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INVERSE SCATTERING PROBLEMS FOR DISPERSIVE EQUATIONS WITH A NON-LOCAL NONLINEARITY

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ABSTRACT. We discuss inverse scattering problems for nonlinear dispersive equations. The nonlinearity of the equations is a non-local term with unknown interaction potentials. We determine the unknown interaction potentials and reconstruct the nonlinearity by the knowledge of the scattering states.

1. INTRODUCTION

In this note, we review the author's works [21, 22, 23] of inverse scattering problems for dispersive equations with a non-local nonlinearity. In particular, we treat the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = V_0 u + F(u) \quad \text{in } \mathbb{R} \times \mathbb{R}^n. \tag{NLS}$$

Here, $u = u(t, x)$ is a complex-valued unknown function, $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, Δ is the Laplacian in $\mathbb{R}^n$, $F(u)$ is a cubic convolution defined by

$$F(u) = \lambda(V_1 * |u|^2)u, \tag{1.1}$$

$V_0(x)$, $V_1(x)$ and $\lambda(x)$ are real-valued measurable functions and $*$ is the convolution in $\mathbb{R}^n$. The equation (NLS), which is initially studied by Chadam–Glassey [7], is approximately derived by the multi body Schrödinger equation (for the detailed derivation, see, e.g., Watanabe [31]). If $\lambda \equiv 1$, then the nonlinearity $F(u)$ means a non-local interaction whose interaction potential is V_1.

1.1. **Definition of the scattering operator.** The inverse scattering problem for the equation (NLS) means to identify unknown $V_0(x)$, $V_1(x)$ and $\lambda(x)$ by using the knowledge of the scattering operator S for (NLS). To treat inverse scattering problems, we first consider the direct scattering problem. For this purpose, we

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give some definitions. Let X be a Hilbert space. For $\delta > 0$, we denote a closed ball $\{\phi \in X; \|\phi\|_X \leq \delta\}$ by $B(\delta; X)$. Let H be a linear operator defined by $H = -\Delta + V_0$. We now assume that H is self-adjoint on X. We remark that for any $\phi \in X$, $e^{-itH}\phi$ solves the linear Schrödinger equation ((NLS) with $F(u) \equiv 0$) whose initial data is ϕ. The *scattering operator* for (NLS)

$$S : B(\delta; X) \ni \phi_- \mapsto \phi_+ \in X$$

is defined if the following property holds for some function space $Z \subset C(\mathbb{R}; X)$ and for some positive δ:

For $\phi_- \in B(\delta; X)$, there exist a function $u \in Z$ and a data $\phi_+ \in X$ such that $u(t)$ is a time-global solution to the integral form of (NLS)

$$u(t) = e^{-itH}\phi_- - i\int_{-\infty}^{t} e^{-i(t-\tau)H} F(u(\tau))d\tau \tag{1.2}$$

and $u(t)$ satisfies

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}\phi_\pm\|_X = 0,$$

respectively.

In Section 2 below, we see that if V_0, V_1 and λ satisfy suitable conditions, then the scattering operator for (NLS) is well-defined in some sense.

1.2. **Known results.** Before reviewing the works [21, 22, 23], we introduce some known results of inverse scattering problems for nonlinear Schrödinger equations. Strauss [25] considered the Schrödinger equation with a power nonlinearity

$$i\partial_t u + \Delta u = V(x)|u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

Suppose that p is an integer satisfying some conditions and $V(x)$ is real-valued continuous and bounded, whose derivatives up to order $l > 3n/4$ are bounded. Then the scattering operator S is well-defined. It was shown that $V(x)$ is recovered from the scattering operator by the following way: For $s \in \mathbb{R}$, let $H^s(\mathbb{R}^n)$ be the Sobolev space $(1-\Delta)^{-s/2}L^2(\mathbb{R}^n)$. For any $\phi \in H^1(\mathbb{R}^n) \cap L^{1+1/p}(\mathbb{R}^n)$, we have

$$V(x_0) = \frac{\lim_{\alpha\to 0} \alpha^{-(n+2)} I[\phi_{\alpha,x_0}]}{\int_{\mathbb{R}}\int_{\mathbb{R}^n} |e^{it\Delta}\phi(x)|^{p+1}dxdt}, \tag{1.3}$$

where $\phi_{\alpha,x_0}(x) = \phi(\alpha^{-1}(x - x_0))$, $\alpha > 0, x, x_0 \in \mathbb{R}^n$,

$$I[\phi] = \lim_{\varepsilon\to 0} \frac{i}{\varepsilon^p} \langle (S - id)(\varepsilon\phi), \phi \rangle,$$

id is the identity mapping and $\langle \cdot, \cdot \rangle$ denotes the $L^2(\mathbb{R}^n)$ inner product. The above limit $I[\phi]$ is called the small amplitude limit. Later, Weder [32, 34, 35, 36, 38] proved that a more general class of nonlinearities is uniquely reconstructed, and moreover, a method is given for the unique reconstruction of the potential that acts as a linear operator and that this problem was not considered in [25].

We next review known results of the inverse scattering problem for (NLS). In particular, we focus on the reconstruction theorem (for the uniqueness theorem, see Watanabe [28] and Sasaki–Watanabe [24]). Remark that even if $V_0 \equiv 0$ and $\lambda \equiv 1$, we can not determine V_1 by the reconstruction formula (1.3). Indeed, we formally conclude

$$\lim_{\alpha \to 0} \alpha^{-(n+2)} I[\phi_{\alpha,x_0}] = \int_{\mathbb{R}} \int_{\mathbb{R}^n} V_1(0) \left| e^{it\Delta}\phi(x) \right|^2 * \left| e^{it\Delta}\phi(x) \right|^2 dx dt$$

whose right hand side ordinarily diverges. Therefore, in order to determine V_1, we have to make some modified version of (1.3) or to find other methods. Watanabe [27] studied the inverse scattering problem for (NLS) with $V_1(x) = |x|^{-\sigma}$. It was proved that if σ is a given number, then we can reconstruct V_0 and λ by the knowledge of the scattering operator. Furthermore, Watanabe [29, 30] determined unknown σ and λ of the term $F(u) = \lambda(|\cdot|^{-\sigma} * |u|^2)u$ if $V_0 \equiv 0$ and λ is a non-zero constant. It was shown by [24] that if $V_0 \equiv 0$, $\lambda \equiv 1$ and V_1 is integrable on $\mathbb{R}^n$, then $\widehat{V_1}(0)$ can be uniquely determined. Here, $\widehat{V_1}$ denotes the Fourier transform of V_1. For the time dependent Hartree-Fock system, Watanabe [31] gave the reconstruction formula for determining interaction potentials.

1.3. **Outline of the author's works.** As we mention before, this note is devoted to introducing the author's works concerned with inverse scattering problems for dispersive equations with a non-local term. Let us focus on the equation (NLS) again. An aim of the works [21, 22, 23] is to make reconstruct formulas for determining V_0, V_1 and λ in the case where we can not determine only by using the above known results. In the rest of this section, we review an outline of such the author's works (the detailed statements are given in Sections 4–6 below).

In [21], it is shown that if $V_0 \equiv 0$, $\lambda \equiv 1$ and unknown $V_1(x)$ is expressed by some radial and smooth function $v(r)$ with $r = |x|$ vanishing at infinity, then we can determine the exact value of $\partial_r^M \widehat{v}(0)$ for any $M = 0, 1, 2, \cdots$. It is also proved by [21] that if $V_0 \equiv 0$, $\lambda \equiv 1$ and $V_1(x)$ is equal to $\sum_{k=1}^N \lambda_k |x|^{-\sigma_k}$ with unknown

$$1 < N < \infty, \quad \{\lambda_k\}_{k=1}^N \subset \mathbb{R}, \quad 2 \le \sigma_N < \sigma_{N-1} < \cdots < \sigma_2 < \sigma_1 \begin{cases} < n & \text{if } n = 3, 4, \\ \le 4 & \text{if } n \ge 5, \end{cases}$$

then we can completely reconstruct V_1.

From [22], assuming that $V_0 \equiv 0$, $\lambda \in (C^1 \cap W^1_\infty)(\mathbb{R}^n)$ and $V_1(x)$ equals to $|x|^{-\sigma}$ with unknown $\sigma \in [2,4] \cap [2,n)$, we can determine λ and σ. Remark that the result [22] is applicable to the case where both λ and σ are unknown.

In [23], it is proved that if we suppose that $n = 3$, $\lambda \equiv 1$ and that V_j, $j = 0, 1$, are expressed by the Yukawa potentials $V_j(x) = Q_j \exp(-\mu_j |x|)/|x|$ with some unknown numbers $Q_j \in \mathbb{R}$ and $\mu_j > 0$, respectively, then we can determine V_0 and V_1.

Furthermore, in [22, 23], the author studied inverse scattering problems for some relativistic equations with the nonlinearity $F(u)$ (for the detail, see Section 7).

The key of the proof of such author's results is to calculate the parameterized small amplitude limit. For example, assuming that V_1 is expressed by the radial function v and by defining one-parameter function $\varphi(\alpha)$ by

$$\varphi(\alpha) = \alpha^{n+2} \lim_{\varepsilon\to 0} \frac{i}{\varepsilon^3}\langle (S - id)(\varepsilon\phi^\alpha), \phi\rangle, \quad \phi^\alpha(x) := \phi(\alpha x),$$

we have for any $M \geq 0$,

$$\partial_r^M \widehat{v}(0) = C_0 \lim_{\beta\to 0}\left(\frac{d^M\varphi}{d\alpha^M}(\beta)\right),$$

where the constant C_0 is given only by M and ϕ.

Introducing the contents of the rest of this note, we close this section. In Section 2, it is shown that if V_0, V_1 and λ satisfy suitable conditions, then the scattering operator S is well-defined on some 0-neighborhood of the Sobolev space $H^s(\mathbb{R}^n)$. The proof of such existence results is based on Stricharz estimates for linear Schrödinger equations. In Section 3, using properties proved in Section 2, we show the small amplitude limit of the scattering operator for (NLS). In Sections 4, 5 and 6, we mention detailed statements concerned with the works [21], [22] and [23], respectively. In Section 7, inverse scattering problems for relativistic equations with $F(u)$ are studied.

2. Direct scattering problem

In this section, we introduce existence results of scattering operators for (NLS). We first define some notation. For $1 \leq p \leq \infty$, $W_p^1(\mathbb{R}^n)$ is the Sobolev space defined by $W_p^1(\mathbb{R}^n) = \{\phi \in L^p(\mathbb{R}^n);\ \nabla\phi \in L^p(\mathbb{R}^n)\}$. Put $X_1 = H^1(\mathbb{R}^n)$, $X_2 = L^2(\mathbb{R}^3)$, $Z_1 = C(\mathbb{R}; X_1) \cap L^3(\mathbb{R}; W^1_{6n/(3n-4)}(\mathbb{R}^n))$ and $Z_2 = C(\mathbb{R}; X_2) \cap L^3(\mathbb{R}; L^{18/5}(\mathbb{R}^3))$. We again define H by $H = -\Delta + V_0$.

Theorem 2.1. *Let $n \geq 3$. We assume that $V_0 \equiv 0$, $\lambda \in (C^1 \cap W^1_\infty)(\mathbb{R}^n)$ and*

$$|V_1(x)| \leq C\left(|x|^{-\sigma_1} + |x|^{-\sigma_2}\right), \quad 2 \leq \sigma_1, \sigma_2 \begin{cases} < n & \text{if } n = 3, 4, \\ \leq 4 & \text{if } n \geq 5 \end{cases}$$

for some positive constant C independent of x. Then there exists a positive number δ such that for any $\phi_- \in B(\delta; X_1)$, the integral equation (1.2) with $H = -\Delta$ has a unique solution $u \in Z_1$ satisfying

$$\|u\|_{Z_1} \leq C\|\phi_-\|_{X_1}, \tag{2.1}$$

$$\|u - e^{it\Delta}\phi_-\|_{Z_1} \leq C\|\phi_-\|^3_{X_1} \tag{2.2}$$

and

$$\lim_{t\to-\infty}\|u(t)-e^{it\Delta}\phi_-\|_{X_1}=0.$$

Furthermore, there exists a data $\phi_+\in X_1$ such that

$$\lim_{t\to+\infty}\|u(t)-e^{it\Delta}\phi_+\|_{X_1}=0.$$

Therefore, the scattering operator $S_1 : B(\delta;X_1)\ni\phi_-\mapsto\phi_+\in X_1$ is well-defined.

