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ON CERTAIN ESTIMATES OF SINGULAR INTEGRALS USEFUL FOR EXTRAPOLATION

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ABSTRACT. We consider several classes of singular integrals with rough kernels. We prove certain L^p estimates $(1 < p < \infty)$ for the singular integrals. As an application, we can prove L^p boundedness of the singular integrals under a minimum size condition on their kernels via an extrapolation argument.

1. INTRODUCTION

Let a function Ω in $L^1(S^{n-1})$ satisfy

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0, \tag{1.1}$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$ and $d\sigma$ is the Lebesgue surface measure on S^{n-1}. We assume $n \geq 2$. We consider singular integrals of the form:

$$T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)K(y)\, dy, \tag{1.2}$$

$$K(x) = h(|x|)\frac{\Omega(x')}{|x|^n}, \qquad x' = x/|x|,$$

where h is a function on $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$.

Now we introduce two function spaces on S^{n-1}.

(1) Let $L\log L(S^{n-1})$ denote the Zygmund class of all functions F on S^{n-1} satisfying

$$\int_{S^{n-1}} |F(\theta)| \log(2 + |F(\theta)|)\, d\sigma(\theta) < \infty.$$

(2) The Hardy space $H^1(S^{n-1})$ is the space of functions $F \in L^1(S^{n-1})$ such that

$$\|F\|_{H^1(S^{n-1})} := \|P^+F\|_{L^1(S^{n-1})} < \infty,$$

where P^+F is the radial maximal function defined as

$$P^+F(\theta) = \sup_{0\leq r<1} \left| \int_{S^{n-1}} F(\omega)P_{r\theta}(\omega)\, d\sigma(\omega) \right|.$$

Here $P_{r\omega}(\theta)$ $(0 \leq r < 1,\ \omega, \theta \in S^{n-1})$ denotes the Poisson kernel:

$$P_{r\omega}(\theta) = c_n \frac{1-r^2}{|r\omega - \theta|^n}.$$

It is known that $L\log L(S^{n-1})$ is a proper subspace of $H^1(S^{n-1})$.

We first assume that h is identically 1; so K is a homogeneous kernel. We write $T = T_\Omega$. Then,

$$(T_\Omega f)\hat{}(\xi) = m(\xi')\hat{f}(\xi),$$

where

$$m(\xi') = -\int_{S^{n-1}} \Omega(\theta)F(\xi',\theta)\,d\sigma(\theta),$$

$$F(\xi',\theta) = \left[i\frac{\pi}{2}\operatorname{sgn}(\langle\xi',\theta\rangle) + \log|\langle\xi',\theta\rangle|\right].$$

This implies, by Young's inequality, that $T_\Omega : L^2 \to L^2$ if $\Omega \in L\log L(S^{n-1})$. Also, by the method of rotations Calderón-Zygmund [4] proved that if Ω belongs to $L^1(S^{n-1})$ and is odd, then $T_\Omega : L^p \to L^p$ for all $1 < p < \infty$ and that if $\Omega \in L\log L(S^{n-1})$, then $T_\Omega : L^p \to L^p$ for all $1 < p < \infty$.

Furthermore, Coifman-Weiss [5], Connett [6] and Ricci-Weiss [15] proved that if $\Omega \in H^1(S^{n-1})$, then $T_\Omega : L^p \to L^p$ for all $1 < p < \infty$ by applying developed versions of the Calderón-Zygmund method of rotations. This is an improvement over the result of Calderón-Zygmund above, since $L\log L(S^{n-1})$ is a proper subspace of $H^1(S^{n-1})$.

Next, we see the case where h is not assumed to be a constant function. For $s \in [1,\infty)$, the space Δ_s is defined as

$$\Delta_s = \{h \text{ on } \mathbb{R}_+ : \|h\|_{\Delta_s} < \infty\},$$

where

$$\|h\|_{\Delta_s} = \sup_{j\in\mathbb{Z}}\left(\int_{2^j}^{2^{j+1}} |h(t)|^s\,dt/t\right)^{1/s}.$$

Here $\mathbb{Z}$ denotes the set of all integers. Also, let $\Delta_\infty = L^\infty(\mathbb{R}_+)$. Then, we can easily see that $\Delta_s \subset \Delta_t$ if $s > t$.

