RIMS Kôkyûroku Bessatsu
B15 (2009), 15–37

Higher-order asymptotic expansions of solutions to a parabolic system of chemotaxis in $\mathbb{R}^n$

By

Tetsuya Yamada *

Abstract

We consider the initial value problem for a system of chemotaxis, and give the higher-order asymptotic expansions of bounded solutions.

§ 1. Introduction

In this article we study the initial value problem of the following system:

$$\text{(KS)} \qquad \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \mathbb{R}^n,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $n \geq 1$ and $u_0, v_0, \partial_j v_0 \in L^1(\mathbb{R}^n) \cap \mathcal{B}(\mathbb{R}^n)\,(1 \leq j \leq n)$. Here and below $\mathcal{B}(\mathbb{R}^n)$ stands for the Banach space of all bounded and uniformly continuous functions on $\mathbb{R}^n$ with the supremum norm.

This system is the mathematical model introduced by Keller-Segel [8], which is describing aggregation phenomena of organisms due to chemotaxis, that is, the directed movement of an organism in response to gradients of a chemical attractant. The function $u(x,t)$ corresponds to the population of the organism at place $x \in \mathbb{R}^n$ and time $t > 0$, and $v(x,t)$ the concentration of the chemical.

For (KS), it is well known that nonnegative solutions are global in time and bounded when $n = 1$, but can blow up in finite time when $n \geq 2$. For example, we refer to

Received Feburary 4, 2009. Revised July 15, 2009. Accepted July 27, 2009.
2000 Mathematics Subject Classification(s): Primary 35K45; Secondary 35B40, 35C20, 92C17
*Department of Mathematics, Graduate School of Science
Hiroshima University, Higashi-Hiroshima, 739-8526, JAPAN
e-mail: `tetsuya-yamada@hiroshima-u.ac.jp`

Childress-Purcus [1], Herrero-Velázquez [4], Horstmann [5], Jäger-Luckhaus [6], Nagai-Senba-Yoshida [12] and reference therein.

As one of further study on (KS), it is an very interesting problem to deduce the asymptotic profiles of bounded solutions to (KS) as $t \to \infty$. Let us recall the previous results for such works. It was shown in Nagai-Yamada [14] (see also Nagai-Syukuinn-Umesako [13]) that every bounded solution (u, v) to (KS) decays to zero as $t \to \infty$ and approaches asymptotically the heat kernel with the self-similarity. These asymptotic profiles in $L^q(\mathbb{R}^n)$ $(1 \leq q \leq \infty)$ space decay at the rate $t^{-n(1-1/q)/2-1/2}$ as $t \to \infty$ if $n \geq 2$, but at the rate $t^{-(1-1/q)/2-1/2} \log t$ as $t \to \infty$ if $n = 1$. The reason is that the L^q estimates of the solution for $n = 1$ might not decay faster than those for $n \geq 2$. It was proved in Kato [7] that under the assumption of Nagai-Yamada [14], the asymptotic rates in $L^q(\mathbb{R}^n)$ $(1 \leq q \leq \infty)$ space for $n = 1$ are improved to the rate $t^{-(1-1/q)/2-1/2}$ by introducing a correction term, and obtained the precise asymptotic expansion of the solutions to (KS). For further study of the asymptotic profile of bounded solutions to (KS) in the case $n = 1$, adjusting the center of the heat kernel by use of a shift which are suitably determined by the initial data and the nonlinear term, Nishihara [15] obtained the decay estimates of difference between the solution and the heat kernel whose center is adjusted. The decay estimates in this result are rather sharp, though he imposes stronger assumptions on the initial date u_0 than ones in Kato [7].