Theorem 2.2. ([23]) *We assume that $\lambda\equiv 1$, $V_0\in X_2$,*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V_0(x)V_0(y)|}{|x-y|^2}dydx<(4\pi)^2,\quad \sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V_0(y)|}{|x-y|}dy<4\pi \tag{2.3}$$

and $V_1\in L^{3/2}(\mathbb{R}^3)$. Then H is absolutely continuous self-adjoint and there exists a positive number δ such that for any $\phi_-\in B(\delta;X_2)$, the integral equation (1.2) has a unique solution $u\in Z_2$ satisfying

$$\|u\|_{Z_2}\le C\|\phi_-\|_{X_2}, \tag{2.4}$$

$$\|u-e^{-itH}\phi_-\|_{Z_2}\le C\|\phi_-\|_{X_2}^3 \tag{2.5}$$

and

$$\lim_{t\to-\infty}\|u(t)-e^{it\Delta}\phi_-\|_{X_2}=0. \tag{2.6}$$

Furthermore, there exists a data $\phi_+\in X_2$ such that

$$\lim_{t\to+\infty}\|u(t)-e^{it\Delta}\phi_+\|_{X_2}=0. \tag{2.7}$$

Therefore, the scattering operator $S_2 : B(\delta;X_2)\ni\phi_-\mapsto\phi_+\in X_2$ is well-defined.

Since the first theorem directly follows from Mochizuki [12], we prove only the second one. For this purpose, we introduce the following Strichartz type estimate:

Proposition 2.3. *Suppose that V_0 satisfies (2.3). Then H is an absolutely continuous self-adjoint operator and we have for any $\phi\in X_2$ and any $f\in(C\cap L^\infty)(\mathbb{R};X_2)$,*

$$\|e^{-itH}\phi\|_{Z_2}\le C\|\phi\|_{X_2} \tag{2.8}$$

and

$$\left\|\int_{-\infty}^t e^{i(t-\tau)H}f(\tau)d\tau\right\|_{Z_2}\le C\|f\|_{L^1(\mathbb{R};X_2)}. \tag{2.9}$$

Proof. By Rodonianski–Schlag [20], it follows from the condition (2.3) that for any $t\neq 0$,

$$\|e^{itH}\phi\|_{L^\infty(\mathbb{R}^3)}\le C|t|^{-3/2}\|\phi\|_{L^1(\mathbb{R}^3)}. \tag{2.10}$$

On the other hand, by the standard argument, we immediately see that H is an absolutely continuous self-adjoint operator (for the detail, see, e.g., [17, 19]). Therefore, for any $t \in \mathbb{R}$, we have

$$\|e^{itH}\phi\|_{X_2} = \|\phi\|_{X_2}. \tag{2.11}$$

From the Riesz-Thorin interpolation theorem, we obtain a decay estimate

$$\|e^{itH}\phi\|_{L^p(\mathbb{R}^3)} \le C|t|^{-3/2+3/p}\|\phi\|_{L^{p'}(\mathbb{R}^3)}. \tag{2.12}$$

Here, $2 \le p \le \infty$ and p' is the Hölder conjugate defined by $1/p' = 1 - 1/p$. Furthermore, the T^*T argument given by Ginibre–Velo [9] implies

$$\|e^{-itH}\phi\|_{L^{q_1}(\mathbb{R};L^{r_1}(\mathbb{R}^3))} \le C\|\phi\|_{X_2},$$

$$\left\|\int_{-\infty}^{t} e^{i(t-\tau)H} f(\tau)d\tau\right\|_{L^{q_2}(\mathbb{R};L^{r_2}(\mathbb{R}^3))} \le C\|f\|_{L^{q_3'}(\mathbb{R};L^{r_3'}(\mathbb{R}^3))}$$

for any $2 \le q_j \le \infty$, $2 \le r_j \le 6$ $(j = 1,2,3)$ with $2/q_j = 3(1/2 - 1/r_j)$. Therefore, we have (2.8) and (2.9). □

We are ready to prove Theorem 2.1.

Proof of Theorem 2.1. By Proposition 2.3, we have already seen that H is an absolutely continuous self-adjoint operator. Put $\|\phi_-\|_{X_2} \le \delta$ and

$$G[u](t) = e^{-itH}\phi_- - i\int_{-\infty}^{t} e^{-i(t-\tau)H}F(u(\tau))d\tau.$$

From (2.8) and (2.9), we obtain

$$\|G[u](t)\|_{Z_2} \le C\left(\|\phi_-\|_{X_2} + \|F(u)\|_{L^1(\mathbb{R};X_2)}\right)$$

We see that

$$\|F(u(t))\|_{X_2} \le \left\|V_1 * |u(t)|^2\right\|_{L^{9/2}(\mathbb{R}^3)} \|u(t)\|_{L^{18/5}(\mathbb{R}^3)} \le C\|u(t)\|^3_{L^{18/5}(\mathbb{R}^3)},$$

where we have used the Hölder-Young inequality in the second inequality. We hence see that

$$\|G[u](t)\|_{Z_2} \le C(\|\phi_-\|_{X_2} + \|u(t)\|^3_{Z_2}).$$

Similarly, we obtain

$$\|G[u](t) - G[\widetilde{u}](t)\|_{Z_2} \le C\|u - \widetilde{u}\|_{Z_2}(\|u\|_{Z_2} + \|\widetilde{u}\|_{Z_2})^2.$$

It is clear that $G[u] \in C(\mathbb{R}; X_2)$. Therefore, we see that there uniquely exists $u \in Z_2$ such that $G[u] = u$ for sufficiently small $\delta > 0$. We can immediately find that the fixed point u solves the equation (1.2). Furthermore, we obtain (2.4) and (2.5). It follows from $u \in L^3(\mathbb{R}; L^{18/5}(\mathbb{R}^3))$ that

$$\|u(t) - e^{-itH}\phi_-\|_{X_2} \le \int_{-\infty}^{t} \|F(u(t))\|_{X_2}dt \le \int_{-\infty}^{t} \|u(t)\|^3_{L^{18/5}(\mathbb{R}^3)}dt \to 0 \quad \text{as } t \to \infty.$$

Furthermore, if we put

$$S_F(\phi_-) = \phi_- + \frac{1}{i}\int_{\mathbb{R}} e^{itH}F(u(t))dt, \tag{2.13}$$

we have

$$\lim_{t\to+\infty} \|u(t) - e^{-itH}S_F(\phi_-)\|_{X_2} = 0.$$

In order to prove the existence of the scattering operator for (1.2), we denote wave operators Ω_- and Ω_+ by

$$\Omega_\pm = s - \lim_{t\to\pm\infty} e^{-it\Delta}e^{-itH},$$

respectively. Since $\Omega_\pm$ are unitary operators in X_2, there exists some $\delta_0 > 0$ such that if $\phi_- \in B(\delta_0, X_2)$, then there uniquely exists $v \in Z_2$ satisfying $G[v] = v$ and

$$\lim_{t\to-\infty} \|v(t) - e^{-itH}\Omega_-(\phi_-)\|_{X_2} = 0. \tag{2.14}$$

Moreover, we have

$$\lim_{t\to\infty} \|v(t) - e^{-itH}S_F\Omega_-(\phi_-)\|_{X_2} = 0. \tag{2.15}$$

By (2.11), we see from (2.14) and (2.15) that

$$\begin{aligned}
&\|v(t) - e^{it\Delta}\phi_-\|_{X_2} \\
&\quad\le \|v(t) - e^{-itH}\Omega_-(\phi_-)\|_{X_2} + \|e^{-itH}\Omega_-(\phi_-) - e^{it\Delta}\phi_-\|_{X_2} \\
&\quad= \|v(t) - e^{-itH}\Omega_-(\phi_-)\|_{X_2} + \|\Omega_-(\phi_-) - e^{itH}e^{it\Delta}\phi_-\|_{X_2} \\
&\quad\to 0 \quad \text{as } t \to -\infty
\end{aligned}$$

and

$$\begin{aligned}
&\|v(t) - e^{it\Delta}\Omega_+^{-1}S_F\Omega_-(\phi_-)\|_{X_2} \\
&\le \|v(t) - e^{-itH}S_F\Omega_-(\phi_-)\|_{X_2} + \|e^{-itH}S_F\Omega_-(\phi_-) - e^{it\Delta}\Omega_+^{-1}S_F\Omega_-(\phi_-)\|_{X_2} \\
&= \|v(t) - e^{-itH}S_F\Omega_-(\phi_-)\|_{X_2} + \|S_F\Omega_-(\phi_-) - e^{itH}e^{it\Delta}\Omega_+^{-1}S_F\Omega_-(\phi_-)\|_{X_2} \\
&\to 0 \quad \text{as } t \to +\infty,
\end{aligned}$$

respectively. Thus, (2.6) and (2.7) hold if we define $S_2 = \Omega_+^{-1}S_F\Omega_-$. This completes the proof. □

3. Small amplitude limit

In this section, we focus on the small amplitude limit of the scattering operator for S_1 derived by Theorems 2.1. The following proposition will be used in Sections 4 and 5 below:

Proposition 3.1. *Let $n \geq 3$. We assume that $V_0 \equiv 0$, $\lambda \in (C^1 \cap W^1_\infty)(\mathbb{R}^n)$ and*

$$|V_1(x)| \leq C\left(|x|^{-\sigma_1} + |x|^{-\sigma_2}\right), \quad 2 \leq \sigma_1, \sigma_2 \begin{cases} < n & \text{if } n = 3, 4, \\ \leq 4 & \text{if } n \geq 5 \end{cases}$$

for some positive constant C independent of x. Then for any $\phi \in X_1$, we have

$$\lim_{\varepsilon \to 0} \frac{i}{\varepsilon^3} \langle (S_1 - id)(\varepsilon\phi), \phi \rangle = \int_{\mathbb{R}} \left\langle F(e^{it\Delta}\phi), e^{it\Delta}\phi \right\rangle dt. \tag{3.1}$$

Remark 3.1. From Theorem 2.1, S_1 is well-defined on $B(\delta; X_1)$ with sufficiently small positive δ. Thus, the left hand side of limit (3.1) makes sense.

