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B14 (2009), 163–173

WELL-POSEDNESS FOR QUADRATIC NONLINEAR SCHRÖDINGER EQUATIONS

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1. Introduction

This is a short review of the result obtained in the paper [6], which is joint work with Nobu Kishimoto.

We consider the Cauchy problem of quadratic nonlinear Schrödinger equations as follows;

$$\begin{cases} (i\partial_t - \partial_x^2)u = N(u), & (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

where unknown function u is complex valued and $N(u) = u^2$, $\bar{u}^2$ or $u\bar{u}$. Our aim is to prove the time local well-posedness of (1.1) with low regularity initial data.

We first assume that $u_0 \in H^s$ and recall the known results. Bourgain [2] introduced the Fourier restriction norm $X^{s,b}$ defined below to study the KdV equation and the nonlinear Schrödinger equation;

$$\|u\|_{X^{s,b}} = \|\langle\xi\rangle^s \langle\tau - \xi^2\rangle^b \widetilde{u}\|_{L^2_{\tau,\xi}},$$

where $\langle\cdot\rangle = 1 + |\cdot|$ and $\widetilde{u}$ is the Fourier transform of u with respect to t and x. Kenig, Ponce and Vega [4] developed this method and obtained the time local well-posedness of (1.1) with $N(u) = u^2, \bar{u}^2$ and $u\bar{u}$ for $s > -3/4, s > -3/4$ and $s > -1/4$, respectively. In the proof, the following bilinear estimate plays an important role;

$$\|N(u)\|_{X^{s,b-1}} \leq C\|u\|_{X^{s,b}}^2.$$

Nakanishi, Takaoka and Tsutsumi [7] proved the counter examples of this estimate with $N(u) = u^2, \bar{u}^2$ and $u\bar{u}$ for $s \leq -3/4, s \leq -3/4$ and $s \leq -1/4$, respectively. This means that we can not improve Kenig, Ponce and Vega's result with the standard Fourier restriction norm method. To overcome this difficulty, Bejenaru and Tao [1] introduced a modified Fourier restriction norm and used a support property of solutions of (1.1), namely, the support of $\widetilde{u}$ is in $\{(\tau,\xi) \in \mathbb{R}^2 | \tau \geq 0\}$ when $N(u) = u^2$ and u satisfies (1.1), to obtain the time local well-posedness of (1.1) with $N(u) = u^2$ for $s \geq -1$. When $N(u) = \bar{u}^2$, the problem is more complicated because this property does not hold. Nevertheless, Kishimoto [5], proved the the time local well-posedness of (1.1) with $N(u) = \bar{u}^2$ for $s \geq -1$ by using a modified Fourier restriction norm with complicated weight functions. The case $N(u) = u\bar{u}$ is totally different from the cases $N(u) = u^2$ or $\bar{u}^2$. For instance, the data-to-solution map $: u_0 \in H^s \to C([0,T] : H^s)$ fails to be C^2 when $s < -1/4$ and $N(u) = u\bar{u}$. This is caused by the Energy flow from high frequency parts to low frequency parts. To overcome this difficulty, we introduce the following function space and we assume $u_0 \in H^{s,a}$.

Put

$$H^{s,a} = \{f \in \mathcal{Z}'(\mathbb{R}) \mid \|f\|_{H^{s,a}} < \infty\},$$
$$\|f\|_{H^{s,a}} = \|\langle\xi\rangle^{s-a}|\xi|^a \widehat{f}\|_{L^2},$$

where $\mathcal{Z}'(\mathbb{R}^n)$ denotes the dual space of

$$\mathcal{Z}(\mathbb{R}^n) := \{f \in \mathcal{S}(\mathbb{R}^n) | D^\alpha \mathcal{F} f(0) = 0 \text{ for every multi-index } \alpha\}.$$

If we apply the standard Fourier restriction norm method to time local well-posedness of (1.1) with $N(u) = u\bar{u}$ in $H^{s,a}$, we need the following bilinear estimate with $b \geq 1/2$;

$$\|u\bar{u}\|_{X^{s,a,b-1}} \leq C\|u\|^2_{X^{s,a,b}} \tag{1.2}$$

where

$$\|u\|_{X^{s,a,b}} = \|\langle\xi\rangle^{s-a}|\xi|^a\langle\tau-\xi^2\rangle^b \widetilde{u}\|_{L^2_{\tau,\xi}}. \tag{1.3}$$

Put

$$\widetilde{u}_N(\tau,\xi) = \begin{cases} 1, & |\xi - N| < 1 \text{ and } |\tau-\xi^2| < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and let $N \in \mathbb{N}$ be sufficiently large. Then, we have

$$\widetilde{u_N\bar{u}_N}(\tau,\xi) = \widetilde{u}_N * \widetilde{\bar{u}}_N(\tau,\xi) \sim \psi_{R_0}(\tau,\xi) \tag{1.4}$$

where ψ_A denotes the characteristic function of the set A and R_0 is the rectangle of dimensions $N \times N^{-1}$ centered at the origin with longest side pointing in the $(1, 2N)$ direction. It follows that

$$R.H.S. \ of \ (1.2) \leq CN^{2s},$$
$$L.H.S. \ of \ (1.2) \geq \Big(\int_{1/2<|\xi|<1}\int \langle\tau-\xi^2\rangle^{2(b-1)}\psi_{R_0}(\tau,\xi)\,d\tau d\xi\Big)^{1/2} \geq cN^{b-1}.$$

Therefore, (1.2) fails for any $a \in \mathbb{R}$, $s < -1/4$ and $b \geq 1/2$.

To overcome this difficulty, we use the weight function defined in (2.1) instead of $\langle\xi\rangle^{s-a}|\xi|^a\langle\tau-\xi^2\rangle^b$ in (1.3) and introduce modified Fourier restriction norms $Z^{s,a}$ and $Y^{s,a}$ (see, Section 2) and prove new bilinear estimates (Proposition 3.1) to obtain the following time local well-posedness result.

Theorem 1.1. *Let $s \geq -(2a+1)/4$ and $1/2 > a > -1/2$. Then, (1.1) with $N(u) = u\bar{u}$ is time locally well-posed in $H^{s,a}$.*

Remark 1.2. Since $H^s \subset H^{s,a}$ when $a \geq 0$, we have the existence of the solution for $u_0 \in H^s$ with $s > -1/2$ by Theorem 1.1. However, the solution $u(t)$ is not in H^s for any $t > 0$ when $-1/4 > s > -1/2$.