If h is not constant, the method of rotations of Calderon-Zygmund does not work in general (see [17]), nevertheless, we have the following:

(a) (R. Fefferman [11]) If $h \in L^\infty$ and Ω satisfies a Lipschitz condition on S^{n-1}, then $T : L^p \to L^p$ for all $1 < p < \infty$.

J. Namazi [13] improved this result by replacing the condition on Ω with the L^q-condition:

(b) Suppose that $h \in L^\infty$ and $\Omega \in L^q(S^{n-1})$ for some $q > 1$. Then $T : L^p \to L^p$ for all $1 < p < \infty$.

J. Duoandikoetxea and J. L. Rubio de Francia [7] further improved this by replacing the condition on h with the Δ_2 condition:

(c) If $h \in \Delta_2$ and $\Omega \in L^q(S^{n-1})$ for some $q > 1$, then $T : L^p \to L^p$ for all $1 < p < \infty$.

The L^q condition on Ω in (c) was relaxed by Fan and Pan [10] as follows:

(d) Suppose that $\Omega \in H^1(S^{n-1})$ and $h \in \Delta_s$ for some $s > 1$. Then $T : L^p \to L^p$ if $|1/p - 1/2| < \min(1/2, 1/s'), s' = s/(s-1)$.

We note the space $L^q(S^{n-1})$, $q > 1$, is a proper subspace of $H^1(S^{n-1})$ and when $s = 2$, the range of p in the conclusion of (d) is $(1,\infty)$.

Also, A. Al-Salman and Y. Pan [2] proved the following:

(e) If $\Omega \in L\log L(S^{n-1})$ and $h \in \Delta_s$ for some $s > 1$, then $T : L^p \to L^p$ for all

$1 < p < \infty$.
In (e) the condition on Ω is slightly stronger than that of (d), but the range of p shrinks to 2 as s approaches 1 in the conclusion of (d), while the range of p is always $(1, \infty)$ in the conclusion of (e), regardless the value of s.

Now, our first result is the following theorem.

Theorem 1. *Let T be as in (1.2) Suppose that $\Omega \in L^q(S^{n-1})$, $h \in \Delta_s$, $q, s \in (1, 2]$. Then*

$$\|T(f)\|_{L^p} \le C_p (q-1)^{-1}(s-1)^{-1}\|\Omega\|_{L^q(S^{n-1})}\|h\|_{\Delta_s}\|f\|_{L^p}$$

for all $p \in (1, \infty)$, where the constant C_p is independent of q, s, Ω and h.

We are interested in this estimate when q and s are near 1. This estimate can be used to prove L^p boundedness of T by extrapolation of Yano under $L \log L$ condition for Ω and a certain condition for h. To state the condition for h, we need to introduce two more function spaces. For h on $\mathbb{R}_+$ and $a > 0$, let

$$L_a(h) = \sup_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} |h(r)| \left(\log(2 + |h(r)|)\right)^a \, dr/r.$$

Define the class $\mathcal{L}_a$ to be the space of all functions h satisfying $L_a(h) < \infty$. Then, we see that if $a < b$, $\mathcal{L}_b \subset \mathcal{L}_a$ and that

$$\bigcup_{s>1} \Delta_s \subsetneq \bigcap_{a>0} \mathcal{L}_a.$$

Also, for h on $\mathbb{R}_+$ and $a > 0$, let

$$N_a(h) = \sum_{m \ge 1} m^a 2^m d_m(h), \quad d_m(h) = \sup_{k \in \mathbb{Z}} 2^{-k}|E(k, m)|,$$

where

$$E(k, m) = \{r \in (2^k, 2^{k+1}] : 2^{m-1} < |h(r)| \le 2^m\}$$

for $m \ge 2$ and

$$E(k, 1) = \{r \in (2^k, 2^{k+1}] : 0 < |h(r)| \le 2\}.$$

We denote by $\mathcal{N}_a$ the class of the functions h on $\mathbb{R}_+$ such that $N_a(h) < \infty$. Then, we can see that $N_a(h) < \infty$ implies $L_a(h) < \infty$. Conversely, if $L_{a+b}(h) < \infty$ for some $b > 1$, then $N_a(h) < \infty$.

Now, we can state an application of Theorem 1.

Theorem 2. *Let $\Omega \in L \log L(S^{n-1})$ and $h \in \mathcal{N}_1$. Then*

$$\|T(f)\|_{L^p} \le C_p\|f\|_{L^p}$$

for all $p \in (1, \infty)$.