Proof of Proposition 3.1. Let ε be a sufficiently small positive number. From the proof of Theorem 2.1, we see that S_1 is expressed by

$$S_1(\varepsilon\phi) = \varepsilon\phi - i \int_{\mathbb{R}} e^{-it\Delta} F(u_\varepsilon(t)) dt,$$

where $u_\varepsilon(t)$ is the solution to (1.2) with $\phi_- = \varepsilon\phi$. Therefore, we obtain

$$(S_1 - id)(\varepsilon\phi) = -i \int_{\mathbb{R}} e^{-it\Delta} F(u_\varepsilon(t)) dt.$$

Set $F(v_1, v_2, v_3) = (V_1 * v_1\overline{v_2})v_3$. Then it follows that

$$\|F(v_1, v_2, v_3)\|_{Z_1} \leq C \prod_{j=1}^{3} \|v_j\|_{Z_1} \tag{3.2}$$

and we see that

$$\begin{aligned} i \langle (S_1 - id)(\varepsilon\phi), \phi \rangle &= \int_{\mathbb{R}} \left\langle F(u_\varepsilon(t)), e^{it\Delta}\phi \right\rangle dt \\ &= \varepsilon^3 \int_{\mathbb{R}} \left\langle F(e^{it\Delta}\phi), e^{it\Delta}\phi \right\rangle dt \\ &\quad + \int_{\mathbb{R}} \left\langle F\left(u_\varepsilon(t) - e^{it\Delta}(\varepsilon\phi), u_\varepsilon(t), u_\varepsilon(t)\right), e^{it\Delta}\phi \right\rangle dt \\ &\quad + \int_{\mathbb{R}} \left\langle F\left(e^{it\Delta}(\varepsilon\phi), u_\varepsilon(t) - e^{it\Delta}(\varepsilon\phi), u_\varepsilon(t)\right), e^{it\Delta}\phi \right\rangle dt \\ &\quad + \int_{\mathbb{R}} \left\langle F\left(e^{it\Delta}(\varepsilon\phi), e^{it\Delta}(\varepsilon\phi), u_\varepsilon(t) - e^{it\Delta}(\varepsilon\phi)\right), e^{it\Delta}\phi \right\rangle dt \\ &=: (I) + (II)_1 + (II)_2 + (II)_3. \end{aligned}$$

By (3.2), (2.1) and (2.2), we obtain for any $j = 1, 2, 3$,

$$|(II)_j| \leq C\|u_\varepsilon(t) - e^{it\Delta}(\varepsilon\phi)\|_{Z_1} \|u_\varepsilon(t)\|^2_{Z_1} \|e^{it\Delta}\phi\|_{X_1} \leq C\|\varepsilon\phi\|^5_{X_1}\|\phi\|_{X_1}.$$

Therefore, we have (3.1). □

4. NLS (Case I)

In this section, we review the work [21] concerned with the inverse scattering problem for (NLS). Let $n \geq 3$ and $\mathcal{S}(\mathbb{R}^n)$ be the Schwatz class. Suppose that, $V_0 \equiv 0$ and $\lambda \equiv 1$. Moreover, assume that V_1 satisfies either the following two conditions:

(A) *V_1 is equal to $\sum_{k=1}^N \lambda_k |x|^{-\sigma_k}$ with*

$$1 < N < \infty, \quad \{\lambda_k\}_{k=1}^N \subset \mathbb{R}, \quad 2 \leq \sigma_N < \sigma_{N-1} < \cdots < \sigma_2 < \sigma_1 \begin{cases} < n & \text{if } n = 3, 4, \\ \leq 4 & \text{if } n \geq 5. \end{cases}$$

(B) *V_1 is radial function belonging to $\mathcal{S}(\mathbb{R}^3)$.*

From Theorem 2.1, we see that the scattering operator $S_1 : B(\delta; X_1) \ni \phi_- \mapsto \phi_- \in X_1$ for (NLS) is well-defined. The goal of [21] is to identify unknown V_1 under the assumption (A) or (B). In order to state main results of [21], we list some notation. For $\phi \in X_1$, let $\mathcal{I}[\phi]$ be the small amplitude limit

$$\mathcal{I}[\phi] = \lim_{\varepsilon \to 0} \frac{i}{\varepsilon^3} \left\langle (S_1 - id)(\varepsilon\phi), \phi \right\rangle. \tag{4.1}$$

For $j = 1, 2, \cdots N$, we put

$$I_j = \int_{\mathbb{R}} \left\langle |\cdot|^{-\sigma_j} * |e^{it\Delta}\phi|^2, |e^{it\Delta}\phi|^2 \right\rangle dt.$$

For $\phi \in X_1$, $\alpha > 0$ and $x \in \mathbb{R}^n$, put $\phi_\alpha(x) = \phi(\alpha^{-1}x)$. We define $\psi(\alpha)$ and $\varphi(\alpha)$ by

$$\psi(\alpha) = \alpha^{-2n-2}\mathcal{I}[\phi_\alpha]$$

and

$$\varphi(\alpha) = \alpha^{n+2}\mathcal{I}[\phi_{\alpha^{-1}}],$$

respectively. The Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ is given by

$$(\mathcal{F}\phi)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\phi(x)dx$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. It is well-known that the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ is defined uniquely. We also define the inverse Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ by

$$(\mathcal{F}^{-1}\phi)(\xi) = (\mathcal{F}\widetilde{\phi})(\xi),$$

where we set $\widetilde{\phi}(x) = \phi(-x)$. We are ready to state the main results.

Theorem 4.1. *Assume that $V_0 \equiv 0$ and $\lambda \equiv 1$ and that V_1 satisfies the condition (A). Put $\phi \in X_1 \setminus \{0\}$. Then we can determine N, $\{\lambda_j\}_{j=1}^N$ and $\{\sigma_j\}_{j=1}^N$ by the following steps:*

(Step I.) We have

$$\sigma_1 = -\lim_{\alpha \to 0} \ln \frac{|\psi(e\alpha)|}{|\psi(\alpha)| + 1}. \tag{4.2}$$

Furthermore, using the given σ_1*, we obtain*

$$\lambda_1 = I_1^{-1} \lim_{\alpha\to 0}\bigl(\alpha^{\sigma_1}\psi(\alpha)\bigr). \tag{4.3}$$

(Step II-1.) Suppose that we have already determined $\{\lambda_j\}_{j=1}^k$ *and* $\{\sigma_j\}_{j=1}^k$*. Put*

$$\psi^k(\alpha) = \psi(\alpha) - \sum_{j=1}^k \alpha^{-\sigma_j}\lambda_j I_j.$$

If $\lim_{\alpha\to 0}|\psi^k(\alpha)| = 0$*, then* $N = k$*.*
(Step II-2.) Suppose that we have already determined $\{\lambda_j\}_{j=1}^k$ *and* $\{\sigma_j\}_{j=1}^k$*. If* $\lim_{\alpha\to 0}|\psi^k(\alpha)| = \infty$*, then we have* $N > k$ *and*

$$\sigma_{k+1} = -\lim_{\alpha\to 0}\ln\frac{|\psi^k(e\alpha)|}{|\psi^k(\alpha)|+1}.$$

Furthermore, using the given σ_{k+1}*, we obtain*

$$\lambda_{k+1} = I_{k+1}^{-1}\lim_{\alpha\to 0}\bigl(\alpha^{\sigma_{k+1}}\psi^k(\alpha)\bigr).$$

Remark 4.1. The value of $\lim_{\alpha\to 0}|\psi^k(\alpha)|$ must be either 0 or ∞.

Theorem 4.2. *Let* $n \geq 3$*. Assume that* $V_0 \equiv 0$ *and* $\lambda \equiv 1$ *and that* V_1 *satisfies the condition (B). If we define* $\mathcal{V}(\rho)$*,* $\rho \geq 0$*, by* $\mathcal{V}(|\xi|) = \mathcal{F}V_1(\xi)$ *and we set* $d = d/d\rho$*, then we have for any* $M = 0, 1, \cdots$ *and any* $\phi \in H^{M/2+n/4-1/2}(\mathbb{R}^n)\setminus\{0\}$*,*

$$(d^M\mathcal{V})(0) = \frac{\lim_{\alpha\to 0}\bigl(d^M\varphi\bigr)(\alpha)}{\displaystyle\int_{\mathbb{R}} \bigl\| |\cdot|^{M/2}\mathcal{F}\bigl(|e^{it\Delta}\phi|^2\bigr)\bigr\|_{L^2(\mathbb{R}^n)}^2 dt}. \tag{4.4}$$

Remark 4.2. Suppose that V_1 satisfy (B) and that we have already seen that $\mathcal{F}V_1$ is analytic. Let $\phi \in \mathcal{S}(\mathbb{R}^n)\setminus\{0\}$. If we obtain $\limsup_{M\to\infty}|(d^M\mathcal{V})(0)|^{1/M} = 0$, then it follows from the Maclaurin theorem that

$$\mathcal{V}(\rho) = \sum_{M=0}^\infty \frac{(d^M\mathcal{V})(0)}{M!}\rho^M.$$

Therefore, for any $x \in \mathbb{R}^n$, we can reconstruct V_1 by

$$V_1(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\sum_{M=0}^\infty \frac{(d^M\mathcal{V})(0)}{M!}e^{ix\cdot\xi}|\xi|^M d\xi.$$

Proof of Theorem 4.1. Suppose that V_1 satisfies (A) and $0 < \alpha < 1$. Since

$$(V_1)_{\alpha^{-1}}(x) = \sum_{j=1}^N \alpha^{-\sigma_j}\lambda_j|x|^{-\sigma_j},$$

we obtain

$$\psi(\alpha) = \int_{\mathbb{R}} \Big\langle \big(\sum_{j=1}^{N} \alpha^{-\sigma_j}\lambda_j|\cdot|^{-\sigma_j}\big) * |e^{it\Delta}\phi|^2, |e^{it\Delta}\phi|^2 \Big\rangle dt.$$

Hence we see that

$$\psi(\alpha) = \sum_{j=1}^{N} \alpha^{-\sigma_j}\lambda_j \int_{\mathbb{R}} \Big\langle \big(|\cdot|^{-\sigma_j}\big) * |e^{it\Delta}\phi|^2, |e^{it\Delta}\phi|^2 \Big\rangle dt = \sum_{j=1}^{N} \alpha^{-\sigma_j}\lambda_j I_j. \tag{4.5}$$

Therefore, we obtain

$$\frac{|\psi(e\alpha)|}{|\psi(\alpha)|+1} = \frac{\Big|\sum_{j=1}^{N} e^{-\sigma_j}\alpha^{-\sigma_j}\lambda_j I_j\Big|}{\Big|\sum_{j=1}^{N} \alpha^{-\sigma_j}\lambda_j I_j\Big|+1} = \frac{\Big|\sum_{j=1}^{N} e^{-\sigma_j}\alpha^{\sigma_1-\sigma_j}\lambda_j I_j\Big|}{\Big|\sum_{j=1}^{N} \alpha^{\sigma_1-\sigma_j}\lambda_j I_j\Big|+\alpha^{\sigma_1}}.$$

By $\sigma_1 > \sigma_j$ for any $j > 1$, it follows from the boundedness of I_j that

$$\lim_{\alpha\to 0} \frac{|\psi(e\alpha)|}{|\psi(\alpha)|+1} = e^{-\sigma_1}.$$

Thus, (4.2) holds. From (4.5), we immediately obtain (4.3).

Since

$$\psi^k(\alpha) = \sum_{j=k+1}^{N} \alpha^{-\sigma_j}\lambda_j I_j,$$

we can determine N, $\{\lambda_j\}_{j=1}^N$ and $\{\sigma_j\}_{j=1}^N$ by using the steps (Step II-1) and (Step II-2). This completes the proof. □

For the proof of Theorem 4.2, we prepare the following lemma:

Lemma 4.3. *Let $n \geq 2$. For any $s > 0$ and any $\phi \in H^{s+n/4-1/2}(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}} \Big\| |\cdot|^s \mathcal{F}\big(|e^{it\Delta}\phi|^2\big)\Big\|_{L^2(\mathbb{R}^n)}^2 dt \leq C\|\phi\|_{H^{s+n/4-1/2}(\mathbb{R}^n)}^4. \tag{4.6}$$

Proof of Theorem 4.2. Suppose that V_1 satisfies the condition (B) and $0 < \alpha < 1$. By

$$\mathcal{F}(V_1)_\alpha = \alpha^{-n}(\mathcal{F}V_1)_{\alpha^{-1}},$$

it follows from (3.1) and the Plancherel theorem that

$$\varphi(\alpha) = \alpha^{n+2}T[\phi_{\alpha^{-1}}] = \int_{\mathbb{R}} \Big\langle (\mathcal{F}V_1)_{\alpha^{-1}}\mathcal{F}\big(|e^{it\Delta}\phi|^2\big), \mathcal{F}\big(|e^{it\Delta}\phi|^2\big)\Big\rangle dt.$$

By (4.6) and $(\mathcal{F}V_1)_{\alpha^{-1}}(\xi) = \mathcal{V}(\alpha|\xi|)$, we obtain

$$\frac{d^M}{d\alpha^M}\varphi(\alpha) = \int_{\mathbb{R}} \Big\langle |\cdot|^M(d^M\mathcal{V})(\alpha\cdot)\mathcal{F}\big(|e^{it\Delta}\phi|^2\big), \mathcal{F}\big(|e^{it\Delta}\phi|^2\big)\Big\rangle dt$$

for any $M = 0, 1, \cdots$. The Lebesgue dominated theorem implies that

$$\lim_{\alpha\to 0}(d^M\varphi)(\alpha) = (d^M\mathcal{V})(0)\int_{\mathbb{R}}\Big\langle |\cdot|^M\mathcal{F}(|e^{it\Delta}\phi|^2), \mathcal{F}(|e^{it\Delta}\phi|^2)\Big\rangle dt.$$

Thus, we have (4.4). This completes the proof. $\square$

5. NLS (Case II)

In this section, we review a part of the work [22] which treats the inverse scattering problem (NLS). Suppose that $n \geq 3$ and that V_0, V_1 and λ satisfy the following condition:

(C) *V_0 and λ satisfy $V_0 \equiv 0$ and $\lambda \in (C^1 \cap W^1_\infty)(\mathbb{R}^n)$, respectively. Furthermore, $\lambda(0) \neq 0$ and $V_1(x)$ is expressed by $V_1(x) = |x|^{-\sigma}$ for some unknown $\sigma \in [2,4]\cap[2,n)$.*

From Theorem 2.1, we see that the scattering operator $S_1 : B(\delta; X_1) \ni \phi_- \mapsto \phi_- \in X_1$ for (NLS) is well-defined. The goal of [22] for (NLS) is to identify unknown λ and σ. In order to state the main result of [22] for (NLS), we list some notation. Let e be the base of the natural logarithm. For $\phi \in X_1$, the small amplitude limit $\mathcal{I}[\phi]$ is defined by 4.1. For $\phi \in X_1$, $\alpha > 0$ and $x, x_0 \in \mathbb{R}^n$, we again put $\phi_{\alpha,x_0}(x) = \phi(\alpha^{-1}(x - x_0))$. In particular, we denote $\phi_{\alpha,0}$ by ϕ_α. We are ready to state the main result.