In Section 2, we give some notations and preliminary lemmas. In Section 3, we prove the main estimates. The proof of Theorem 1.1 follows from a standard argument and these estimates (see, e.g. [5]). So, we omit the proof.

2. NOTATIONS AND PRELIMINARY LEMMAS

Throughout this paper $C > 0$ denotes various constants. The notation $P \lesssim Q$ denote the estimate $P \leq CQ$. We use $P \sim Q$ to denote $P \lesssim Q \lesssim P$.

Put

$$P_1 = \{(\tau, \xi) \in \mathbb{R}^2 \,|\, |\tau - \xi^2| \leq |\xi|/4 \textit{ and } |\xi| \geq 1\},$$
$$P_2 = \{(\tau, \xi) \in \mathbb{R}^2 \,|\, |\tau - \xi^2| \geq |\xi|/4 \textit{ or } |\xi| < 1\},$$

and

$$w_{s,a}(\tau, \xi) = \begin{cases} \langle \xi \rangle^s \langle \tau - \xi^2 \rangle, & (\tau, \xi) \in P_1, \\ \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s}, & (\tau, \xi) \in P_2. \end{cases} \tag{2.1}$$

Note that

$$w_{s,a}(\tau, \xi) \sim \min\{\langle \xi \rangle^{s-a} |\xi|^a \langle \tau - \xi^2 \rangle, \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s}\}.$$

We define function spaces $Z^{s,a}$ and $Y^{s,a}$ as follows;

$$Z^{s,a} = \{u \in \mathcal{Z}'(\mathbb{R}^2) \,|\, \|u\|_{Z^{s,a}} < \infty\},$$
$$Y^{s,a} = \{u \in \mathcal{Z}'(\mathbb{R}^2) \,|\, \|u\|_{Y^{s,a}} < \infty\},$$

where

$$\|u\|_{Z^{s,a}} = \|w_{s,a} \widetilde{u}\|_{L^2_{\tau,\xi}}, \quad \|u\|_{Y^{s,a}} = \| \int \langle \xi \rangle^{s-a} |\xi|^a \widetilde{u} \, d\tau \|_{L^2_\xi}.$$

Put

$$Q_1 = \{(\tau, \xi) \in \mathbb{R}^2 \,|\, |\tau + \xi^2| \leq |\xi|/4 \textit{ and } |\xi| \geq 1\},$$
$$Q_2 = \{(\tau, \xi) \in \mathbb{R}^2 \,|\, |\tau + \xi^2| \geq |\xi|/4 \textit{ or } |\xi| < 1\},$$
$$w'_{s,a}(\tau, \xi) = \begin{cases} \langle \xi \rangle^s \langle \tau + \xi^2 \rangle, & (\tau, \xi) \in Q_1, \\ \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau + \xi^2 \rangle^{1/2+s}, & (\tau, \xi) \in Q_2, \end{cases}$$

and

$$\|u\|_{\bar{Z}^{s,a}} = \|w'_{s,a} \widetilde{u}\|_{L^2_{\tau,\xi}}.$$

Note that $P_j(\tau, \xi) = Q_j(-\tau, -\xi)$ and $\|\bar{u}\|_{Z^{s,a}} = \|u\|_{\bar{Z}^{s,a}}$.

The following lemmas are basic tools of the Fourier restriction norm method.

Lemma 2.1. *Let $0 \leq p \leq q$ and $p + q > 1$. Then the following estimate holds for all $a, b \in \mathbb{R}$;*

$$\int \langle \tau - a \rangle^{-p} \langle \tau - b \rangle^{-q} \, d\tau \lesssim \langle a - b \rangle^{-r}$$

where $r = p - [1 - q]_+$. (We recall that $[\lambda]_+ = \lambda$ if $\lambda > 0$, $= \varepsilon > 0$ if $\lambda = 0$ and $= 0$ if $\lambda < 0$).

For the proof of this lemma, see Lemma 4.2 in [3].

For a subset $\Omega \subset \mathbb{R}^4$, we define the characteristic function χ_Ω as follows;

$$\chi_\Omega(\tau, \xi, \tau_1, \xi_1) = \begin{cases} 1, & \text{for } (\tau, \xi, \tau_1, \xi_1) \in \Omega \\ 0, & \text{for } (\tau, \xi, \tau_1, \xi_1) \notin \Omega \end{cases}$$

and put

$$\widetilde{B_\Omega(u,v)} := \int_{\mathbb{R}^2} \chi_\Omega \widetilde{u}(\tau-\tau_1, \xi-\xi_1)\widetilde{v}(\tau_1,\xi_1)\, d\tau_1 d\xi_1.$$

Lemma 2.2. *If*

$$\sup_{\tau,\xi} \int_{\mathbb{R}^2} \chi_\Omega w_1^{-2}(\tau,\xi) w_2^{-2}(\tau-\tau_1,\xi-\xi_1) w_3^{-2}(\tau_1,\xi_1)\, d\tau_1 d\xi_1 \lesssim 1$$

or

$$\sup_{\tau_1,\xi_1} \int_{\mathbb{R}^2} \chi_\Omega w_1^{-2}(\tau,\xi) w_2^{-2}(\tau-\tau_1,\xi-\xi_1) w_3^{-2}(\tau_1,\xi_1)\, d\tau d\xi \lesssim 1$$

hold for measurable functions w_1, w_2 and w_3 on $\mathbb{R}^2$, then we have

$$\|w_1^{-1}\widetilde{B_\Omega(u,v)}\|_{L^2_{\tau,\xi}} \lesssim \|w_2\widetilde{u}\|_{L^2_{\tau,\xi}} \|w_3\widetilde{v}\|_{L^2_{\tau,\xi}}.$$

Lemma 2.3. *If*

$$\sup_{\xi} \int_{\mathbb{R}^2} \chi_\Omega w_1^{-2}(\tau,\xi) w_2^{-2}(\tau-\tau_1,\xi-\xi_1) w_3^{-2}(\tau_1,\xi_1)\, d\tau_1 d\xi_1 d\tau \lesssim 1$$

or

$$\sup_{\xi_1} \int_{\mathbb{R}^2} \chi_\Omega w_1^{-2}(\tau,\xi) w_2^{-2}(\tau-\tau_1,\xi-\xi_1) w_3^{-2}(\tau_1,\xi_1)\, d\tau d\xi d\tau_1 \lesssim 1$$

hold for measurable functions w_1, w_2 and w_3 on $\mathbb{R}^2$, then we have

$$\| \int w_1^{-1}\widetilde{B_\Omega(u,v)}\, d\tau \|_{L^2_\xi} \lesssim \|w_2\widetilde{u}\|_{L^2_{\tau,\xi}} \|w_3\widetilde{v}\|_{L^2_{\tau,\xi}}.$$

For the proof of Lemmas 2.2, 2.3, see Section 3 in [3].