Theorem 2 follows from Theorem 1 and Yano's extrapolation (see [26]). Al-Salman-Pan [2] proved L^p boundedness of T under the condition that $\Omega \in L \log L(S^{n-1})$ and $h \in \Delta_s$ for some $s > 1$. Theorem 2 improves that result by replacing the assumption on h with the condition $h \in \mathcal{N}_1$. Here we recall that the condition $h \in \mathcal{N}_1$ follows if $h \in \mathcal{L}_a$ for some $a > 2$.

Proofs of Theorems 1 and 2 can be found in [18]. To prove Theorem 1 we apply the method of J. Duoandikoetxea and J. L. Rubio de Francia [7] involving the Littlewood-Paley theory. A new element of our proof is to apply a Littlewood-Paley decomposition adapted to a suitable lacunary sequence depending on q and s for which $\Omega \in L^q(S^{n-1})$ and $h \in \Delta_s$. The method of appropriately choosing a lacunary sequence has been already used in a different way from ours by A. Al-Salman and Y. Pan [2]. In the following sections, we shall see that Theorem 1 can extend to several classes of singular integrals.

2. Singular Radon transforms

Let

$$T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - P(y))K(y)\, dy = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} f(x - P(y))K(y)\, dy,$$

where

$$K(y) = h(|y|)\Omega(y')|y|^{-n}, \quad y' = |y|^{-1}y,$$

$\Omega \in L^1(S^{n-1})$ satisfies (1.1), f is an appropriate function on $\mathbb{R}^d$ (d may be different from n) and $P(y) = (P_1(y), P_2(y), \dots, P_d(y))$ is a polynomial mapping (each P_j is a real-valued polynomial on $\mathbb{R}^n$).

We assume that $P(-y) = -P(y)$. Then we have the following theorem.

Theorem 3. *Suppose that $\Omega \in L^q(S^{n-1})$, $h \in \Delta_s$, $q, s \in (1, 2]$. Then*

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \le C_p(q-1)^{-1}(s-1)^{-1}\|\Omega\|_{L^q(S^{n-1})}\|h\|_{\Delta_s}\|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$, where the constant C_p is independent of q, s, Ω and h. Also, the constant C_p is independent of polynomials P_j if we fix $\deg(P_j)$ *($j = 1, 2, \dots, d$).*

By Theorem 3 and extrapolation we have the following result.

Theorem 4. *Let $\Omega \in L\log L(S^{n-1})$ and $h \in \mathcal{N}_1$. Then*

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \le C_p\|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$, where C_p is independent of polynomials P_j if the polynomials are of fixed degree.

Previous results are as follows.
(a) (E. M. Stein [23]) If $h = 1$ and $\Omega \in C^1(S^{n-1})$, then $T : L^p \to L^p$ for all $1 < p < \infty$.
(b) (D. Fan and Y. Pan [10]) Suppose that $\Omega \in H^1(S^{n-1})$ and $h \in \Delta_s$ for some $s > 1$. Then $T : L^p \to L^p$ if $|1/p - 1/2| < \min(1/2, 1/s')$.
(c) (A. Al-Salman and Y. Pan [2]) Suppose that $\Omega \in L\log L(S^{n-1})$, $h \in \Delta_s$ for some $s > 1$ and $P(-y) = -P(y)$. Then $T : L^p \to L^p$ for all $1 < p < \infty$.

Theorem 4 improves the result (c) by replacing the assumption on h with $h \in \mathcal{N}_1$. For Theorems 3 and 4, see [18]. Relevant results can be found in [14], [24].

3. Singular integrals associated with functions of finite type

We consider a singular Radon transform of the form:

$$T(f)(x) = \text{p.v.} \int_{B(0,1)} f(x - \Phi(y))K(y)\, dy,$$

where $K(y) = \Omega(y')|y|^{-n}$, $\Omega \in L^1(S^{n-1})$, $y' = |y|^{-1}y$, $\Phi : B(0,1) \to \mathbb{R}^d$ is a smooth function, $B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}$. We assume that Ω satisfies (1.1) and that Φ is of finite type at the origin, where Φ is said to be of finite type at the origin if for any $\xi \in S^{d-1}$ there exists a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $|\alpha| \geq 1$ and

$$\partial_x^\alpha \langle \Phi(x), \xi \rangle|_{x=0} \neq 0.$$

Then we have the following theorem.