Theorem 5.1. *Let $n \geq 3$. Assume that V_0, V_1 and λ satisfy the condition (C). Then for any $\phi \in X_1$, the unknown σ is given by*

$$\sigma = 2n + 2 - \lim_{\alpha\to 0}\ln\frac{|\mathcal{I}[\phi_{e\alpha}]|}{|\mathcal{I}[\phi_\alpha]| + \alpha^{2n+2}}. \tag{5.1}$$

Furthermore, using the given σ, we have for any $x_0 \in \mathbb{R}^n$,

$$\lambda(x_0) = \frac{\lim_{\alpha\to 0}\alpha^{-(2n+2-\sigma)}\mathcal{I}[\phi_{\alpha,x_0}]}{\displaystyle\int |y|^{-\sigma}|e^{it\Delta}\phi(x-y)|^2|e^{it\Delta}\phi(x)|^2 d(t,x,y)}. \tag{5.2}$$

Proof. By (3.1), it follows that

$$\mathcal{I}[\phi] = \int_{\mathbb{R}^{1+n+n}_{(t,x,y)}}\lambda(x)|y|^{-\sigma}|e^{it\Delta}\phi(x-y)|^2|e^{it\Delta}\phi(x)|^2 d(t,x,y). \tag{5.3}$$

Having in mind that

$$e^{it\Delta}\phi_\alpha = (e^{it\alpha^{-2}\Delta}\phi)_\alpha,$$

we see that

$$\mathcal{I}[\phi_\alpha] = \alpha^{2n+2-\sigma}\int_{\mathbb{R}^{1+n+n}_{(t,x,y)}}\lambda(\alpha x)R(\phi)(t,x,y)d(t,x,y),$$

where

$$R(\phi)(t,x,y) = |y|^{-\sigma}|e^{it\Delta}\phi(x-y)|^2|e^{it\Delta}\phi(x)|^2.$$

We see that $R(\phi)$ is integrable. Hence it follows that

$$\int_{\mathbb{R}^{1+n+n}_{(t,x,y)}} \lambda(\beta x)R(\phi)(t,x,y)d(t,x,y)$$
$$\to \int_{\mathbb{R}^{1+n+n}_{(t,x,y)}} \lambda(0)R(\phi)(t,x,y)d(t,x,y) \neq 0 \quad \text{as } \beta \to 0$$

if $\phi \neq 0$. Hence we obtain

$$\frac{|\mathcal{I}[\phi_{e\alpha}]|}{|\mathcal{I}[\phi_\alpha]| + \alpha^{2n+2}} = \frac{e^{2n+2-\sigma}|\int \lambda(e\alpha x)R(\phi)d(t,x,y)|}{|\int \lambda(\alpha x)R(\phi)d(t,x,y)| + \alpha^\sigma} \to e^{2n+2-\sigma} \quad \text{as } \alpha \to 0.$$

Thus, we have (5.1). We immediately see that the remainder (5.2) follows from (3.1). This completes the proof. □

6. NLS (Case III)

In this section, we review a part of the work [23] which is concerned with the inverse scattering problem for (NLS). Suppose that $n = 3$ and that V_0, V_1 and λ satisfy the following condition:

(D). *λ satisfy $\lambda \equiv 1$. Furthermore, V_j, $j = 0, 1$, are expressed by the Yukawa potentials*

$$V_j(x) = \frac{Q_j \exp(-\mu_j|x|)}{|x|}, \quad x \in \mathbb{R}^3. \tag{6.1}$$

Here, $Q_j \in \mathbb{R}$ and $\mu_j > 0$ are unknown parameters with

$$|Q_0| < \mu_0. \tag{6.2}$$

Since the condition (6.2) implies (2.3), we see from Theorem 2.2 that the scattering operator $S_2 : B(\delta; X_2) \ni \phi_- \mapsto \phi_- \in X_2$ for (NLS) is well-defined. The goal of [23] is to identify unknown Q_j and μ_j. In order to state the main result of [23] for (NLS), we list some notation and proposition. We set $r(x) = |x|$. we again put wave operators $\Omega_\pm = s - \lim_{t\to\pm\infty} e^{-it\Delta}e^{-itH}$ and remark that $\Omega_\pm$ are bijective functions from X_2 into itself. We define a mapping S_{V_0} by

$$S_{V_0} = \Omega_+^{-1}\Omega_- : X_2 \to X_2.$$

The operator S_{V_0} is the scattering operator for the linear Schrödinger equation ((NLS) with $F(u) \equiv 0$). Using the method of [27, 32, 34, 35, 36], we see that S_{V_0} can be determined from the knowledge of S_1.

Theorem 6.1. *([27, 32, 34, 35, 36]) Assume that (6.1) and (6.2) hold. For any $\phi \in X_2 \setminus \{0\}$, we have*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} S_2(\varepsilon\phi) = S_{V_0}(\phi) \quad \text{in } X_2. \tag{6.3}$$

Once we have determined S_{V_0}, we can reconstruct V_0, $e^{\pm itH}$, $\Omega_\pm$, $\Omega_\pm^{-1}$ by Enss–Weder [6]. The main result of [23] concerned with (NLS) is to determine the remainder Q_1 and μ_1.

Theorem 6.2. *Assume that (6.1) and (6.2) hold and that $\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ satisfies $\phi \neq 0$ and $(\Delta^2+1)^{-1}\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$.*

(i) We have

$$\frac{Q_1}{\mu_1^2} = \frac{\lim_{\alpha\to\infty} i\alpha^4 \left\langle (\Omega_+ S_2 \Omega_-^{-1} - id)(\alpha^{-3}\phi_\alpha), \phi_\alpha \right\rangle}{4\pi \| e^{it\Delta}\phi\|^4_{L^4(\mathbb{R};L^4(\mathbb{R}^3))}}. \tag{6.4}$$

(ii) Suppose that $Q_1 \neq 0$. Put

$$b = \left|\frac{Q_1}{\mu_1^2}\right|^{1/2}, \qquad H(b) = -\Delta - bQ_0 \frac{\exp(-b\mu_0 r)}{r},$$

$$\Psi_1(\alpha) = \int_{\mathbb{R}} \left\langle \frac{\alpha \exp(-\sqrt{|\alpha|}r)}{r} * \left|e^{-itH(b)}\phi\right|^2, \left|e^{-itH(b)}\phi\right|^2 \right\rangle dt, \quad \alpha \in \mathbb{R},$$

$$a = \lim_{\varepsilon\to 0} i\varepsilon^{-3} b^{-7} \left\langle (\Omega_+ S_2 \Omega_-^{-1} - id)(\varepsilon\phi_b), \phi_b \right\rangle,$$

$$m_0 = \max\{m \in \mathbb{Z}_{\geq 0}; \Psi_1(m) \leq |a|\},$$

$$q_1 = \max\left\{q = 0,1; \Psi_1\left(m_0 + \frac{q}{2}\right) \leq |a|\right\},$$

$$q_{j+1} = \max\left\{q = 0,1; \Psi_1\left(m_0 + \sum_{k=1}^{j} \frac{q_k}{2^k} + \frac{q}{2^{j+1}}\right) \leq |a|\right\}, \quad j = 1,2,\cdots.$$

Then we have

$$Q_1 = \operatorname{sign}\left(\frac{Q_1}{\mu_1^2}\right)\left(m_0 + \sum_{j=1}^{\infty} \frac{q_j}{2^j}\right). \tag{6.5}$$

6.1. **Proof of Theorem 6.2.** In the rest of this section, we give the proof of Theorem 6.2. For this purpose, we first list some notation and propositions. For $\alpha > 0$ and $y \in \mathbb{R}^3$, let $H(\alpha, y)$ be a linear operator on X_2 defined by

$$D(H(\alpha,y)) = D(-\Delta), \quad H(\alpha,y) = -\Delta + \alpha^2 V_0(\alpha x - y).$$

Furthermore, we put

$$H(\alpha) = H(\alpha, 0).$$

The operator $H(\alpha,y)$ becomes a self-adjoint operator on X_2. We denote norms $\|\cdot\|_{X_2}$, $\|\cdot\|_{L^q(\mathbb{R}^3)}$ ($q \in [1,\infty]$) and $\|\cdot\|_{L^4(\mathbb{R};L^4(\mathbb{R}^3))}$ by $\|\cdot\|$, $\|\cdot\|_q$ and $\|\cdot\|_{(4,4)}$, respectively. For $E \subset \mathbb{R}^3$, let $C_c^\infty(E)$ be the set of all smooth functions with compact support in E.

Proposition 6.3. *Let $\alpha > 0$ and $\phi \in X_2$. Then we have the following identities:*

(i)

$$\left(e^{-itH}\phi_\alpha\right)(x) = \left(e^{-i\alpha^{-2}tH(\alpha)}\phi\right)(\alpha^{-1}x). \tag{6.6}$$

(ii)

$$\left(e^{-itH(\alpha)}\phi\right)(x-\alpha^{-1}y) = \left(e^{-itH(\alpha,y)}\tau_{\alpha^{-1}y}\phi\right)(x), \tag{6.7}$$

where $\tau_z\phi(x) := \phi(x-z)$, $z \in \mathbb{R}^3$.

Proposition 6.4. *If* $\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$, *then we have*

$$\lim_{\alpha\to\infty} \left\|\left(e^{-itH(\alpha,y)} - e^{it\Delta}\right)\left((-\Delta+i)(-\Delta-i)\phi\right)\right\| = 0. \tag{6.8}$$

Proposition 6.5. *For any* $\alpha > 0$ *and for any* $\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$, *we have*

$$\left\|e^{-itH(\alpha)}\phi\right\|_6 \le C(\phi)(1+|t|)^{-1/2}. \tag{6.9}$$

Here, the constant $C(\phi)$ *is independent of* α *and* t.

We omit the proof of Proposition 6.3. We now show Proposition 6.4. The proof of is essentially similar to that of Theorem VIII.20 in [16].