For the higher-order asymptotic expansions in higher-dimensional case, Yamada [18] introduced the correction term $R(x, t)$ mentioned in the result of Kato [7] for the case $n = 1$, and gave the higher-order asymptotic expansions up to n-th order for the solutions to (KS) with certain space-time decay properties. To state this result in more detail, we denote the heat kernel by

$$G(x, t) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)} \tag{1.1}$$

and use the notations M_α, $E_{\alpha,p}$ and $F_{\alpha,p}$ given by

$$M_\alpha = \int_{\mathbb{R}^n} y^\alpha u_0 \, dy, \tag{1.2}$$

$$E_{\alpha,p} = \int_0^\infty \int_{\mathbb{R}^n} y^\alpha (1+s)^p u \nabla v \, dyds, \tag{1.3}$$

$$F_{\alpha,p} = \int_0^\infty \int_{\mathbb{R}^n} y^\alpha (1+s)^p \{(u\nabla v)(s) - M_0^2 (G\nabla G)(1+s)\} \, dyds, \tag{1.4}$$

respectively. Also, we introduce the correction term $R(x, t)$ appearing in the higher-order asymptotic expansions as

$$
R(x,t) = \begin{cases} W(x,t) - P_0(x,t)\log(1+t) & \text{if } n = 2, \\ W(x,t) - \displaystyle\sum_{i=0}^{(n-3)/2} \frac{2}{n-2-2i} P_i(x,t) & \text{if } n \text{ is odd with } n \geq 3, \\ W(x,t) - \displaystyle\sum_{i=0}^{(n-4)/2} \frac{2}{n-2-2i} P_i(x,t) - P_{(n-2)/2}(x,t)\log(1+t) & \\ & \text{if } n \text{ is even with } n \geq 4, \end{cases} \tag{1.5}
$$

where

$$
W(x,t) = M_0^2 \int_0^t \int_{\mathbb{R}^n} \nabla G(x-y,t-s) \cdot (G\nabla G)(y,1+s)\, dyds, \tag{1.6}
$$

$$
P_i(x,t) = \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{|\alpha|+p=i} \frac{(-1)^p}{2^{|\alpha|}\alpha! p!} \Delta \partial_x^{2\alpha} \partial_t^p G(x,1+t). \tag{1.7}
$$

Here we note that this correction term given by (1.5) is estimated as follows (see the assertion (ii) of Lemma 5 in Yamada [18]):

$$
\|R(t)\|_{L^q} \leq C M_0^2 (1+t)^{-n(1-1/q)/2-n/2} \quad (t > 0,\ 1 \leq q \leq \infty). \tag{1.8}
$$

Theorem 1.1 ([18]). *Let $n \geq 2$, $1 \leq q \leq \infty$ and (u,v) be the solution to* (KS) *on $\mathbb{R}^n \times [0,\infty)$ satisfying*

$$
\sup_{\substack{x\in\mathbb{R}^n,\, t>0 \\ 0\leq\mu\leq n}} (1+|x|)^{n-\mu}(1+t)^{\mu/2}(|u(x,t)| + |v(x,t)|) < \infty. \tag{1.9}
$$

Then, under the conditions $|x|^n u_0 \in L^1(\mathbb{R}^n)$ and $\sup_{x\in\mathbb{R}^n}(1+|x|)^n(|u_0(x)| + |v_0(x)|) < \infty$, the following assertions hold:

(i) If n is odd, then the integral $E_{\alpha,p}$ given by (1.3) *converges for $|\alpha| + 2p \leq n-1$ and*

$$
\lim_{t\to\infty} t^{n(1-1/q)/2+n/2} \Bigg\| u(t) - \sum_{|\alpha|+2p\leq n} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \partial_x^\alpha \partial_t^p G(1+t) M_\alpha + \sum_{|\alpha|+2p\leq n-1} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot E_{\alpha,p} + R(t) \Bigg\|_{L^q} = 0. \tag{1.10}
$$