Let $\widehat{P_l f} = \widehat{f}|_{|\xi|<1}$ and $\langle\cdot,\cdot\rangle_{L^2}$ be the inner product in L^2. The following lemma is a variant of the Sobolev inequality.

Lemma 2.4. *(i) Let $b_1+b_2+b_3 > 1/2, b_1 \geq 0, b_2 \geq 0, b_3 \geq 0$ and $\alpha,\beta,\gamma \in \mathbb{R}$. Then, we have*

$$\langle fg, h\rangle_{L^2_x} \lesssim \|\langle\xi-\alpha\rangle^{b_1}\widehat{f}\|_{L^2_\xi} \|\langle\xi-\beta\rangle^{b_2}\widehat{g}\|_{L^2_\xi} \|\langle\xi-\gamma\rangle^{b_3}\widehat{h}\|_{L^2_\xi} \tag{2.2}$$

where implicit constant depends only on b_1, b_2 and b_3.

(ii) Let $s_1+s_2+s_3 > 1/2, s_1+s_2 \geq 0, s_2+s_3 \geq 0$ and $s_3+s_1 \geq 0$. Then, we have

$$\langle fg, h\rangle_{L^2_x} \lesssim \|\langle\xi\rangle^{s_1}\widehat{f}\|_{L^2_\xi} \|\langle\xi\rangle^{s_2}\widehat{g}\|_{L^2_\xi} \|\langle\xi\rangle^{s_3}\widehat{h}\|_{L^2_\xi} \tag{2.3}$$

where implicit constant depends only on s_1, s_2 and s_3.

(iii)Let $-1/2 < a < 1/2$. Then, we have

$$\langle (P_l f)g, h\rangle_{L^2_x} \lesssim \||\xi|^a\widehat{f}\|_{L^2_\xi} \|\widehat{g}\|_{L^2_\xi} \|\widehat{h}\|_{L^2_\xi}, \tag{2.4}$$

$$\langle (P_l f)(P_l g), h\rangle_{L^2_x} \lesssim \||\xi|^a\widehat{f}\|_{L^2_\xi} \||\xi|^{-a}\widehat{g}\|_{L^2_\xi} \|\widehat{h}\|_{L^2_\xi}, \tag{2.5}$$

$$\langle (P_l f)(P_l g), P_l h\rangle_{L^2_x} \lesssim \||\xi|^a\widehat{f}\|_{L^2_\xi} \||\xi|^{a}\widehat{g}\|_{L^2_\xi} \||\xi|^{-a}\widehat{h}\|_{L^2_\xi}, \tag{2.6}$$

where the implicit constants depend only on a.

Proof. By the Plancherel theorem, the Hölder inequality and the Young inequality, we have

$$\begin{aligned}\langle fg,h\rangle_{L^2_x} \sim \langle \widehat{f}*\widehat{g},\widehat{h}\rangle_{L^2_\xi} &\lesssim \|\widehat{f}\|_{L^{p_1}_\xi}\|\widehat{g}\|_{L^{p_2}_\xi}\|\widehat{h}\|_{L^{p_3}_\xi} \\ &\lesssim \|\langle\xi-\alpha\rangle^{-b_1}\|_{L^{q_1}_\xi}\|\langle\xi-\beta\rangle^{-b_2}\|_{L^{q_2}_\xi}\|\langle\xi-\gamma\rangle^{-b_3}\|_{L^{q_3}_\xi} \\ &\quad\times\|\langle\xi-\alpha\rangle^{b_1}\widehat{f}\|_{L^2_\xi}\|\langle\xi-\beta\rangle^{b_2}\widehat{g}\|_{L^2_\xi}\langle\xi-\gamma\rangle^{b_3}\widehat{h}\|_{L^2_\xi},\end{aligned}$$

for any $1 \le p_j \le 2$ and $2 \le q_j \le \infty$ satisfying $1/p_1+1/p_2+1/p_3 = 2$ and $1/q_j + 1/2 = 1/p_j$. Since $b_1+b_2+b_3 > 1/2$ and $1/q_1+1/q_2+1/q_3 = 1/2$, we can take q_j such that $q_j > 1/b_j$ for $b_j > 0$ and $q_j = \infty$ for $b_j = 0$. Thus, we obtain (2.2).

For the proof of (2.3), we can assume $s_1 \ge s_2 \ge s_3$ without loss of generality. Since the case $s_3 \ge 0$ follows from (2.2), we only need to show the case $s_2 \ge 0 > s_3$. By using the triangle inequality $\langle\xi\rangle \le \langle\xi_1\rangle + \langle\xi-\xi_1\rangle$ and the Plancherel theorem, we have

$$\begin{aligned}\langle fg,h\rangle_{L^2_x} \sim&\Big\langle \int \widehat{f}(\xi-\xi_1)\widehat{g}(\xi_1)\,d\xi_1, \widehat{h}(\xi)\Big\rangle_{L^2_\xi} \\ \lesssim&\Big\langle \int \widehat{f}(\xi-\xi_1)\langle\xi_1\rangle^{-s_3}\widehat{g}(\xi_1)\,d\xi_1, \langle\xi\rangle^{s_3}\widehat{h}(\xi)\Big\rangle_{L^2_\xi} \\ &+\Big\langle \int \langle\xi-\xi_1\rangle^{-s_3}\widehat{f}(\xi-\xi_1)\widehat{g}(\xi_1)\,d\xi_1, \langle\xi\rangle^{s_3}\widehat{h}(\xi)\Big\rangle_{L^2_\xi}.\end{aligned}$$

Therefore, this case also follows from (2.2).