Proof of Proposition 6.4. We give the proof dividing two steps.
(Step I.) For $\alpha \in \mathbb{R}$ and $m \in \mathbb{Z}_{\ge 0}$, let $f(\alpha) = e^{-it\alpha}$ and $g_m(\alpha) = e^{-\alpha^2/m^2}$. Let $\psi \in X_2$ and $\varepsilon > 0$. Then there exists some $m_0 \in \mathbb{Z}_{\ge 0}$ such that

$$\|g_m(-\Delta)\psi - \psi\| \le \varepsilon \tag{6.10}$$

for any $m \ge m_0$. Henceforth, we assume that $m = m_0$. Since g_m is a continuous function vanishing at infinity, it follows from the Stone-Weierstrass theorem (see, e.g., [16]) that there exists some two-parameter polynomial $P(\alpha,\beta)$ such that

$$\sup_{\alpha\in\mathbb{R}} \left|g_m(\alpha) - P\left((\alpha+i)^{-1}, (\alpha-i)^{-1}\right)\right| \le \varepsilon.$$

Therefore, for any self-adjoint operator A, we have

$$\left\|g_m(A) - P\left((A+i)^{-1}, (A-i)^{-1}\right)\right\| \le \varepsilon. \tag{6.11}$$

Thus, we obtain

$$\begin{aligned}
&\|(g_m(H(\alpha,y)) - g_m(-\Delta))\psi\| \\
&\le \left\|g_m(H(\alpha,y)) - P\left((H(\alpha,y)+i)^{-1}, (H(\alpha,y)-i)^{-1}\right)\right\| \|\psi\| \\
&\quad + \left\|g_m(-\Delta) - P\left((-\Delta+i)^{-1}, (-\Delta-i)^{-1}\right)\right\| \|\psi\| \\
&\quad + \left\|P\left((H(\alpha,y)+i)^{-1}, (H(\alpha,y)-i)^{-1}\right)\psi - P\left((-\Delta+i)^{-1}, (-\Delta-i)^{-1}\right)\psi\right\| \\
&\le 2\varepsilon\|\psi\| + \left\|P\left((H(\alpha,y)+i)^{-1}, (H(\alpha,y)-i)^{-1}\right)\psi - P\left((-\Delta+i)^{-1}, (-\Delta-i)^{-1}\right)\psi\right\|.
\end{aligned} \tag{6.12}$$

Here, we have used the property (6.11) in the last inequality. Since it follows that

$$P(\alpha_1,\beta_1)-P(\alpha_2,\beta_2)=\sum_{k,l}C_{k,l}\left\{(\alpha_1-\alpha_2)\widetilde{P}^{k-1}(\alpha_1,\alpha_2)\beta_1^l+(\beta_1-\beta_2)\widetilde{P}^{l-1}(\beta_1,\beta_2)\alpha_2^k\right\}$$

for some two-parameter polynomial $\widetilde{P}^k$, $k=-1,0,1,2,\cdots$, we obtain

$$\begin{aligned}&\left\|P\left((H(\alpha,y)+i)^{-1},(H(\alpha,y)-i)^{-1}\right)\psi-P\left((-\Delta+i)^{-1},(-\Delta-i)^{-1}\right)\psi\right\|\\&\quad\le C\left\{\left\|\left((H(\alpha,y)+i)^{-1}-(-\Delta+i)^{-1}\right)\psi\right\|+\left\|\left((H(\alpha,y)-i)^{-1}-(-\Delta-i)^{-1}\right)\psi\right\|\right\}.\end{aligned}\tag{6.13}$$

Henceforth, we put $\psi=(-\Delta+i)(-\Delta-i)\phi$, $\phi\in C_c^\infty(\mathbb{R}^3\setminus\{0\})$. Then we see that

$$\begin{aligned}&\left\|\left((H(\alpha,y)\pm i)^{-1}-(-\Delta\pm i)^{-1}\right)\psi\right\|\\&\quad\le\left\|\left((H(\alpha,y)\pm i)^{-1}-(-\Delta\pm i)^{-1}\right)(-\Delta+i)(-\Delta-i)\phi\right\|\\&\quad=\left\|(H(\alpha,y)\pm i)^{-1}\alpha^2V_0(\cdot-\alpha^{-1}y)(-\Delta\mp i)\phi\right\|\\&\quad\le\left\|\alpha^2V_0(\cdot-\alpha^{-1}y)(-\Delta\mp i)\phi\right\|.\end{aligned}$$

We set $\eta=\mathrm{dist}(\mathrm{supp}\phi,0)$. If $\alpha>0$ is sufficiently large, then we have

$$\left\|\left((H(\alpha,y)\pm i)^{-1}-(-\Delta\pm i)^{-1}\right)\psi\right\|\le C|Q_0|\alpha^2\frac{\exp(-\mu_0|\alpha\eta-y|)}{|\alpha\eta-y|}\|(-\Delta\mp i)\phi\|.\tag{6.14}$$

We see from (6.12)–(6.14) that

$$\lim_{\alpha\to\infty}\|g_m(H(\alpha,y))\psi-g_m(-\Delta)\psi\|=0.\tag{6.15}$$

(Step II.) Since fg_m is a continuous function vanishing at infinity, it follows from the same argument of the proof of (6.12) that

$$\begin{aligned}&\|(f(H(\alpha,y))-f(-\Delta))\psi\|\\&\quad\le\|(fg_m)(H(\alpha,y))\psi-f(H(\alpha,y))\psi\|+\|(fg_m)(-\Delta)\psi-f(-\Delta)\psi\|\\&\qquad+\|(fg_m)(H(\alpha,y))\psi-(fg_m)(-\Delta)\psi\|\\&\quad\le\|f(H(\alpha,y))\|\|g_m(H(\alpha,y))\psi-\psi\|+\|f(-\Delta)\|\|g_m(-\Delta)\psi-\psi\|+2\varepsilon\|\psi\|\\&\qquad+\left\|\acute{P}\left((H(\alpha,y)+i)^{-1},(H(\alpha,y)-i)^{-1}\right)\psi-\acute{P}\left((-\Delta+i)^{-1},(-\Delta-i)^{-1}\right)\psi\right\|\\&\quad\le\|g_m(H(\alpha,y))\psi-g_m(-\Delta)\psi\|+2\|g_m(-\Delta)\psi-\psi\|+2\varepsilon\|\psi\|\\&\qquad+\left\|\acute{P}\left((H(\alpha,y)+i)^{-1},(H(\alpha,y)-i)^{-1}\right)\psi-\acute{P}\left((-\Delta+i)^{-1},(-\Delta-i)^{-1}\right)\psi\right\|\end{aligned}\tag{6.16}$$

for some two-parameter polynomial $\acute{P}$. Here, we have used the unitarity of $f(A)$ in the last inequality. By (6.10) and (6.13)–(6.15), (6.8) holds. □

We next show Proposition 6.5.

Proof of Proposition 6.5 . Let C_b be the best constant for the embedding $\dot{W}_2^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$:

$$C_b = \sup_{\psi \in \dot{H}^1 \setminus \{0\}} \frac{\|\psi\|_6}{\|\nabla \psi\|}.$$

Then we obtain

$$\begin{aligned}
\left\|e^{-itH(\alpha)}\phi\right\|_6^2 &\le C_b^2 \left\|\nabla e^{-itH(\alpha)}\phi\right\|^2 \\
&\le C_b^2 \left\langle \nabla e^{-itH(\alpha)}\phi, \nabla e^{-itH(\alpha)}\phi \right\rangle \\
&\le C_b^2 \left\langle (-\Delta) e^{-itH(\alpha)}\phi, e^{-itH(\alpha)}\phi \right\rangle \\
&\le C_b^2 \left\langle H(\alpha) e^{-itH(\alpha)}\phi, e^{-itH(\alpha)}\phi \right\rangle - C_b^2 \left\langle \alpha^2 (V_0)_{\alpha^{-1}} e^{-itH(\alpha)}\phi, e^{-itH(\alpha)}\phi \right\rangle \\
&\le C_b^2 \left\langle H(\alpha)\phi, \phi \right\rangle + C_b^2 \|\alpha^2 (V_0)_{\alpha^{-1}}\|_{3/2} \left\|e^{-itH(\alpha)}\phi\right\|_6^2 \\
&\le C_b^2 \|\nabla \phi\|^2 + C_b^2 \left|\left\langle \alpha^2 (V_0)_{\alpha^{-1}}\phi, \phi \right\rangle\right| + C_b^2 \|V_0\|_{3/2} \left\|e^{-itH(\alpha)}\phi\right\|_6^2 .
\end{aligned}$$

It follows that

$$\left|\left\langle \alpha^2 (V_0)_{\alpha^{-1}}\phi, \phi \right\rangle\right| \le |Q_0| \alpha \frac{\exp(-\mu_0 \eta \alpha)}{\eta} \|\phi\|^2 \le \frac{|Q_0| \|\phi\|^2}{e \mu_0 \eta^2},$$

where $\eta = \operatorname{dist}(\operatorname{supp}\phi, 0)$. By Aubin and Talenti [1, 26], the best constant C_b is explicitly given by $C_b = 3^{-1} 2^{1/3} \pi^{-4/3}$. Therefore, we have

$$C_b^2 \left\| \frac{e^{-r}}{r} \right\|_{3/2} < 1. \tag{6.17}$$

We hence see that

$$\left\|e^{-itH(\alpha)}\phi\right\|_6 \le \sqrt{\frac{\|\nabla \phi\|^2 + \dfrac{|Q_0|}{e \mu_0 \eta^2} \|\phi\|^2}{C_b^{-2} - \dfrac{|Q_0|}{\mu_0} \left\| \dfrac{e^{-r}}{r} \right\|_{3/2}}}. \tag{6.18}$$

Furthermore, by [20] and (6.6), we have

$$\begin{aligned}
\left\|e^{-itH(\alpha)}\phi\right\|_6 &= \left\| \left(e^{-it\alpha^2 H(\alpha)} \phi_\alpha \right)_{\alpha^{-1}} \right\|_6 \\
&= \alpha^{-1/2} \left\| e^{-it\alpha^2 H(\alpha)} \phi_\alpha \right\|_6 \\
&\le C \alpha^{-1/2} |t\alpha^2|^{-3(1/2-1/6)} \|\phi_\alpha\|_{6/5} \\
&= C \alpha^{-1/2-2+5/2} t^{-1} \|\phi\|_{6/5} \\
&= C t^{-1} \|\phi\|_{6/5}.
\end{aligned} \tag{6.19}$$

From (6.18) and (6.19), we have (6.9). □

We are now ready to state the proof of Theorem 6.2.

Proof of Theorem 6.2. We first show (6.4). Remark that for any $\phi \in X_2 \setminus \{0\}$ and any $\alpha > \|\phi\|^{3/2}\delta^{-1}$, $(\Omega_+ S_2 \Omega_-^{-1} - id)(\alpha^{-3}\phi_\alpha)$ is well-defined because we have

$$\|\alpha^{-3}\phi_\alpha\| \leq \alpha^{-3/2}\|\phi\|.$$

Let u_α be the time-global solution to (1.2) with $\phi_- = \alpha^{-3}\phi_\alpha$. Put

$$u_\alpha^0 = e^{-itH}(\alpha^{-3}\phi_\alpha), \quad \widetilde{u_\alpha^0} = e^{-itH}(\phi_\alpha), \quad u_\alpha^1 = u_\alpha - u_\alpha^0.$$

Then we obtain

$$i\alpha^4\Big\langle(\Omega_+ S_2 \Omega_-^{-1} - id)(\alpha^{-3}\phi_\alpha), \phi_\alpha\Big\rangle = (I)_\alpha + (II)_\alpha^1 + (II)_\alpha^2 + (II)_\alpha^3,$$

where

$$(I)_\alpha = \alpha^4 \int_{\mathbb{R}} \Big\langle \big(V_1 * u_\alpha^0 \overline{u_\alpha^0}\big) u_\alpha^0, \widetilde{u_\alpha^0}\Big\rangle dt,$$

$$(II)_\alpha^1 = \alpha^4 \int_{\mathbb{R}} \Big\langle \big(V_1 * u_\alpha^1 \overline{u_\alpha^0}\big) u_\alpha^0, \widetilde{u_\alpha^0}\Big\rangle dt,$$

$$(II)_\alpha^2 = \alpha^4 \int_{\mathbb{R}} \Big\langle \big(V_1 * u_\alpha \overline{u_\alpha^1}\big) u_\alpha^0, \widetilde{u_\alpha^0}\Big\rangle dt,$$

$$(II)_\alpha^3 = \alpha^4 \int_{\mathbb{R}} \Big\langle \big(V_1 * u_\alpha \overline{u_\alpha}\big) u_\alpha^1, \widetilde{u_\alpha^0}\Big\rangle dt.$$