(ii) If n is even, then the integral $E_{\alpha,p}$ given by (1.3) *converges for $|\alpha| + 2p \leq n-2$ and*

the one $F_{\alpha,p}$ *given by* (1.4) *is well-deified for* $|\alpha|+2p=n-1$, *and*

$$\begin{aligned}(1.11)\qquad \lim_{t\to\infty} t^{n(1-1/q)/2+n/2}\Bigg\| u(t) &- \sum_{|\alpha|+2p\le n} \frac{(-1)^{|\alpha|+p}}{\alpha!p!}\partial_x^\alpha\partial_t^p G(1+t)M_\alpha \\ &+ \sum_{|\alpha|+2p\le n-2} \frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot E_{\alpha,p} + P_{(n-2)/2}(t)\log(1+t) \\ &+ \sum_{|\alpha|+2p=n-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot F_{\alpha,p} + R(t)\Bigg\|_{L^q} = 0,\end{aligned}$$

where $P_{(n-2)/2}(t)$ *is the one given by* (1.7) *with* $i=(n-2)/2$.

(*iii*) v *also has the same asymptotic profile as* u.

Remark. (i) For small initial data, there exists a unique solution to (KS) with (1.9) by using the contraction mapping principle (see Theorem 3 in Yamada [18]).

(ii) In the case $n=2$, the following lower bounds of decay rates for $R(t)$ defined by (1.5) is shown

$$\|R(t)\|_{L^\infty} \ge cM_0^2(1+t)^{-2} \quad \text{for} \quad t\ge 2. \tag{1.12}$$

For the proof of (1.12), see the assertion (iii) of Lemma 5 in Yamada [18].

Now we refer to several works, closely related to (KS). For the drift-diffusion system related to (KS), it was shown in Kobayashi-Kawashima [9] that in the case $n\ge 3$, the solutions converge to the heat kernel as $t\to\infty$, and the asymptotic rates in $L^q(\mathbb{R}^n)\,(1\le q\le\infty)$ space are $t^{-n(1-1/q)/2-1/2}$ if $n\ge 4$, and $t^{-3(1-1/q)/2-1/2}\log t$ if $n=3$. Ogawa-Yamamoto [16] showed that the asymptotic rates in $L^q(\mathbb{R}^n)\,(1\le q\le\infty)$ space for $n=3$ given in Kobayashi-Kawashima [9] is improved to the rate $t^{-3(1-1/q)/2-1/2}$ by introducing a correction term on the basis of the argument used in Kato [7], and obtained the precise asymptotic expansions up to second order for the solutions. The situation is rather similar to Theorem 1.1 above.

On the other hand, Fujigaki-Miyakawa [2] proved that under small initial data, a solution of the incompressible Navier-Stokes equations with the decay properties like (1.9) admits the higher-order asymptotic expansion in terms of the space-time derivatives of Gaussian-like functions. The solution treated in Fujigaki-Miyakawa [2] decays sufficiently faster because for the initial data in $L^1(\mathbb{R}^n)$, the average of initial data is naturally zero by use of the divergence free condition for the initial data (see Miyakawa [11]). Thus, the correction term does not appear in the asymptotic expansion in contrast to (KS) and the drift-diffusion system.

The aim of this article is to show that in the case $n\ge 2$, bounded solutions to (KS) admit the same higher-order asymptotic expansions as the solutions treated in Theorem 1.1. Our result is the following.

Theorem 1.2. *Assume $n \geq 2$, $1 \leq q \leq \infty$, and let (u, v) be the solution to* (KS) *on $\mathbb{R}^n \times [0, \infty)$ satisfying*

$$\sup_{t>0} (\|u(t)\|_{L^r} + \|v(t)\|_{L^r}) < \infty \quad \textit{for} \quad r = 1, \infty. \tag{1.13}$$

Then, under the condition $|x|^n u_0 \in L^1(\mathbb{R}^n)$, the assertions of Theorem 1.1 hold.

Remark. (i) In the case $n \geq 2$, the existence of bounded solutions to (KS) is assured if $\|u_0\|_{L^1}$, $\|\nabla v_0\|_{L^1}$, $\|\nabla v_0\|_{L^\infty}$ are small enough, but $\|u_0\|_{L^\infty}$ not necessary small (see Nagai-Syukuinn-Umesako [13]).

