_0 > 0$ such that for $\|u_0\|_{\dot{B}^0_{1,2}} < \varepsilon_0$, there exists a unique global solution u to (3.1) such that $u \in C([0,\infty); \dot{B}^0_{1,2}) \cap L^2(0,\infty; \dot{B}^1_{1,2}) \cap C((0,\infty); \dot{B}^2_{1,2}) \cap C^1((0,\infty); \dot{B}^0_{1,2})$.*

We should emphasize that the functional (3.8) in general make no sense in the whole space if we consider the positive solution $u(t,x)$ since $\psi(t,x)$ behaves like the fundamental solution of the two dimensional Laplacian and therefore $\|\nabla\psi\|_2 = \infty$ in general. To ensure the above mentioned a priori estimate, we require the solution should be in $\psi \in \dot{H}^1$ and $u \in L\log L$ at least. Indeed, if $u \in \dot{B}^0_{1,2}$ then by the Plancherel identity,

$$\|\nabla\psi\|_2^2 = \|\nabla(-\Delta)^{-1}u\|_2^2 = c_2 \int_{\mathbb{R}^2} \frac{|\hat{u}(\xi)|^2}{|\xi|^2} d\xi$$

and we observe that

which is required estimate for the a priori estimate. It is also possible to show that a variant of the John-Nirenberg type estimate for the function in the dual Besov space is known: There exists two constants $C > 0$ and $\gamma > 0$ such that for any $f \in \dot{B}^0_{\infty,2}$,

$$\int_{\mathbb{R}^2} \Big[\exp|f(x)| - 1 - |f(x)|\Big] dx \le C\exp\left(\frac{1}{\gamma}(\|f\|_2 + \|f\|_{\dot{B}^0_{\infty,2}}) - 1\right). \tag{3.13}$$

From this inequality, we may show that the exponential integrability of the function in the dual Besov space $\dot{B}^0_{\infty,2}$ and this shows that the embedding from the Besov space $\dot{B}^0_{1,2}(\mathbb{R}^2)$ into the Orlitz space $L\log L_{loc}(\mathbb{R}^2)$ is possible.

4 Multiple Existence Result

In this section, we consider the elliptic-parabolic system modeling chemotaxis with a perturbed nonlinearity:

$$\begin{cases} \partial_t u - \Delta u + \nabla\cdot(u\nabla v) = 0, & t > 0, \quad x \in \mathbb{R}^2, \\ -\Delta v + v - v^p = u, & t > 0, \quad x \in \mathbb{R}^2, \\ u(0,x) = u_0(x) \ge 0, & x \in \mathbb{R}^2. \end{cases} \tag{4.14}$$

This type of the parabolic system is called as the perturbed Keller-Segel system and it is introduced as a mathematical model of chemotaxis collapse with a different sign in the nonlinear term. When the diffusion of the chemical substances is much slower than that of chemotaxis ameba then the dynamics of chemotaxis is described by the following simplified system:

$$\begin{cases} \partial_t u - \Delta u + \nabla\cdot(u\nabla v) = 0, & t > 0, \quad x \in \mathbb{R}^2, \\ -\Delta v + v = u, & t > 0, \quad x \in \mathbb{R}^2, \\ u(0,x) = u_0(x) \ge 0, & x \in \mathbb{R}^2. \end{cases} \tag{4.15}$$

It is well known that the existence of the finite time blow up of the solution for (4.15) which corresponds to the concentration of ameba. Chen-Zhong [9] introduced a perturbed system of

(4.15): For $p > 1$,

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla v) = 0, & t > 0, \quad x \in \mathbb{R}^2, \\ -\Delta v + v + v^p = u, & t > 0, \quad x \in \mathbb{R}^2, \\ u(0,x) = u_0(x) \geq 0, & x \in \mathbb{R}^2. \end{cases} \tag{4.16}$$

This model is considered as a model of the chemotaxis with a nonlinear diffusion for the chemical substance. It has been proved that the solution of this system (4.16) has a similar behavior to the original system (4.15). In fact, one can show the local existence theory and finite time blow up with mass concentration phenomena as is shown for the case (4.15), see Chen-Zhong [7] and Kurokiba-Senba-Suzuki [35].