By the Plancherel theorem, the Hölder inequality and the Young inequality, we have

$$\langle (P_lf)g,h\rangle_{L^2_x} \sim \langle \widehat{P_lf}*\widehat{g},\widehat{h}\rangle_{L^2_\xi} \lesssim \|\widehat{P_lf}\|_{L^1_\xi}\|\widehat{g}\|_{L^2_\xi}\|\widehat{h}\|_{L^2_\xi}.$$

Since $\|\widehat{P_lf}\|_{L^1_\xi} \le \||\xi|^{-a}\|_{L^2_\xi(-1,1)}\||\xi|^a\widehat{f}\|_{L^2_\xi} \lesssim \||\xi|^a\widehat{f}\|_{L^2_\xi}$, we obtain (2.4).

For the proof of (2.5), we can assume $a \ge 0$ without loss of generality. From (2.4), we have

$$\langle (P_lf)(P_lg),h\rangle_{L^2_x} \lesssim \||\xi|^a\widehat{f}\|_{L^2_\xi}\|\widehat{P_lg}\|_{L^2_\xi}\|\widehat{h}\|_{L^2_\xi}.$$

Since $\|\widehat{P_lg}\|_{L^2_\xi} \le \||\xi|^{-a}\widehat{g}\|_{L^2_\xi}$, we obtain (2.5).

For the proof of (2.6), we can assume $a \ge 0$ without loss of generality. From the Plancherel theorem, we have

$$\langle (P_lf)(P_lg),P_lh\rangle_{L^2_x} \sim \Big\langle \int \widehat{P_lf}(\xi-\xi_1)\widehat{P_lg}(\xi_1)\,d\xi_1, \widehat{P_lh}(\xi)\Big\rangle_{L^2_\xi}.$$

Since $\max\{|\xi-\xi_1|^a, |\xi_1|^a\} \gtrsim |\xi|^a$, (2.6) follows from (2.4). □

From this lemma, we obtain the following space time estimates.

Proposition 2.5. *Let $b_1+b_2+b_3 > 1/2, b_1 \ge 0, b_2 \ge 0, b_3 \ge 0$ and $i,j,k = 1$ or -1. (i) Moreover, we assume that $s_1+s_2+s_3 > 1/2, s_1+s_2 \ge 0, s_2+s_3 \ge 0$ and $s_3+s_1 \ge 0$. Then, we have*

$$\begin{aligned}&\langle fg,h\rangle_{L^2_{t,x}} \\ &\lesssim \|\langle\xi\rangle^{s_1}\langle\tau-i\xi^2\rangle^{b_1}\widetilde{f}\|_{L^2_{\tau,\xi}}\|\langle\xi\rangle^{s_2}\langle\tau-j\xi^2\rangle^{b_2}\widetilde{g}\|_{L^2_{\tau,\xi}}\|\langle\xi\rangle^{s_3}\langle\tau-k\xi^2\rangle^{b_3}\widetilde{h}\|_{L^2_{\tau,\xi}}.\end{aligned} \tag{2.7}$$

(ii) Moreover, we assume $-1/2 < a < 1/2$. Then, we have

$$(2.8)\qquad \begin{aligned}&\langle (P_l f)g, h\rangle_{L^2_{t,x}}\\ &\lesssim \||\xi|^a\langle \tau - i\xi^2\rangle^{b_1}\widetilde{f}\|_{L^2_{\tau,\xi}}\|\langle \tau - j\xi^2\rangle^{b_2}\widetilde{g}\|_{L^2_{\tau,\xi}}\|\langle \tau - k\xi^2\rangle^{b_3}\widetilde{h}\|_{L^2_{\tau,\xi}},\end{aligned}$$

$$(2.9)\qquad \begin{aligned}&\langle (P_l f)(P_l g), h\rangle_{L^2_{t,x}}\\ &\lesssim \||\xi|^a\langle \tau - i\xi^2\rangle^{b_1}\widetilde{f}\|_{L^2_{\tau,\xi}}\||\xi|^{-a}\langle \tau - j\xi^2\rangle^{b_2}\widetilde{g}\|_{L^2_{\tau,\xi}}\|\langle \tau - k\xi^2\rangle^{b_3}\widetilde{h}\|_{L^2_{\tau,\xi}},\end{aligned}$$

$$(2.10)\qquad \begin{aligned}&\langle (P_l f)(P_l g), P_l h\rangle_{L^2_{t,x}}\\ &\lesssim \||\xi|^a\langle \tau - i\xi^2\rangle^{b_1}\widetilde{f}\|_{L^2_{\tau,\xi}}\||\xi|^{a}\langle \tau - j\xi^2\rangle^{b_2}\widetilde{g}\|_{L^2_{\tau,\xi}}\||\xi|^{-a}\langle \tau - k\xi^2\rangle^{b_3}\widetilde{h}\|_{L^2_{\tau,\xi}}.\end{aligned}$$

Proof. Fix $\xi, \xi_1 \in \mathbb{R}$. Then, from (2.2), we have

$$\begin{aligned}&\int \widetilde{f}(\tau_1,\xi_1)\widetilde{g}(\tau-\tau_1,\xi-\xi_1)\widetilde{h}(\tau,\xi)\,d\tau_1 d\tau\\ \lesssim&\|\langle \cdot - i\xi_1^2\rangle^{b_1}\widetilde{f}(\cdot,\xi_1)\|_{L^2}\|\langle \cdot - j(\xi-\xi_1)^2\rangle^{b_2}\widetilde{g}(\cdot,\xi-\xi_1)\|_{L^2}\|\langle \cdot - k\xi^2\rangle^{b_3}\widetilde{h}(\cdot,\xi)\|_{L^2}\end{aligned}$$

where implicit constant does not depend on ξ, ξ_1. Therefore, the left-hand side of (2.7) is bounded by

$$\int \|\langle \cdot - i\xi_1^2\rangle^{b_1}\widetilde{f}(\cdot,\xi_1)\|_{L^2}\|\langle \cdot - j(\xi-\xi_1)^2\rangle^{b_2}\widetilde{g}(\cdot,\xi-\xi_1)\|_{L^2}\|\langle \cdot - k\xi^2\rangle^{b_3}\widetilde{h}(\cdot,\xi)\|_{L^2}\,d\xi_1 d\xi,$$

which is bounded by the right-hand side of (2.7) by (2.3). In the same manner, (2.8)–(2.10) follow from (2.2), (2.4)–(2.6). □