The proof of Theorem 1.2 can be basically obtained by applying techniques used in Theorem 1.1. By the definition of solutions to (KS) (see Definition 2.1), we can write u as

$$u(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \nabla \cdot (u \nabla v)(s)\, ds =: I_1(t) - I_2(t),$$

where $e^{t\Delta}$ is the semigroup associated with the heat equation. Under the suitable moment condition, the asymptotic expansion of $I_1(t)$ is obtained as follows:

$$I_1(t) = \sum_{|\alpha|+2p \leq n} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \partial_x^\alpha \partial_t^p G(1+t) M_\alpha + o(t^{-n(1-1/q)/2-n/2}) \quad (t \to \infty), \tag{1.14}$$

where M_α is the one given by (1.2) (for example, see Fujigaki-Miyakawa [2]), and is not depending on whether n is odd or even. Thus, as mentioned in Theorem 1.1, it's believed that the difference of asymptotic expansions in the odd and even dimensional cases is caused by $I_2(t)$.

Hereafter, let us consider only the asymptotic expansion of $I_2(t)$. To give this expansion, we expand the integral kernel of $\nabla e^{(t-s)\Delta}$ with respect to the space-time variable by virtue of Taylor's formula. Then, the integral $E_{\alpha,p}$ given by (1.3) appears in the asymptotic expansion of $I_2(t)$. This integral converges if $|\alpha| + 2p \leq n - 2$, but the convergence of it is not assured if $|\alpha| + 2p = n - 1$. The reason is that the estimate

$$\int_{\mathbb{R}^n} |y|^{|\alpha|} |u \nabla v|\, dy \leq C(1+s)^{-n/2-1/2+|\alpha|/2} \quad (s > 0,\ |\alpha| \leq n) \tag{1.15}$$

is satisfied. Therefore, we can't expect to obtain the asymptotic expansions in Theorem 1.1.

Here, by noting that the term $-\nabla \cdot (u \nabla v)$ is approximating the one $-M_0^2 \nabla \cdot (G \nabla G)$, where $M_0 = \int_{\mathbb{R}^n} u_0\, dy$ (see (2.11) of Lemma 2.6), the integral $F_{\alpha,p}$ given by (1.4) converges for $|\alpha| + 2p = n - 1$ due to the estimate

$$\begin{aligned} &\int_{\mathbb{R}^n} |y|^{|\alpha|} |(u \nabla v)(y, s) - M_0^2 (G \nabla G)(y, 1+s)|\, dy \\ &\leq C(1+s)^{-n/2-1+|\alpha|/2} \quad (s > 0,\ |\alpha| \leq n). \end{aligned} \tag{1.16}$$

Hence, using this fact and (1.14), and adding the correction term $R(x,t)$ given by (1.5), we estimate the error term in $L^q(\mathbb{R}^n)$ $(1 \le q \le \infty)$ space of asymptotic expansion up to n-th order for $u(t)$ as follows:

(1.17)

$$u(t) = \sum_{|\alpha|+2p\le n} \frac{(-1)^{|\alpha|+p}}{\alpha!p!}\partial_x^\alpha\partial_t^p G(1+t)M_\alpha - \sum_{|\alpha|+2p\le n-2} \frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot E_{\alpha,p}$$
$$- \sum_{|\alpha|+2p=n-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot F_{\alpha,p} - N_n(t;t) - R(t) + o(t^{-n(1-1/q)/2-n/2})$$

as $t \to \infty$, where $E_{\alpha,p}$, $F_{\alpha,p}$ and $N_n(t;t)$ are the ones given by (1.3), (1.4) and (2.4) with $l = n$, $z = t > 0$, respectively (see Proposition 4.2). Then, there exists a difference between the third and fourth terms on the right-hand side of (1.17) in the odd and even dimensional cases since $N_n(t;z) = 0\,(t, z > 0)$ if n is odd, and

$$N_n(t;t) = P_{(n-2)/2}(t)\log(1+t)$$

if n is even (see Lemma 2.3). As a consequence, this causes the difference between the asymptotic expansions of solutions to (KS) in the odd and even dimensional cases.