Note that the nonlinear term v^p in the second equation in (4.14) has a different sign compare to (4.16). According to this difference, the behavior of the solution for (4.14) is much different from the one of (4.16). Indeed, the nonhomogeneous elliptic problem corresponding to the second equation of (4.14):

$$-\Delta v + v - v^p = f, \quad x \in \mathbb{R}^2 \tag{4.17}$$

admits at least two positive solutions when f is a sufficiently small nonnegative nontrivial function, while

$$-\Delta v + v + v^p = f, \quad x \in \mathbb{R}^2$$

has only one solution. Moreover, it is also known that if the external force f is large in H^{-1} sense, then there is no positive solution for the equation (4.17). Hence it is an interesting question to ask whether the finite time blow up of the solution occur in the case (4.14). Or more primitively, whether the time local solution exists properly and the system is well posed in some sense or not. In this sense, the structure of the time dependent positive solution of (4.14) seems to be very much different from that of the original system (4.15) or perturbed system (4.16).

We present here the existence of multiple solutions of (4.14) in the following sense:

$$\begin{aligned} u &\in C([0,\infty); L^2(\mathbb{R}^2)) \cap C^1((0,\infty); L^2(\mathbb{R}^2)) \cap C((0,\infty); \dot{H}^2(\mathbb{R}^2)), \\ v &\in C((0,\infty); H^1(\mathbb{R}^2)) \cap C((0,\infty); W^{2,2}(\mathbb{R}^2)). \end{aligned}$$

Kurokiba-Ogawa-Takahashi [34] proved that, for a small nonnegative initial data, there exists a global-in-time solution for (4.14) which is, in a sense, "small" one. On the other hand, as is mentioned above, the perturbed nonlinear elliptic equation (4.17) which corresponds to the second equation in (4.14) admits at least two positive solutions for small and nonnegative f. Therefore it is natural to ask whether the time dependent equation (4.14) also has the second positive solution. The main issue of this paper is to show the existence of *two positive time dependent solutions of (4.14)* under the radially symmetric setting.

Theorem 4.1 (Multiple existence) (Ishiwata-Ogawa-Takahashi [25]) *Let $1 < p < \infty$. Then there exists a constant $C_{**} > 0$ such that, if the radially symmetric nonnegative initial data $u_0 \in L^2$ satisfies*

$$\|u_0\|_2 \leq C_{**},$$

then there exist two positive radial pair of solutions $(u_1(t), v_1(t))$ and $(u_2(t), v_2(t))$ for (4.14). *One of them is different from the solution obtained in* [34].

The main idea to construct the second time dependent solution heavily relies on the variational structure of the elliptic part of the system. The v-component of the solution obtained in [34] corresponds to the solution of (4.17) which is bifurcated from the trivial solution of the elliptic problem (4.17) with $f = 0$. On the other hand, it has been known that the problem (4.17) with $f = 0$ has a unique positive solution w (see Gidas-Ni-Nirenberg [19] and Kwong [36]). This solution is obtained as a mountain pass critical point of

$$I_0(v) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} |v|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^2} |v|^{p+1} dx.$$

Then, if the second variation of the functional I_0 at w is non-degenerate and if f is small, then we may construct the solution v of (4.17) bifurcated from the mountain pass solution w. This is not always possible, since the kernel of the Hessian $(\nabla^2 I_0)_w$ of I_0 at w is nontrivial. If we restrict the class of initial data, however, there is a possibility of succeeding in constructing the second local-in-time solution of (4.14) . In this paper, we shall show this under the radial symmetry.

It should be also noted that our problem is related to the unconditional uniqueness problem in the general nonlinear evolution equations. Let X be a Banach space. If the initial value problem admits the uniqueness of the solution in the class $C([0,T);X)$ whose initial data is belonging to X, then this is called as the *unconditional uniqueness*. For example, if our problem (4.14) admits the unique local or global solution for the initial data belonging to some space like $L^2(\mathbb{R}^2)$, the unconditional uniqueness requires the regularity of the solution at most $u(t) \in C([0,T);L^2(\mathbb{R}^2))$. If the class of the solution is reasonably restricted, the well-posed problem is expected to have the unconditional uniqueness. For our problem (4.14), however, there is no possibility to have the unconditional uniqueness by means of the regularity. Namely no matter how the class of the solution is restricted by the regularity point of view, the solution exists at least two. The uniqueness of the solution is obtained only when the variational characterization of the second component of the solution v is given and the function space does not distinguish the uniqueness class. In this sense, there is no unconditional uniqueness holds for (4.14). This kind of phenomena may occur for a general nonlinear problem. In our particular setting, there exists at least two time dependent solutions and are uniquely continued in time each other under the variational restriction.

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