Theorem 5. *Let $q \in (1, 2]$ and $\Omega \in L^q(S^{n-1})$. Then*

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \leq C_p(q-1)^{-1}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$, where the constant C_p is independent of q and Ω.

By Theorem 5 and extrapolation, we can give a different proof for the following result of Al-Salman-Pan [2]:

Theorem A. *Suppose that $\Omega \in L \log L(S^{n-1})$. Then*

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \leq C_p\|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$.

Relevant results are in [8]. See [20] for Theorem 5.

4. Singular integrals along surfaces of revolution

Let

$$\Gamma : [0, \infty) \to \mathbb{R}^m$$

be a continuous mapping satisfying $\Gamma(0) = 0$. We define a singular integral operator along the surface $(y, \Gamma(|y|))$ by

$$Tf(x, z) = \text{p.v.} \int_{\mathbb{R}^n} f(x - y, z - \Gamma(|y|))K(y)\, dy$$

where $K(y) = h(|y|)\Omega(y')|y|^{-n}$. We assume that $\Omega \in L \log L(S^{n-1})$ satisfies (1.1).

Let

$$M_\Gamma g(z) = \sup_{R>0} R^{-1} \int_0^R |g(z - \Gamma(t))|\, dt.$$

We assume that $M_\Gamma : L^p(\mathbb{R}^m) \to L^p(\mathbb{R}^m)$ for all $p > 1$. An example of such Γ is a polynomial mapping. Γ may have infinite order contact with its tangent at the origin. Under this condition on Γ, we have the following theorem.

Theorem 6. *Suppose that $\Omega \in L^q(S^{n-1})$ for some $q \in (1, 2]$ and $h \in \Delta_s$ for some $s > 1$. Then,*

$$\|Tf\|_{L^p(\mathbb{R}^{n+m})} \leq C_p(q-1)^{-1}\|\Omega\|_q\|h\|_{\Delta_s}\|f\|_{L^p(\mathbb{R}^{n+m})}$$

if $|1/p - 1/2| < \min(1/s', 1/2)$, where the constant C_p is independent of q and Ω.

An extrapolation implies the following.

Theorem 7. *Suppose* $\Omega \in L\log L(S^{n-1})$ *and* $h \in \Delta_s$ *for some* $s > 1$. *Then,* T *is bounded on* $L^p(\mathbb{R}^{n+m})$ *if* $|1/p - 1/2| < \min(1/s', 1/2)$.

When $m = 1$ and Γ is a C^2, convex, increasing function, Theorem 7 was proved by Al-Salman and Pan [2]. In that case, M_Γ is bounded on $L^p(\mathbb{R}^1)$ for all $p \in (1, \infty)$. See [19] for Theorems 6 and 7.

5. Littlewood-Paley functions

We consider the Littlewood-Paley function on $\mathbb{R}^n$ defined by

$$S_\psi(f)(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where ψ is in $L^1(\mathbb{R}^n)$, $\psi_t(x) = t^{-n}\psi(t^{-1}x)$. We assume that

$$\int_{\mathbb{R}^n} \psi(x)\,dx = 0.$$

One of the well-known sufficient conditions for L^p boundedness of S_ψ is the following:

Theorem B. Suppose that there exists $\epsilon > 0$ such that

$$|\psi(x)| \le C(1 + |x|)^{-n-\epsilon},$$

$$\int_{\mathbb{R}^n} |\psi(x-y) - \psi(x)|\,dx \le C|y|^\epsilon.$$

Then the operator S_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.

This is due to Benedek, Calderón and Panzone [3]. It is known that the second assumption in Theorem B is not needed (see [16] and also [9]):

Theorem C. Suppose that

$$|\psi(x)| \le C(1 + |x|)^{-n-\epsilon} \quad \text{for some } \epsilon > 0.$$

Then

$$S_\psi : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \quad \text{for all } p \in (1, \infty).$$

Let

$$\psi(x) = |x|^{-n+1}\Omega(x')\chi_{(0,1]}(|x|), \quad x' = x/|x|,$$

where $\Omega \in L^1(S^{n-1})$ satisfies (1.1) and χ_E denotes the characteristic function of E. Define

$$\mu_\Omega(f) = S_\psi(f).$$

Then, $\mu_\Omega(f)$ is called the Marcinkiewicz integral (see Stein [22] and also Hörmander [12]). T.Walsh [25] proved the following result.