Now, we should emphasize that there exists a difference between the proofs of (1.15) and (1.16) in Theorems 1.1 and 1.2 which plays an important role in discussing the higher-order asymptotic expansion. In Theorem 1.1, if the solution (u,v) to (KS) satisfies the condition (1.9), then the L^1-norm for the solution is bounded on $[0,\infty)$, and the decay estimates for the solution are obtained, which together with $\sup_{x\in\mathbb{R}^n,t>0}(1+|x|)^n|u(x,t)| < \infty$ gives (1.15) and (1.16). On the other hand, the solution (u,v) to (KS) with (1.13) treated in Theorem 1.2 satisfies the moment estimate (3.1) given in Proposition 3.1 below under the condition $|x|^n u_0 \in L^1(\mathbb{R}^n)$. Hence, the estimates (1.15) and (1.16) follow from the decay estimates for the solution and (3.1).

Before closing this section, for simplicity we use the following notation.

$$\mathbb{Z}_+ = \mathbb{N}\cup\{0\}, \quad \alpha = (\alpha_1,\cdots,\alpha_n)\in\mathbb{Z}_+^n, \quad |\alpha| = \alpha_1+\cdots+\alpha_n,$$
$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_j = \frac{\partial}{\partial x_j}, \quad \partial_x^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}, \quad \nabla = (\partial_1,\cdots,\partial_n).$$

For $k \in \mathbb{N}$ and $1 \le q \le \infty$, we denote by $\|\cdot\|_{L^q}$ the usual $L^q(\mathbb{R}^n)$-norm, and by $W^{k,q}(\mathbb{R}^n)$ the usual Sobolev space. The function $f(\cdot,t)$ with respect to $x \in \mathbb{R}^n$ is defined by $f(t)$.

§ 2. Preliminaries

The aim of this section is to give the definition of solutions to (KS) and to collect known results which will be used often in the proof of Theorem 1.2. Firstly, we begin with the definition of solutions to (KS). For this purpose, we define $e^{t\Delta}f(x)$ by

$$e^{t\Delta}f(x) = \int_{\mathbb{R}^n} G(x-y,t)f(y)\,dy,$$

where $G(x,t)$ is the heat kernel given by (1.1).

Definition 2.1. A function (u,v) on $\mathbb{R}^n \times [0,T]$ $(0 < T < \infty)$ is said to be a solution to (KS) on $\mathbb{R}^n \times [0,T]$ if u, v satisfy

$$u, v, \partial_j v \in C([0,T]; L^1(\mathbb{R}^n)) \cap C([0,T]; \mathcal{B}(\mathbb{R}^n)) \ (1 \le j \le n),$$

and for all $0 < t \le T$,

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)\,ds, \tag{2.1}$$

$$v(t) = e^{-t}e^{t\Delta}v_0 + \int_0^t e^{-(t-s)}e^{(t-s)\Delta}u(s)\,ds. \tag{2.2}$$

Also, (u,v) is said to be a solution to (KS) on $\mathbb{R}^n \times [0,\infty)$ if (u,v) is a solution to (KS) on $\mathbb{R}^n \times [0,T]$ for all $0 < T < \infty$.

Remark. Making use of the standard regularity argument for parabolic equations (for example, see Ladyzhenskaya-Solonnikov-Uralt'seva [10]), we see that (u,v) is a classical solution to (KS) on $\mathbb{R}^n \times (0,T]$, which satisfies

$$u, v \in C((0,T); W^{2,q}(\mathbb{R}^n)) \cap C^1((0,T); L^q(\mathbb{R}^n))$$

for all $1 < q < \infty$, and

$$\partial_j u, \Delta v \in C((0,T); L^\infty(\mathbb{R}^n)) \ (1 \le j \le n).$$

The following lemma is the L^q-estimates for the heat kernel defined by (1.1).