3. Bilinear estimates

Proposition 3.1. *Let $0 > s \geq -(2a+1)/4$ and $1/2 > a > -1/2$. Then the following estimates hold;*

$$(3.1)\qquad \|\mathcal{F}^{-1}\langle \tau - \xi^2\rangle^{-1}\widetilde{u\bar{v}}\|_{Z^{s,a}} \lesssim \|u\|_{Z^{s,a}}\|v\|_{Z^{s,a}},$$

$$(3.2)\qquad \|\mathcal{F}^{-1}\langle \tau - \xi^2\rangle^{-1}\widetilde{u\bar{v}}\|_{Y^{s,a}} \lesssim \|u\|_{Z^{s,a}}\|v\|_{Z^{s,a}}.$$

Moreover, the same estimates hold with $u\bar{v}$ replaced by uv or $\bar{u}\bar{v}$.

We prove only the case $u\bar{u}$ because the case uv and $\bar{u}\bar{v}$ are easier.

Proof. We first consider (3.1), which is equivalent to

$$\|\mathcal{F}^{-1}\langle \tau - \xi^2\rangle^{-1}\widetilde{uv}\|_{Z^{s,a}} \lesssim \|u\|_{Z^{s,a}}\|v\|_{\bar{Z}^{s,a}}.$$

Put

$$\Omega_{i,j,k} = \{(\tau,\xi,\tau_1,\xi_1) | (\tau,\xi) \in P_i, (\tau-\tau_1,\xi-\xi_1) \in P_j, (\tau_1,\xi_1) \in Q_k\}$$

for $i, j, k = 1$ or 2. Then, we have

$$B_{\mathbb{R}^4}(u,v) = \sum_{i,j,k} B_{\Omega_{i,j,k}}(u,v).$$

Therefore, we only need to show

$$(3.3)\qquad \|\mathcal{F}^{-1}\langle \tau - \xi^2\rangle^{-1}\widetilde{B_\Omega(u,v)}\|_{Z^{s,a}} \lesssim \|u\|_{Z^{s,a}}\|v\|_{\bar{Z}^{s,a}}$$

with $\Omega = \Omega_{i,j,k}$ for $i, j, k = 1$ or 2. Put $M_1 = \max\{|\tau-\xi^2|,\ |\tau-\tau_1-(\xi-\xi_1)^2|,\ |\tau_1+\xi_1^2|\}$. Then, we have the following algebraic property;

$$M_1 \geq (|\tau - \xi^2| + |\tau - \tau_1 - (\xi - \xi_1)^2| + |\tau_1 + \xi_1^2|)/3 \geq 2|\xi\xi_1|/3,$$

which plays an important role in our proof.

(a-1) We prove that $\Omega_{1,1,1}$ is empty. If $M_1 = |\tau - \xi^2|$ and $(\tau, \xi) \in P_1$, then $2|\xi\xi_1|/3 \leq M_1 \leq |\xi|/4$. Therefore, we have $|\xi_1| \leq 3/8$, which contradicts $(\tau_1, \xi_1) \in Q_1$. If $M_1 = |\tau_1 + \xi_1^2|$ and $(\tau_1, \xi_1) \in Q_1$, then $2|\xi\xi_1|/3 \leq M_1 \leq |\xi_1|/4$. Therefore, we have $|\xi| \leq 3/8$, which contradicts $(\tau, \xi) \in P_1$. If $M_1 = |\tau - \tau_1 + (\xi - \xi_1)^2|$ and $(\tau - \tau_1, \xi - \xi_1) \in P_1$, then $2|\xi\xi_1|/3 \leq M_1 \leq |\xi - \xi_1|/4 \leq \max\{|\xi|, |\xi_1|\}/2$. Therefore, we have $|\xi| \leq 3/4$ or $|\xi_1| \leq 3/4$, which contradicts $(\tau, \xi) \in P_1$ and $(\tau_1, \xi_1) \in Q_1$. Thus, we obtain (3.3) with $\Omega = \Omega_{1,1,1}$.

(a-2) (3.3) with $\Omega = \Omega_{2,1,1}$ is equivalent to

$$\begin{aligned} &\|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau - \xi^2\rangle^{-1/2+s}\widetilde{B_{\Omega_{2,1,1}}(u, v)}\|_{L^2_{\tau,\xi}} \\ &\lesssim \|\langle\xi\rangle^s\langle\tau - \xi^2\rangle\widetilde{u}\|_{L^2_{\tau,\xi}}\|\langle\xi\rangle^s\langle\tau + \xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}}. \end{aligned} \tag{3.4}$$

We devide $\Omega_{2,1,1}$ into two parts;

$$\begin{aligned} A_1 &= \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,1,1} |\, |\xi| < 1\}, \\ A_2 &= \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,1,1} |\, |\xi| \geq 1\}. \end{aligned}$$

From Lemma 2.2, (3.4) with $\Omega_{2,1,1}$ replaced by A_1 can be reduced to

$$\sup_{\tau_1,\xi_1} \int \frac{\chi_{A_1}\langle\xi_1\rangle^{-2s}|\xi|^{2a}\langle\xi - \xi_1\rangle^{-2s}}{\langle\tau_1 + \xi_1^2\rangle^2\langle\tau - \xi^2\rangle^{1-2s}\langle\tau - \tau_1 - (\xi - \xi_1)^2\rangle^2}\, d\tau d\xi \lesssim 1.$$

Since $\langle M_1\rangle \sim \langle\xi\xi_1\rangle$ and $\langle\xi_1\rangle \sim \langle\xi - \xi_1\rangle \sim |\xi_1|$, from Lemma 2.1, the left hand side is bounded by

$$\int \frac{|\xi|^{2a}|\xi_1|^{-4s}}{\langle M_1\rangle^{1-2s}}\, d\xi \lesssim \int \frac{|\xi\xi_1|^{-4s-1}}{\langle\xi\xi_1\rangle^{1-2s}}|\xi_1|\, d\xi \lesssim \int \frac{|p|^{-4s-1}}{\langle p\rangle^{1-2s}}\, dp \lesssim 1.$$

Here, we put $p = \xi\xi_1$ and used $2a \geq -4s - 1$ and $1 - 2s > -4s$.