Following the proof of Proposition 3.1, we see that for $j = 1, 2, 3$,

$$\big|(II)_\alpha^j\big| \leq C\alpha^{-7/2}\|\phi\|^6 \to 0 \quad \text{as } \alpha \to \infty.$$

By proposition (6.3), we obtain

$$\begin{aligned}
(I)_\alpha &= \alpha^{-5} \int_{\mathbb{R}^7} Q_1 \frac{\exp(-\mu_1|y|)}{|y|} \big|e^{-i\alpha^{-2}tH(\alpha)}\phi(\alpha^{-1}x - \alpha^{-1}y)\big|^2 \big|e^{-i\alpha^{-2}tH(\alpha)}\phi(\alpha^{-1}x)\big|^2 d(t,x,y) \\
&= \int_{\mathbb{R}^7} Q_1 \frac{\exp(-\mu_1|y|)}{|y|} \big|e^{-itH(\alpha,y)}\tau_{\alpha^{-1}y}\phi(x)\big|^2 \big|e^{-itH(\alpha)}\phi(x)\big|^2 d(t,x,y) \\
&= \int_{\mathbb{R}^3} Q_1 \frac{\exp(-\mu_1|y|)}{|y|} \Phi(\alpha, y)\, dy,
\end{aligned}$$

where

$$\Phi(\alpha, y) = \int_{\mathbb{R}^{1+3}} \big|e^{-itH(\alpha,y)}\tau_{\alpha^{-1}y}\phi(x)\big|^2 \big|e^{-itH(\alpha)}\phi(x)\big|^2 d(t,x).$$

For the function Φ, we have the following property:

Lemma 6.6. *Assume that $\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ satisfies $\phi \neq 0$ and $(\Delta^2 + 1)^{-1}\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$.*

(i) For any $y \in \mathbb{R}^3$, we have

$$\lim_{\alpha \to \infty} \Phi(\alpha, y) = \|e^{it\Delta}\phi\|_{(4,4)}^4.$$

(ii) For any $\alpha > 0$ and $y \in \mathbb{R}^3$, we have

$$|\Phi(\alpha, y)| \leq C(\phi),$$

where the constant $C(\phi)$ is independent of α and y.

Proof of Lemma 6.6. It follows from the Hölder inequality that

$$\begin{aligned}
&\left|\Phi(\alpha,y) - \|e^{it\Delta}\phi\|_{(4,4)}^4\right| \\
&\leq \int_{\mathbb{R}^{1+3}} \left(\left|e^{-itH(\alpha,y)}\tau_{\alpha^{-1}y}\phi(x)\right|^2\left|e^{-itH(\alpha)}\phi(x)\right|^2 - |e^{it\Delta}\phi(x)|^4\right) d(t,x) \\
&\leq \int_{\mathbb{R}^{1+3}} \left|e^{-itH(\alpha,y)}\tau_{\alpha^{-1}y}\phi(x) - e^{it\Delta}\phi(x)\right|\left|e^{-itH(\alpha,y)}\tau_{\alpha^{-1}y}\phi(x)\right|\left|e^{-itH(\alpha)}\phi(x)\right|^2 d(t,x) \\
&\quad + \int_{\mathbb{R}^{1+3}} \left|e^{it\Delta}\phi(x)\right|\left|e^{-itH(\alpha,y)}\tau_{\alpha^{-1}y}\phi(x) - e^{it\Delta}\phi(x)\right|\left|e^{-itH(\alpha)}\phi(x)\right|^2 d(t,x) \\
&\quad + \int_{\mathbb{R}^{1+3}} \left|e^{it\Delta}\phi(x)\right|^2\left|e^{-itH(\alpha)}\phi(x) - e^{it\Delta}\phi(x)\right|\left|e^{-itH(\alpha)}\phi(x)\right| d(t,x) \\
&\quad + \int_{\mathbb{R}^{1+3}} \left|e^{it\Delta}\phi(x)\right|^3\left|e^{-itH(\alpha)}\phi(x) - e^{it\Delta}\phi(x)\right| d(t,x) \\
&\leq \int_{\mathbb{R}} \left\|e^{-itH(\alpha,y)}\tau_{\alpha^{-1}y}\phi - e^{it\Delta}\phi\right\| \left\|e^{-itH(\alpha)}\phi\right\|_6^3 dt \\
&\quad + \int_{\mathbb{R}} \left\|e^{it\Delta}\phi\right\|_6 \left\|e^{-itH(\alpha,y)}\tau_{\alpha^{-1}y}\phi - e^{it\Delta}\phi\right\| \left\|e^{-itH(\alpha)}\phi\right\|_6^2 dt \\
&\quad + \int_{\mathbb{R}} \left\|e^{it\Delta}\phi\right\|_6^2 \left\|e^{-itH(\alpha)}\phi - e^{it\Delta}\phi\right\| \left\|e^{-itH(\alpha)}\phi\right\|_6 dt \\
&\quad + \int_{\mathbb{R}} \left\|e^{it\Delta}\phi\right\|_6^3 \left\|e^{-itH(\alpha)}\phi - e^{it\Delta}\phi\right\| dt
\end{aligned}$$

where we have used the equality

$$\left\|e^{-itH(\alpha,y)}\tau_{\alpha^{-1}y}\phi\right\|_6 = \left\|e^{-itH(\alpha)}\phi\right\|_6,$$

which is given by (6.7), in the last inequality. We can easily see that

$$\left\|e^{it\Delta}\phi\right\|_6 \leq (1+|t|)^{-1}\left(\|\nabla\phi\| + \|\phi\|_{6/5}\right).$$

Therefore, by Propositions 6.4 and 6.5 and by applying the Lebesgue dominated theorem with respect to the variable t, we obtain (i). Similarly, we have (ii). □

Let us go back to the proof of Theorem 6.2. Henceforth, we suppose that $\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ satisfies $\phi \neq 0$ and $(\Delta^2+1)^{-1}\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$. Using the above Lemma 6.6, we see from the Lebesgue dominated theorem with respect to the variable y that

$$\lim_{\alpha\to\infty}(I)_\alpha = \|e^{it\Delta}\phi\|_{(4,4)}^4 \int_{\mathbb{R}^3} Q_1 \frac{\exp(-\mu_1|y|)}{|y|}dy$$
$$= 4\pi\frac{Q_1}{\mu_1^2}\|e^{it\Delta}\phi\|_{(4,4)}^4,$$

which implies (6.4).

We next show (6.5). Suppose that $Q_1 \neq 0$. Recall the definition of b, Ψ_1, m_0 and q_j, $j = 1, 2, \cdots$. It follows from Propositions 3.1 and 6.3, that

$$\begin{aligned}
a = b^{-7}K[\phi_b] &= b^{-7}Q_1 \int_{\mathbb{R}} \left\langle \frac{\exp(-\mu_1 r)}{r} * \left|e^{-itH}\phi_b\right|^2, \left|e^{-itH}\phi_b\right|^2 \right\rangle dt \\
&= b^{-7}Q_1 \int_{\mathbb{R}} \left\langle \frac{\exp(-\mu_1 r)}{r} * \left|\left(e^{-itb^{-2}H(b)}\phi\right)_b\right|^2, \left|\left(e^{-itb^{-2}H(b)}\phi\right)_b\right|^2 \right\rangle dt \\
&= b^{-7+2+3+3}Q_1 \int_{\mathbb{R}} \left\langle \frac{\exp(-b\mu_1 r)}{br} * \left|e^{-itH(b)}\phi\right|^2, \left|e^{-itH(b)}\phi\right|^2 \right\rangle dt \\
&= Q_1 \int_{\mathbb{R}} \left\langle \frac{\exp(-\sqrt{|Q_1|}r)}{r} * \left|e^{-itH(b)}\phi\right|^2, \left|e^{-itH(b)}\phi\right|^2 \right\rangle dt \\
&= \Psi_1(Q_1).
\end{aligned}$$

By the Plancherel theorem, we have

$$\Psi_1(\alpha) = 4\pi \int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{\alpha}{|\alpha| + |\xi|^2} \left| \left(\mathcal{F}\left|e^{-itH(b)}\phi\right|^2\right)(\xi) \right|^2 d\xi dt.$$

Therefore, $\Psi_1 : \mathbb{R} \to \Psi_1(\mathbb{R})$ is odd, continuous, bijective and monotonically increasing. Thus, we obtain

$$\Psi_1\left(m_0 + \sum_{k=1}^{j} \frac{q_k}{2^k}\right) \leq \Psi_1(|Q_1|) < \Psi_1\left(m_0 + \sum_{k=1}^{j} \frac{q_k}{2^k} + \frac{1}{2^j}\right)$$

and

$$m_0 + \sum_{k=1}^{j} \frac{q_k}{2^k} \leq |Q_1| < m_0 + \sum_{k=1}^{j} \frac{q_k}{2^k} + \frac{1}{2^j}.$$

Hence (6.5) holds. □

7. APPLICATIONS

In this section, dividing two subsections, we review a part of the works [22] and [23] which is concerned with the inverse scattering problem for relativistic equations with a non-local term. In the first subsection, we consider the nonlinear Klein-Gordon equation

$$\partial_t^2 u - \Delta u + u = \lambda \left(V_1 * |u|^2\right) u, \quad \text{in } \mathbb{R} \times \mathbb{R}^n. \tag{NLKG}$$

Here, V_1 and λ satisfy the condition (C) appeared in Section 5. The existence of the scattering operator for (NLKG) is directly proved by [12]. We introduce the result of [22], which is concerned with determining unknown σ and λ by using the scattering states.

On the other hand, in the second subsection, we consider the semi-relativistic Hartree equation

$$i\partial_t u - (1-\Delta)^{1/2} u = \left(\frac{Q_1 \exp(-\mu_1|x|)}{|x|} * |u|^2\right) u, \quad \text{in } \mathbb{R} \times \mathbb{R}^3. \tag{SRH}$$

Here, the real number Q_1 and the positive number μ_1 are unknown. The existence of the scattering operator for (SRH) is directly proved by Cho–Ozawa [5]. We introduce a part of the result of [23], which is concerned with determining unknown Q_1 and μ_1 by using the scattering states.

Before treating the inverse scattering problem for the above two relativistic equations, we first review known results for nonlinear Klein-Gordon equations. Morawetz–Strauss [11] initially studied the inverse scattering problem for the Klein-Gordon equation with power nonlinearity. Later, Bachelot [2] considered more general cases. Weder [33, 37] proved that a more general class of nonlinearities is uniquely reconstructed, and moreover, a method is given for the unique reconstruction of the potential that acts as a linear operator and that this problem was not considered in [11, 2]. The inverse scattering problem for the Klein-Gordon equation with a cubic convolution

$$\partial_t^2 w - \Delta w + w = \left(V * |w|^2\right) w, \quad \text{in } (t,x) \in \mathbb{R}^{1+n}$$

was initially studied by Sasaki–Watanabe [24]. As far as the author knows, there is no result except [23] of the inverse scattering problem for semi-relativistic Hartree equation.

7.1. **Nonlinear Klein-Gordon equation.** Let us focus on the inverse scattering problem for the nonlinear Klein-Gordon equation (NLKG). We here suppose that V_1 and λ satisfy the condition (C) appeared in Section 5. We first mention the existence theorem for the scattering operator for (NLKG).