Theorem D. If $\Omega \in L(\log L)^{1/2}(S^{n-1})$, then

$$\mu_\Omega : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

Here $\Omega \in L(\log L)^{1/2}(S^{n-1})$ means

$$\int_{S^{n-1}} |\Omega(\theta)| \left[\log(2 + |\Omega(\theta)|)\right]^{1/2} \, d\sigma(\theta) < \infty.$$

Al-Salman, Al-Qassem, Cheng and Pan [1] extended Theorem D to all L^p $(1 < p < \infty)$ spaces.

Theorem E. If $\Omega \in L(\log L)^{1/2}(S^{n-1})$, then

$$\mu_\Omega : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \quad \text{for all } p \in (1, \infty).$$

We can give a different proof of Theorem E by extrapolation and the following result (see [21]).

Theorem 8. *If $\Omega \in L^q(S^{n-1})$ for some $q \in (1, 2]$, we have*

$$\|\mu_\Omega(f)\|_p \le C_p (q-1)^{-1/2} \|\Omega\|_q \|f\|_p$$

for $p \in (1, \infty)$, where the constant C_p is independent of q and Ω.

REFERENCES

[1] A. Al-Salman, H. Al-Qassem, L. C. Cheng and Y. Pan, *L^p bounds for the function of Marcinkiewicz*, Math. Res. Lett. **9** (2002), 697–700.
[2] A. Al-Salman and Y. Pan, *Singular integrals with rough kernels in $L\log L(S^{n-1})$*, J. London Math. Soc. (2) **66** (2002), 153–174.
[3] A. Benedek, A. P. Calderón and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U. S. A. **48** (1962), 356–365.
[4] A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309.
[5] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.
[6] W. Connett, *Singular integrals near L^1*, Proc. Symp. Pure Math. **35**, Part 1 (1979), 163–165.
[7] J. Duoandikoetxea and J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. **84** (1986), 541–561.
[8] D. Fan, K. Guo and Y. Pan, *Singular integrals along submanifolds of finite type*, Michigan Math. J. **44** (1997), 135–142.
[9] D. Fan and S. Sato, *Remarks on Littlewood-Paley functions and singular integrals*, J. Math. Soc. Japan **54** (2002), 565–585.
[10] D. Fan and Y. Pan, *Singular integral operators with rough kernels supported by subvarieties*, Amer. J. Math. **119** (1997), 799–839.
[11] R. Fefferman, *A note on singular integrals*, Proc. Amer. Math. Soc. **74** (1979), 266–270.
[12] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–139.
[13] J. Namazi, *On a singular integral*, Proc. Amer. Math. Soc. **96** (1986), 421–424.
[14] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals, I*, J. Func. Anal. **73** (1987), 179–194.
[15] F. Ricci and G. Weiss, *A characterization of $H^1(\Sigma_{n-1})$*, Proc. Symp. Pure Math. **35**, Part 1 (1979), 289–294.
[16] S. Sato, Remarks on square functions in the Littlewood-Paley theory, Bull. Austral. Math. Soc. **58** (1998), 199–211.
[17] S. Sato, *Singular integrals and Littlewood-Paley functions*, Selected papers on differential equations and analysis, Translations. Series 2. **215** (2005), 57-78, American Mathematical Society, Providence, RI.
[18] S. Sato, *Estimates for singular integrals and extrapolation*, Studia Math. 192 (2009), 219–233, arXiv:0704.1537v1 [math.CA].

[19] S. Sato, *Estimates for singular integrals along surfaces of revolution*, to appear in J. Aust. Math. Soc.

[20] S. Sato, *Estimates for singular integrals associated with manifolds of finite type and extrapolation*, preprint.

[21] S. Sato, *Estimates for Littlewood-Paley functions and extrapolation*, Integr. equ. oper. theory **62** (2008), 429-440.

[22] E. M. Stein, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430–466.

[23] E. M. Stein, *Problems in harmonic analysis related to curvature and oscillatory integrals*, Proceedings of International Congress of Mathematicians, Berkeley (1986), 196–221.

[24] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993.

[25] T. Walsh, *On the function of Marcinkiewicz*, Studia Math. **44** (1972), 203–217.

[26] A. Zygmund, *Trigonometric Series*, 2nd ed., Cambridge Univ. Press, Cambridge, London, New York and Melbourne, 1977.

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