From Lemma 2.2, (3.4) with $\Omega_{2,1,1}$ replaced by A_2 can be reduced to

$$\sup_{\tau,\xi} \int \frac{\chi_{A_2}|\xi|\langle\xi_1\rangle^{-2s}\langle\xi - \xi_1\rangle^{-2s}}{\langle\tau - \xi^2\rangle^{1-2s}\langle\tau_1 + \xi_1^2\rangle^2\langle\tau - \tau_1 - (\xi - \xi_1)^2\rangle^2}\, d\tau_1 d\xi_1 \lesssim 1.$$

In the same manner as (a-1), it follows that $M_1 = \langle\tau - \xi^2\rangle \sim \langle\xi\xi_1\rangle$ from $(\tau - \tau_1, \xi - \xi_1) \in P_1, (\tau_1, \xi_1) \in Q_1$ and $|\xi| \geq 1$. Therefore, from Lemma 2.1, the left hand side is bounded by

$$\int \frac{\langle\xi_1\rangle^{-2s}\langle\xi - \xi_1\rangle^{-2s}}{\langle\xi\xi_1\rangle^{1-2s}\langle\tau - \xi^2 + 2\xi\xi_1\rangle^2}|\xi|\, d\xi_1 \lesssim \int \frac{\langle p\rangle^{-4s}}{\langle p\rangle^{1-2s}\langle\tau - \xi^2 + 2p\rangle^2}\, dp \lesssim 1.$$

Here, we put $p = \xi\xi_1$ and used $1 - 2s \geq -4s$.

(a-3) (3.3) with $\Omega = \Omega_{1,2,1}$ is equivalent to

$$\|\langle\xi\rangle^s\widetilde{B_{\Omega_{1,2,1}}(u, v)}\|_{L^2_{\tau,\xi}} \lesssim \|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau - \xi^2\rangle^{1/2+s}\widetilde{u}\|_{L^2_{\tau,\xi}}\|\langle\xi\rangle^s\langle\tau + \xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}}. \tag{3.5}$$

We devide $\Omega_{1,2,1}$ into two parts;

$$A_1 = \{(\tau,\xi,\tau_1,\xi_1) \in \Omega_{1,2,1} | \, |\xi-\xi_1| < 1\},$$
$$A_2 = \{(\tau,\xi,\tau_1,\xi_1) \in \Omega_{1,2,1} | \, |\xi-\xi_1| \geq 1\}.$$

Since $\langle\xi\rangle \sim \langle\xi_1\rangle$ and $\langle\xi-\xi_1\rangle \sim 1$ in A_1, (3.5) with $\Omega_{1,2,1}$ replaced by A_1 can be reduced to

$$\|\widetilde{(P_l u)v}\|_{L^2_{\tau,\xi}} \lesssim \||\xi|^a \langle\tau-\xi^2\rangle^{1/2+s}\widetilde{u}\|_{L^2_{\tau,\xi}} \|\langle\tau+\xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}},$$

which follows from the duality argument and (2.8) in Proposition 2.5. Since $\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle \gtrsim \langle\xi-\xi_1\rangle$ in A_2, (3.5) with $\Omega_{1,2,1}$ replaced by A_2 can be reduced to

$$\|\langle\xi\rangle^s\widetilde{uv}\|_{L^2_{\tau,\xi}} \lesssim \|\langle\xi\rangle^{1+s}\widetilde{u}\|_{L^2_{\tau,\xi}} \|\langle\xi\rangle^s\langle\tau+\xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}},$$

which follows from the duality argument and (2.7) in Proposition 2.5.

(a-4) (3.3) with $\Omega = \Omega_{2,2,1}$ is equivalent to

$$\begin{aligned} &\|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau-\xi^2\rangle^{-1/2+s}\widetilde{B_{\Omega_{2,2,1}}(u,v)}\|_{L^2_{\tau,\xi}} \\ &\lesssim \|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau-\xi^2\rangle^{1/2+s}\widetilde{u}\|_{L^2_{\tau,\xi}} \|\langle\xi\rangle^s\langle\tau+\xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}}. \end{aligned} \tag{3.6}$$

We devide $\Omega_{2,2,1}$ into four parts;

$$A_1 = \{(\tau,\xi,\tau_1,\xi_1) \in \Omega_{2,2,1} | \, |\xi| < 1, |\xi-\xi_1| < 1\},$$
$$A_2 = \{(\tau,\xi,\tau_1,\xi_1) \in \Omega_{2,2,1} | \, |\xi| < 1, |\xi-\xi_1| \geq 1\},$$
$$A_3 = \{(\tau,\xi,\tau_1,\xi_1) \in \Omega_{2,2,1} | \, |\xi| \geq 1, |\xi-\xi_1| < 1\},$$
$$A_4 = \{(\tau,\xi,\tau_1,\xi_1) \in \Omega_{2,2,1} | \, |\xi| \geq 1, |\xi-\xi_1| \geq 1\}.$$

Since $\langle\xi\rangle \sim \langle\xi_1\rangle \sim \langle\xi-\xi_1\rangle \sim 1$ in A_1, (3.6) with $\Omega_{2,2,1}$ replaced by A_1 can be reduced to

$$\||\xi|^a\langle\tau-\xi^2\rangle^{-1/2+s}P_l\widetilde{\{(P_l u)v\}}\|_{L^2_{\tau,\xi}} \lesssim \||\xi|^a\langle\tau-\xi^2\rangle^{1/2+s}\widetilde{u}\|_{L^2_{\tau,\xi}} \|\langle\tau+\xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}},$$

which follows from the duality argument and (2.9) in Proposition 2.5.

Since $\langle\xi\rangle \sim 1$ and $\langle\xi_1\rangle \sim \langle\xi-\xi_1\rangle$ in A_2, (3.6) with $\Omega_{2,2,1}$ replaced by A_2 can be reduced to

$$\||\xi|^a\langle\tau-\xi^2\rangle^{-1/2+s}\widetilde{P_l(uv)}\|_{L^2_{\tau,\xi}} \lesssim \|\langle\xi\rangle^{1/2+s}\langle\tau-\xi^2\rangle^{1/2+s}\widetilde{u}\|_{L^2_{\tau,\xi}} \|\langle\tau+\xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}},$$

which follows from (2.8) in Proposition 2.5.