Theorem 7.1. *Let $n \geq 3$, $2 \leq \sigma \leq 4$ and $\sigma < n$. Put $X_3 = H^1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ and $Z_3 = C(\mathbb{R}; H^1(\mathbb{R}^n)) \cap C^1(\mathbb{R}; L^2(\mathbb{R}^n)) \cap L^3(\mathbb{R}; H_{q_3}^{1-\rho_3})$ with $1/q_3 = 1/2 - 2/3(n-1+\theta)$, $\rho_3 = (n+1+\theta)/3(n-1+\theta)$ and $\theta = 2 - \sigma/2$. There exists some $\delta > 0$ such that for any $\psi_- = {}^t(\psi_-^1, \psi_-^2) \in B(\delta; X_3)$, we uniquely have $w \in Z_3$ and $\psi_+ \in X_3$ such* *that*

$$
\begin{aligned}
&W = {}^t(w, \partial_t w) \in C^1(\mathbb{R}; H^1(\mathbb{R}^n)) \oplus C(\mathbb{R}; L_2(\mathbb{R}^n)),\\
&W(t) = U(t)\phi_- + \frac{1}{i}\int_{-\infty}^t U(t-\tau)\begin{pmatrix} 0 \\ iF(w(\tau))\end{pmatrix} d\tau,\\
&\|W(t) - U(t)\psi_\pm\|_{X_3} \to 0 \quad \text{as } t \to \pm\infty,\\
&\|w\|_{Z_3} \le C\|\psi_-\|_{H^1(\mathbb{R}^n)}, && (7.1)\\
&\left\|w - \left(\cos(t\omega)\psi_-^1 + \omega^{-1}\sin(t\omega)\psi_-^2\right)\right\|_{Z_3} \le C\|\psi_-\|_{X_3}^3, && (7.2)\\
&\left\|\int_{\mathbb{R}} e^{it\omega}(V_1 * (w_1\overline{w_2}))w_3 dt\right\|_{L^2(\mathbb{R}^n)} \le C\Pi_{j=1}^3\|w_j\|_{Z_3}.\\
&\qquad\qquad\qquad\qquad \text{for } w_j \in Z_3,\ j = 1,2,3. && (7.3)
\end{aligned}
$$

Moreover, we can define the scattering operator for (NLKG)

$$S : B(\delta; X_3) \ni \psi_- \mapsto \psi_+ = \psi_- + \frac{1}{i}\int_{\mathbb{R}} U(-t)\begin{pmatrix} 0 \\ iF(w(t))\end{pmatrix} dt \in X_3.$$

The goal of [22] is to identify unknown λ and σ. In order to state the main result of [22], we list some notation. For $\phi \in L^2(\mathbb{R}^n)$, the small amplitude $\mathcal{K}[\phi]$ denotes

$$\mathcal{K}[\phi] = \lim_{\varepsilon\to 0}\frac{i}{\varepsilon^3}\left\langle (S-I)(\varepsilon\,{}^t(\phi,0)),\ {}^t(0,\phi)\right\rangle_{X_3}.$$

For $s \in \mathbb{R}$ and $q \in [1,\infty]$, we denote by H_q^s the Sobolev space $H_q^s(\mathbb{R}^3) = (1-\Delta)^{-s/2}L^q(\mathbb{R}^n)$. For $q \in [1,\infty]$, q' means the Hölder conjugate of q. We are ready to state the main result.

Theorem 7.2. *Suppose that $n \ge 3$, $2 \le \sigma \le 4$, $\sigma < n$,*

$$\phi \in H^1 \cap \bigcap_{r\in I} H_{r'}^{(n+1)(1/2-1/r)}$$

and $\phi \ne 0$. Let $I = (6(n-1)/(3n-5), 2n/(n-2)]$. Then we have the formula for determining σ

$$\sigma = 2n + 1 - \lim_{\alpha\to 0}\ln\frac{|\mathcal{K}[\phi_{e\alpha}]|}{|\mathcal{K}[\phi_\alpha]| + \alpha^{2n+1}}. \tag{7.4}$$

Furthermore, the reconstruction formula for λ can be also given by

$$\lambda(x_0) = \frac{\lim_{\alpha\to 0}\alpha^{-(2n+1-\sigma)}\mathcal{K}[\phi_{\alpha,x_0}]}{\displaystyle\int |y|^{-\sigma}|w_0(t,x-y)|^2|w_0(t,x)|^2 d(t,x,y)}, \tag{7.5}$$

where $w_0 = \cos(t\sqrt{-\Delta})\phi$.

Now we prove Theorem 7.2. To derive (7.4), we follow the line of the proof of Theorem 5.1. Let $\sigma \in [2,4] \cap [2,n)$.

We here assume that

$$\phi \in \Lambda := H^1 \cap \bigcap_{r \in I} H_{r'}^{(n+1)(1/2-1/r)}$$

and $\phi \neq 0$, where $I = (6(n-1)/(3n-5), 2n/(n-2)]$. By Theorem 7.1, and $\Lambda \subset H^1$, $\mathcal{K}[\phi]$ is well-defined. By (7.1)–(7.3), it follows from the proof of (3.1) that

$$\mathcal{K}[\phi] = \int_{\mathbb{R}^{1+n+n}_{(t,x,y)}} \lambda(x)|y|^{-\sigma}|\Gamma^1(t,x-y)|^2|\Gamma^1(t,x)|^2 d(t,x,y),$$

where $\Gamma^m(t) = \cos\bigl(t\sqrt{m^2-\Delta}\bigr)\phi$.

Having in mind that

$$e^{it\sqrt{1-\Delta}}\phi_\alpha = (e^{it\alpha^{-1}\sqrt{\alpha^2-\Delta}}\phi)_\alpha,$$

we also have

$$\mathcal{K}[\phi_\alpha] = \alpha^{2n+1-\sigma}\int_{\mathbb{R}^{1+n+n}_{(t,x,y)}} \lambda(\alpha x)Q^\alpha(\phi)(t,x,y)d(t,x,y),$$

where

$$Q^m(\phi)(t,x,y) = |y|^{-\sigma}|\Gamma^m(t,x-y)|^2|\Gamma^m(t,x)|^2.$$

In order to see the convergence of $|\mathcal{K}[\phi_{e\alpha}]|/(|\mathcal{K}[\phi_\alpha]| + \alpha^{2n+1})$, we give the following lemma:

Lemma 7.3. *Let $I = (6(n-1)/(3n-5), 2n/(n-2)]$ and $0 \le \beta \le 1$. For $n/(n-1) < \sigma \le 4$, $\sigma < n$, and*

$$\phi \in H^1 \cap \bigcap_{r \in I} H_{r'}^{(n+1)(1/2-1/r)},$$

then $Q^\beta(\phi)(t,x,y)$ is integrable on $\mathbb{R}^{1+n+n}_{(t,x,y)}$. Moreover, we have

$$\int Q^\beta(\phi)d(t,x,y) \to \int Q^0(\phi)d(t,x,y) \quad \text{as } \beta \to 0. \tag{7.6}$$

Proof. We first state $L^p - L^{p'}$ estimates (see, e.g., [14, 15])

$$\|e^{it\sqrt{-\Delta}}\phi\|_r \le C|t|^{-(n-1)(1/2-1/r)}\|\phi\|_{\dot{H}_{r'}^{(n+1)(1/2-1/r)}} \tag{7.7}$$

and

$$\|e^{it\sqrt{1-\Delta}}\phi\|_r \le C|t|^{-(n-1)(1/2-1/r)}\|\phi\|_{H_{r'}^{(n+1)(1/2-1/r)}}, \tag{7.8}$$

where $2 < r < \infty$ and $\dot{H}_p^s$ is the homogeneous Sobolev space (for the definition, see, e.g., [3, 8]).

By the embedding $H^s_{r'} \hookrightarrow \dot{H}^s_{r'}$ and $H^{n(1/2-1/r)} \hookrightarrow L^r(\mathbb{R}^n)$, it follows from (7.7) that

$$\|e^{it\sqrt{-\Delta}}\phi\|_r \leq C(1+|t|)^{-(n-1)(1/2-1/r)}N_r(\phi), \tag{7.9}$$

where

$$N_r(\phi) = \max\{\|\phi\|_{H^{n(1/2-1/r)}}, \|\phi\|_{H^{(n+1)(1/2-1/r)}_{r'}}\}.$$

On the other hand, Using the identity

$$e^{it\sqrt{m^2-\Delta}}\phi = (e^{itm\sqrt{1-\Delta}}\phi_m)_{m^{-1}},$$

and (7.8), we obtain

$$\begin{aligned}
\|e^{it\sqrt{m^2-\Delta}}\phi\|_r &= m^{-n/r}\|e^{itm\sqrt{1-\Delta}}\phi_m\|_r \\
&\leq Cm^{-n/r}|tm|^{-(n-1)(1/2-1/r)}\|\phi_m\|_{H^{(n+1)(1/2-1/r)}_{r'}} \\
&\leq Cm^{(n+1)(1/2-1/r)}(1+m^{-(n+1)(1/2-1/r)}) \\
&\qquad\qquad \times |t|^{-(n-1)(1/2-1/r)}\|\phi\|_{H^{(n+1)(1/2-1/r)}_{r'}} \\
&\leq C|t|^{-(n-1)(1/2-1/r)}\|\phi\|_{H^{(n+1)(1/2-1/r)}_{r'}}
\end{aligned}$$

if $0 < m \leq 1$. By the embedding $H^{n(1/2-1/r)} \hookrightarrow L^r$, it follows from (7.9) that

$$\|e^{it\sqrt{\beta^2-\Delta}}\phi\|_r \leq C(1+|t|)^{-(n-1)(1/2-1/r)}N_r(\phi). \tag{7.10}$$

Let $s \in [0,1]$. By the Hardy-Littlewood-Sobolev inequality, the Hölder inequality and the embedding $L^{(1/2+s/n)^{-1}}(\mathbb{R}^n) \hookrightarrow H^{-s}$, we have

$$\begin{aligned}
&\|Q^m(\phi)-Q^0(\phi)\|_{L^1(\mathbb{R}^{2n}_{(x,y)})} \\
&\quad\leq \|(|\cdot|^{-\sigma} * (\Gamma^m-\Gamma^0)\overline{\Gamma^m})\Gamma^m\overline{\Gamma^m}\|_{L^1(\mathbb{R}^n)} \\
&\qquad + \|(|\cdot|^{-\sigma} * \Gamma^0(\overline{\Gamma^m-\Gamma^0}))\Gamma^m\overline{\Gamma^m}\|_{L^1(\mathbb{R}^n)} \\
&\qquad + \|(|\cdot|^{-\sigma} * \Gamma^0\overline{\Gamma^0})(\Gamma^m-\Gamma^0)\overline{\Gamma^m}\|_{L^1(\mathbb{R}^n)} \\
&\qquad + \|(|\cdot|^{-\sigma} * \Gamma^0\overline{\Gamma^0})\Gamma^0(\overline{\Gamma^m-\Gamma^0})\|_{L^1(\mathbb{R}^n)} \\
&\quad\leq \sum_{\beta_1,\beta_2\in\{0,1\}} \|(|\cdot|^{-\sigma} * |\Gamma^{\beta_1}|^2)|\Gamma^{\beta_2}||\Gamma^m-\Gamma^0|\|_{L^1(\mathbb{R}^n)} \\
&\quad\leq C\|\Gamma^m-\Gamma^0\|_{H^s} \sum_{\beta_1,\beta_2\in\{0,1\}} \|(|\cdot|^{-\sigma} * |\Gamma^{\beta_1}|^2)|\Gamma^{\beta_2}|\|_{H^{-s}} \\
&\quad\leq C\|\Gamma^m-\Gamma^0\|_{H^1} \sum_{\beta_1,\beta_2\in\{0,1\}} \|(|\cdot|^{-\sigma} * |\Gamma^{\beta_1}|^2)|\Gamma^{\beta_2}|\|_{(\frac{1}{2}+\frac{s}{n})^{-1}} \\
&\quad\leq C\|\Gamma^m-\Gamma^0\|_{H^1}\{\|\Gamma^m\|_r^3 + \|\Gamma^0\|_r^3\},
\end{aligned}$$

where we have used the equality

$$\int (|\cdot|^{-\sigma} * v_1)v_2 = \int (|\cdot|^{-\sigma} * v_2)v_1$$

for the second inequality, and r satisfies

$$\frac{3}{2} + \frac{s}{n} = \frac{\sigma}{n} + \frac{3}{r}, \quad 2 < r < \infty.$$