Since $|\tau-\xi^2| \sim |\xi| \sim |\xi_1| \gtrsim 1$, $\langle\xi-\xi_1\rangle \sim 1$ in A_3, (3.6) with $\Omega_{2,2,1}$ replaced by A_3 can be reduced to

$$\|\widetilde{uv}\|_{L^2_{\tau,\xi}} \lesssim \||\xi|^a\langle\tau-\xi^2\rangle^{1/2+s}\widetilde{u}\|_{L^2_{\tau,\xi}} \|\langle\tau+\xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}},$$

which follows from (2.8) in Proposition 2.5.

Since $|\xi| \sim |\xi_1| \gtrsim 1$, $\langle\tau-\xi^2\rangle \gtrsim \langle\xi\rangle$ and $\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle \gtrsim \langle\xi-\xi_1\rangle$ in A_4, (3.6) with $\Omega_{2,2,1}$ replaced by A_4 can be reduced to

$$\|\langle\xi\rangle^s\widetilde{uv}\|_{L^2_{\tau,\xi}} \lesssim \|\langle\xi\rangle^{1+s}\widetilde{u}\|_{L^2_{\tau,\xi}} \|\langle\xi\rangle^s\langle\tau+\xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}},$$

which follows from (2.7) in Proposition 2.5.

(a-5) We can prove (3.3) with $\Omega = \Omega_{1,1,2}$ in the same manner as (a-3).

(a-6) We can prove (3.3) with $\Omega = \Omega_{2,1,2}$ in the same manner as (a-4).

(a-7) Since $\langle\xi\rangle^s \le \langle\tau-\xi^2\rangle^s$ in $\Omega_{1,2,2}$, (3.3) with $\Omega=\Omega_{1,2,2}$ can be reduced to

$$\begin{aligned}(3.7)\qquad &\|\langle\tau-\xi^2\rangle^s \widetilde{B_{\Omega_{1,2,2}}(u,v)}\|_{L^2_{\tau,\xi}}\\ &\lesssim \|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau-\xi^2\rangle^{1/2+s}\tilde{u}\|_{L^2_{\tau,\xi}}\|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau+\xi^2\rangle^{1/2+s}\tilde{v}\|_{L^2_{\tau,\xi}}.\end{aligned}$$

We devide $\Omega_{1,2,2}$ into three parts;

$$\begin{aligned}A_1&=\{(\tau,\xi,\tau_1,\xi_1)\in\Omega_{1,2,2}|\,|\xi-\xi_1|<1/2,|\xi_1|\ge 1/2\},\\ A_2&=\{(\tau,\xi,\tau_1,\xi_1)\in\Omega_{1,2,2}|\,|\xi-\xi_1|\ge 1/2,|\xi|<1/2\},\\ A_3&=\{(\tau,\xi,\tau_1,\xi_1)\in\Omega_{1,2,2}|\,|\xi-\xi_1|\ge 1/2,|\xi|\ge 1/2\}.\end{aligned}$$

(3.7) with $\Omega_{1,2,2}$ replaced by A_1 or A_2 follow from (2.8) in Proposition 2.5 and (3.7) with $\Omega_{1,2,2}$ replaced by A_3 follows from (2.7) in Proposition 2.5.

(a-8) (3.3) with $\Omega=\Omega_{2,2,2}$ is equivalent to

$$\begin{aligned}(3.8)\qquad &\|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau-\xi^2\rangle^{-1/2+s}\widetilde{B_{\Omega_{2,2,2}}(u,v)}\|_{L^2_{\tau,\xi}}\\ &\lesssim \|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau-\xi^2\rangle^{1/2+s}\tilde{u}\|_{L^2_{\tau,\xi}}\|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau+\xi^2\rangle^{1/2+s}\tilde{v}\|_{L^2_{\tau,\xi}}.\end{aligned}$$

We devide $\Omega_{2,2,2}$ into seven parts;

$$\begin{aligned}A_1&=\{(\tau,\xi,\tau_1,\xi_1)\in\Omega_{2,2,2}|\,|\xi|<1,|\xi-\xi_1|<1/2,|\xi_1|<1/2\},\\ A_2&=\{(\tau,\xi,\tau_1,\xi_1)\in\Omega_{2,2,2}|\,|\xi|<1,|\xi-\xi_1|\ge 1/2,|\xi_1|<1/2\},\\ A_3&=\{(\tau,\xi,\tau_1,\xi_1)\in\Omega_{2,2,2}|\,|\xi|<1,|\xi-\xi_1|<1/2,|\xi_1|\ge 1/2\},\\ A_4&=\{(\tau,\xi,\tau_1,\xi_1)\in\Omega_{2,2,2}|\,|\xi|<1,|\xi-\xi_1|\ge 1/2,|\xi_1|\ge 1/2\},\\ A_5&=\{(\tau,\xi,\tau_1,\xi_1)\in\Omega_{2,2,2}|\,|\xi|\ge 1,|\xi-\xi_1|\ge 1/2,|\xi_1|<1/2\},\\ A_6&=\{(\tau,\xi,\tau_1,\xi_1)\in\Omega_{2,2,2}|\,|\xi|\ge 1,|\xi-\xi_1|<1/2,|\xi_1|\ge 1/2\},\\ A_7&=\{(\tau,\xi,\tau_1,\xi_1)\in\Omega_{2,2,2}|\,|\xi|\ge 1,|\xi-\xi_1|\ge 1/2,|\xi_1|\ge 1/2\}.\end{aligned}$$

(3.8) with $\Omega_{2,2,2}$ replaced by A_1 follows from (2.10) in Proposition 2.5, (3.8) with $\Omega_{2,2,2}$ replaced by A_2 or A_3 follow from (2.9) in Proposition 2.5 and (3.8) with $\Omega_{2,2,2}$ replaced by A_4 or A_5 or A_6 follow from (2.8) in Proposition 2.5. Since $\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau-\xi^2\rangle^{-1/2+s} \le \langle\tau-\xi^2\rangle^s$ in A_7, (3.8) with $\Omega_{2,2,2}$ replaced by A_7 can be reduced to

$$\|\langle\tau-\xi^2\rangle^s\widetilde{uv}\|_{L^2_{\tau,\xi}} \lesssim \|\langle\xi\rangle^{1/2}\langle\tau-\xi^2\rangle^{1/2+s}\tilde{u}\|_{L^2_{\tau,\xi}}\|\langle\xi\rangle^{1/2}\langle\tau+\xi^2\rangle^{1/2+s}\tilde{v}\|_{L^2_{\tau,\xi}},$$

which follows from (2.7) in Proposition 2.5.

We next consider (3.2), which is equivalent to

$$\|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{uv}\|_{Y^{s,a}} \lesssim \|u\|_{Z^{s,a}}\|v\|_{\bar{Z}^{s,a}}$$

Because

$$\|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{B_\Omega(u,v)}\|_{Y^{s,a}} \lesssim \|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{B_{\Omega_{1,j,k}}(u,v)}\|_{X^{s,a}}$$

for $\Omega=\Omega_{1,j,k}$ with $j,k=1$ or 2, we only need to show

$$(3.9)\qquad \|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{B_\Omega(u,v)}\|_{Y^{s,a}} \lesssim \|u\|_{Z^{s,a}}\|v\|_{\bar{Z}^{s,a}}$$

for $\Omega=\Omega_{2,j,k}$ with $j,k=1$ or 2.

(b-1) We devide $\Omega_{2,1,1}$ into two parts;

$$A_1 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,1,1} | \, |\xi| < 1\},$$
$$A_2 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,1,1} | \, |\xi| \geq 1\}.$$

Since $\langle \xi_1 \rangle \sim \langle \xi - \xi_1 \rangle \sim |\xi_1| \gtrsim 1$ in A_1, from Lemma 2.3, (3.9) with $\Omega = A_1$ can be reduced to

$$\sup_{\xi_1} \int_{A_1} \frac{|\xi|^{2a}|\xi_1|^{-4s}}{\langle \tau - \xi^2 \rangle^2 \langle \tau_1 + \xi_1^2 \rangle^2 \langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^2} \, d\tau_1 d\tau d\xi \lesssim 1.$$

Since $\langle M_1 \rangle \sim \langle \xi \xi_1 \rangle$, from Lemma 2.1, the left hand side is bounded by

$$\int \frac{|\xi|^{2a}|\xi_1|^{-4s}}{\langle M_1 \rangle^2} \, d\xi \lesssim \int \frac{|\xi\xi_1|^{-4s-1}}{\langle \xi\xi_1 \rangle^2} \frac{1}{|\xi_1|} \, d\xi \lesssim \int \frac{|p|^{-4s-1}}{\langle p \rangle^2} \, dp \lesssim 1.$$

Here, we put $p = \xi\xi_1$ and used $2a \geq -4s - 1$ and $2 > -4s$.

From Lemma 2.3, (3.9) with $\Omega = A_2$ can be reduced to

$$\sup_{\xi_1} \int_{A_2} \frac{\langle \xi_1 \rangle^{-2s} \langle \xi \rangle^{2s} \langle \xi - \xi_1 \rangle^{-2s}}{\langle \tau - \xi^2 \rangle^2 \langle \tau_1 + \xi_1^2 \rangle^2 \langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^2} \, d\tau_1 d\xi d\tau \lesssim 1.$$

Since $\langle M_1 \rangle \sim \langle \xi \xi_1 \rangle$, from Lemma 2.1, the left hand side is bounded by

$$\int \frac{\langle \xi_1 \rangle^{-2s} \langle \xi \rangle^{2s} (\langle \xi_1 \rangle^{-2s} + \langle \xi \rangle^{-2s})}{\langle \xi\xi_1 \rangle^2} \, d\xi \lesssim \int \frac{\langle \xi_1 \rangle^{-4s} \langle \xi \rangle^{2s}}{\langle \xi\xi_1 \rangle^2} \, d\xi + \int \frac{\langle \xi_1 \rangle^{-2s}}{\langle \xi\xi_1 \rangle^2} \, d\xi \lesssim 1.$$

(b-2) (3.9) with $\Omega = \Omega_{2,2,1}$ is equivalent to

$$\begin{aligned} &\| \int \langle \xi \rangle^{s-a} |\xi|^a \langle \tau - \xi^2 \rangle^{-1} \widetilde{B_{\Omega_{2,2,1}}(u, v)} \, d\tau \|_{L^2_\xi} \\ &\lesssim \| \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s} \widetilde{u} \|_{L^2_{\tau,\xi}} \| \langle \xi \rangle^s \langle \tau + \xi^2 \rangle \widetilde{v} \|_{L^2_{\tau,\xi}}, \end{aligned}$$

which follows from Proposition 2.5 in the same manner as (a-4) because the left-hand side is bounded by

$$\| \langle \xi \rangle^{s-a} |\xi|^a \langle \tau - \xi^2 \rangle^{-1/2+\varepsilon} \widetilde{B_{\Omega_{2,2,1}}(u, v)} \|_{L^2_{\tau,\xi}}$$

for any $\varepsilon > 0$.

(b-3) For $\Omega = \Omega_{2,1,2}$, we can prove the estimate in the same manner as (b-2).

(b-4) For $\Omega = \Omega_{2,2,2}$, we only need to show

$$\begin{aligned} &\| \int \langle \xi \rangle^{s-a} |\xi|^a \langle \tau - \xi^2 \rangle^{-1} \widetilde{B_{\Omega_{2,2,2}}(u, v)} \, d\tau \|_{L^2_\xi} \\ &\lesssim \| \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s} \widetilde{u} \|_{L^2_\xi} \| \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau + \xi^2 \rangle^{1/2+s} \widetilde{v} \|_{L^2_\xi}, \end{aligned}$$

which follows from Proposition 2.5 in the same manner as (a-8) because the left-hand side is bounded by

$$\| \langle \xi \rangle^{s-a} |\xi|^a \langle \tau - \xi^2 \rangle^{-1/2+\varepsilon} \widetilde{B_{\Omega_{2,2,2}}(u, v)} \|_{L^2_{\tau,\xi}}$$

for any $\varepsilon > 0$.

$\square$

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