Using (7.10), we see that

$$\|Q^m(\phi) - Q^0(\phi)\|_{L^1(\mathbb{R}^{2n}_{(x,y)})} \leq C(1+|t|)^{-3(n-1)(1/2-1/r)}\big(N_r(\phi)\big)^3\|\Gamma^m - \Gamma^0\|. \quad (7.11)$$

Since $n/(n-1) < \sigma \leq 4$ and $\sigma < n$, we can put $r \in I$. Thus, we have $n(1/2-1/r) \leq 1$ and $-3(n-1)(1/2-1/r) < -1$. Hence, $Q^\beta(\phi)$ is integrable, and the right hand side of (7.11) is bounded by some integrable function which is independent of m. For all $t \in \mathbb{R}$, we can easily show that

$$\|\Gamma^m(t) - \Gamma^0(t)\|_{H^s} \to 0 \quad \text{as } m \to 0.$$

Thus, applying the Lebesgue dominate theorem on $\mathbb{R}_t$, we have (7.6). This completes the proof. □

Let us go back to the proof of Theorem 7.2. We again assume $\phi \in \Lambda$. From Lemma 7.3, we see that

$$\int_{\mathbb{R}^{1+n+n}_{(t,x,y)}} \lambda(\alpha x)Q^\alpha(\phi)(t,x,y)d(t,x,y)$$
$$\to \int_{\mathbb{R}^{1+n+n}_{(t,x,y)}} \lambda(0)Q^0(\phi)(t,x,y)d(t,x,y) \neq 0 \quad \text{as } \alpha \to 0.$$

So we obtain

$$\frac{|\mathcal{K}[\phi_{e\alpha}]|}{|\mathcal{K}[\phi_\alpha]| + \alpha^{2n+1}} = \frac{e^{2n+1-\sigma}|\int \lambda(e\alpha x)Q^0(\phi)(t,x,y)|}{|\int \lambda(a\alpha x)Q^0(\phi)(t,x,y)| + \alpha^\sigma} \to e^{2n+1-\sigma} \quad \text{as } \alpha \to 0.$$

Thus, we have (7.4).

It remains to show the reconstruction formula (7.5). We put $\phi \in \Lambda$. Then we have

$$\mathcal{K}[\phi_{\alpha,x_0}] = \alpha^{2n+1-\sigma}\int_{\mathbb{R}^{1+n+n}_{(t,x,y)}} \lambda(\alpha x + x_0)Q^\alpha(\phi)(t,x,y)d(t,x,y).$$

Here, σ is a known number which is determined in the above. Since λ is bounded and continuous, it follows from Lemma 7.3 that

$$\lim_{\alpha\to 0} \alpha^{-(2n+1-\sigma)}\mathcal{K}[\phi_{\alpha,x_0}] = \int_{\mathbb{R}^{1+n+n}_{(t,x,y)}} \lambda(x_0)Q^0(\phi)(t,x,y)d(t,x,y).$$

Thus, we have (7.5). This completes the proof of Theorem 7.2.

7.2. **Semi-relativistic Hartree equation.** Let us focus on the inverse scattering problem for the semi-relativistic Hartree equation (SRH). We first mention the existence theorem for the scattering operator for (SRH).

Theorem 7.4. *Let $s \geq 5/6$. Put $U_2(t) = e^{-it\sqrt{1-\Delta}}$, $X_4 = H^s(\mathbb{R}^3)$ and $Z_4 = C(\mathbb{R}; X_4) \cap L^2(\mathbb{R}; H_6^{s-5/6}(\mathbb{R}^3))$. Then there exists some $\delta > 0$ satisfying the following properties:*

If $\phi_- \in B(\delta; X_4)$, then there uniquely exist $w \in Z_4$ and $\phi_+ \in X_4$ such that

$$\begin{aligned}
w(t) &= U_2(t)\phi_- + \frac{1}{i}\int_{-\infty}^{t} U_2(t-\tau)F(w(\tau))d\tau, \\
\phi_+ &= \phi_- + \frac{1}{i}\int_{\mathbb{R}} U_2(-t)F(w(t))dt, \\
\|w\|_{Z_4} &\leq C\|\phi_-\|_{X_4}, \\
\|w - U_2(t)\phi_-\|_{Z_4} &\leq C\|\phi_-\|_{X_4}^3, \\
\lim_{t\to\pm\infty} \|w(t) - U_2(t)\phi_\pm\|_{X_4} &= 0.
\end{aligned}$$

Therefore, we can define the scattering operator for (SRH)

$$S_4 : B(\delta; X_4) \ni \phi_- \mapsto \phi_+ \in X_4.$$

The goal of [23] for (SRH) is to identify unknown λ and σ. The main result of [23] is the following state:

Theorem 7.5. *Let s be a positive number given by Theorem 7.4. Let $1 \leq p < 12/7$ and $k > 11/12$. Assume that*

$$\phi \in \left(H^s(\mathbb{R}^3) \cap H_p^k(\mathbb{R}^3)\right) \setminus \{0\}.$$

(i) we have

$$\frac{Q_1}{\mu_1^2} = \frac{\lim_{\alpha\to\infty} i\alpha^4 \left\langle (S_4 - id)(\alpha^{-3}\phi_\alpha), \phi_\alpha \right\rangle}{4\pi \|e^{i\frac{t}{2}\Delta}\phi\|_{(4,4)}^4}. \tag{7.12}$$

(ii) Put

$$\begin{aligned}
d &= \left|\frac{Q_1}{\mu_1^2}\right|^{1/2}, \\
\Psi_2(\alpha) &= \int_{\mathbb{R}} \left\langle \frac{\alpha\exp(-\sqrt{|\alpha|}r)}{r} * \left|e^{-it\sqrt{d^2-\Delta}}\phi\right|^2, \left|e^{-it\sqrt{d^2-\Delta}}\phi\right|^2 \right\rangle dt, \quad \alpha\in\mathbb{R}, \\
h &= \lim_{\varepsilon\to 0} i\varepsilon^{-3}d^{-6}\left\langle (S_4 - id)(\varepsilon\phi_d), \phi_d\right\rangle, \\
l_0 &= \max\{l\in\mathbb{Z}_{\geq 0}; \Psi_2(l)\leq |h|\}, \\
p_1 &= \max\left\{p=0,1; \Psi_2\left(l_0+\frac{p}{2}\right)\leq |h|\right\}, \\
p_{j+1} &= \max\left\{p=0,1; \Psi_2\left(l_0+\sum_{k=1}^{j}\frac{p_k}{2^k}+\frac{p}{2^{j+1}}\right)\leq |h|\right\}, \quad j=1,2,\cdots.
\end{aligned}$$

Then we have

$$Q_1 = \operatorname{sign}\left(\frac{Q_1}{\mu_1^2}\right)\left(l_0+\sum_{j=1}^{\infty}\frac{p_j}{2^j}\right). \tag{7.13}$$

In order to show Theorem 7.2, we first prepare the following lemma:

Lemma 7.6. *For $\alpha > 0$, let*

$$U^\alpha(t) = e^{it\alpha^2 - it\alpha\sqrt{\alpha^2-\Delta}}, \quad U^\infty(t) = e^{i\frac{t}{2}\Delta}.$$

Assume that $1\leq p < 12/7$ and $k > 11/12$. If $\phi\in H^s(\mathbb{R}^3)\cap H_p^k(\mathbb{R}^3)$, then we have

$$\lim_{\alpha\to\infty}\|U^\alpha(t)\phi - U^\infty(t)\phi\|_{(4,4)} = 0. \tag{7.14}$$

Proof. We again denote the norm $\|\cdot\|_{L^q(\mathbb{R}^3)}$ ($q\in[1,\infty]$) by $\|\cdot\|_q$. From the embedding $H^{3/4}(\mathbb{R}^3)\hookrightarrow L^4(\mathbb{R}^3)$ and the Plancherel theorem we obtain

$$\begin{aligned}
\|U^\alpha(t)\phi - U^\infty(t)\phi\|_4 &\leq C\|U^\alpha(t)\phi - U^\infty(t)\phi\|_{H^{3/4}(\mathbb{R}^3)} \\
&\leq C\left\|\left(e^{it\alpha^2-it\alpha\sqrt{\alpha^2+|\xi|^2}} - e^{-it|\xi|^2/2}\right)(1+|\xi|)^{3/4}\mathcal{F}\phi\right\|_{L^2(\mathbb{R}^3_\xi)}.
\end{aligned}$$

Since $\langle\xi\rangle^{3/4}\mathcal{F}\phi\in X_2$ and

$$\alpha^2 - \alpha\sqrt{\alpha^2+|\xi|^2} = \frac{-|\xi|^2}{1+\sqrt{1+|\xi|^2/\alpha^2}} \to \frac{-|\xi|^2}{2} \quad \text{as } \alpha\to\infty$$

for any $\xi\in\mathbb{R}^3$, it follows from the Lebesgue dominated theorem that

$$\lim_{\alpha\to\infty}\|U^\alpha(t)\phi - U^\infty(t)\phi\|_4 = 0 \quad \text{for any } t\in\mathbb{R}. \tag{7.15}$$

Now we put $\alpha > 1$. By the $L^p - L^q$ estimate for the free Klein-Gordon equation in [4], we obtain

$$\begin{aligned}\|U^\alpha(t)\phi\|_4 &= \left\| \left(e^{-it\alpha^2\sqrt{1-\Delta}}\phi_\alpha\right)_{\alpha^{-1}} \right\|_4 \\ &= \alpha^{-3/4}\|e^{-it\alpha^2\sqrt{1-\Delta}}\phi_\alpha\|_4 \\ &\le C\alpha^{-3/4}|t\alpha^2|^{-3/4}\|\phi\|_{H^{5/4}_{4/3}} \\ &\le C|t|^{-3/4}\|\phi\|_{H^{5/4}_{4/3}}. \end{aligned} \tag{7.16}$$

Using the complex interpolation method for the linear operator $U^\alpha(t)$, we see from

$$\|U^\alpha(t)\phi\|_4 \le C\|U^\alpha(t)\phi\|_{H^{3/4}} \le C\|\phi\|_{H^{3/4}}$$

and (7.16) that

$$\|U^\alpha(t)\phi\|_4 \le C(1+|t|)^{-3\theta/4}\|\phi\|_{A_\theta}, \tag{7.17}$$

where

$$A_\theta = H^{3/4}(\mathbb{R}^3) \cap H^{k_\theta}_{p_\theta}(\mathbb{R}^3), \quad k_\theta = \frac{3}{4} + \frac{\theta}{2}, \quad p_\theta = \frac{1}{2} + \frac{\theta}{4}, \quad 0 \le \theta \le 1.$$

Thus, we can easily see that the left hand side of (7.17) belongs $L^4(\mathbb{R})$ if $1/3 < \theta \le 1$. Therefore, we obtain

$$\|U^\alpha(t)\phi - U^\infty(t)\phi\|_4 \le Cg(t),$$

where $g \in L^4(\mathbb{R})$ is some suitable function independent of α. By (7.15), it follows from the Lebesgue dominated theorem with respect to time t that (7.14) holds. □

We are ready to prove Theorem 7.2.

Proof of Theorem 7.2. Following the line of the proof of (6.4), we obtain

$$\begin{aligned} &\lim_{\alpha\to\infty} i\alpha^4\left\langle (S_4 - id)(\alpha^{-3}\phi_\alpha), \phi_\alpha \right\rangle \\ &= \lim_{\alpha\to\infty} i\alpha^{-5}\int_{\mathbb{R}} \left\langle F(U_2(t)\phi_\alpha), U_2(t)\phi_\alpha \right\rangle dt \\ &= \lim_{\alpha\to\infty} \int_{\mathbb{R}^{1+2\cdot 3}} Q_1 \frac{\exp(-\mu_1|y|)}{|y|} \left|U^\alpha(t)\phi(x-\alpha^{-1}y)\right|^2 \left|U^\alpha(t)\phi(x)\right|^2 d(t,x,y). \end{aligned}$$

By (7.14), we have (7.12). The remaining formula (7.13) can be shown by the same argument as the proof of (7.12). □

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