Lemma 2.2. *Let $\alpha_j, \beta \in \mathbb{Z}_+$ $(1 \le j \le n)$ and $1 \le q \le \infty$. Then*

$$\|\partial_x^\alpha \partial_t^\beta G(t)\|_{L^q} \le Ct^{-n(1-1/q)/2-|\alpha|/2-\beta} \quad (t > 0), \tag{2.3}$$

where C is a positive constant depending on α, β, n, q.

The following lemma is needed to obtain the higher-order asymptotic expansion of u in more detail. The proof can be given by a direct calculation (see Yamada [18]).

Lemma 2.3 ([18]). *Let $1 \le l \le n$ be a positive integer and set*

$$\begin{aligned} N_l(t;z) :=& M_0^2 \sum_{|\alpha|+2p=l-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot \\ &\cdot \int_0^z \int_{\mathbb{R}^n} y^\alpha(1+s)^p (G\nabla G)(y,1+s)\,dyds \quad (t,z>0). \end{aligned} \tag{2.4}$$

Then,

$$
\text{(2.5)} \qquad N_l(t;z) = \begin{cases} 0 & \textit{if } l \textit{ is odd,} \\ P_{(l-2)/2}(t) \int_0^z (1+s)^{-n/2+l/2-1}\, ds & \textit{if } l \textit{ is even} \end{cases}
$$

for $t, z > 0$, *where* M_0 *and* $P_i(t)$ *are the ones defined by* (1.2) *with* $\alpha = 0$ *and* (1.7) *with* $i = (l-2)/2$, *respectively.*

The following lemma is concerned with well-known $L^r - L^q$ estimates of $e^{t\Delta}f$, which are proved by the Young's inequality for convolution (see Y. Giga-M.-H. Giga [3]).

Lemma 2.4 ([3]). *Let* $1 \le q \le r \le \infty$ *and* $\alpha_j, \beta \in \mathbb{Z}_+$ $(1 \le j \le n)$. *Then*

$$
\text{(2.6)} \qquad \|\partial_x^\alpha \partial_t^\beta e^{t\Delta} f\|_{L^r} \le C t^{-n(1/q-1/r)/2-|\alpha|/2-\beta} \|f\|_{L^q} \quad \textit{for} \quad f \in L^q(\mathbb{R}^n),
$$

where C *is a positive constant depending on* α, β, n, r, q.

Let (u, v) be the solution to (KS) with (1.13). Then the L^q-estimates for the solution have shown in Nagai-Yamada [14].

Lemma 2.5 ([14]). *Let* $n \ge 1$. *Then the following estimates hold:*

$$
\text{(2.7)} \qquad \sup_{t>0} (1+t)^{n(1-1/q)/2} \|u(t)\|_{L^q} < \infty \qquad (1 \le q \le \infty),
$$

$$
\text{(2.8)} \qquad \sup_{t>0} (1+t)^{n(1-1/q)/2+1/2} \|\nabla v(t)\|_{L^q} < \infty \quad (1 < q \le \infty).
$$

Furthermore, the following lemma, which is a key one to prove Theorem 1.2, gives some L^q-estimates for the solution. The proof is obtained by the calculations similar to those in Kato [7] (see also Yamada [18]).

Lemma 2.6 ([7, 18]). *Let* $n \ge 1$, $1 \le q \le \infty$ *and* (u, v) *be the solution to* (KS) *with* (1.13). *Then, under the condition* $|x|u_0 \in L^1(\mathbb{R}^n)$, *the following estimates hold:*

$$
\text{(2.9)} \qquad \sup_{t>0} (1+t)^{n(1-1/q)/2+1/2} \|u(t) - M_0 G(1+t)\|_{L^q} < \infty,
$$

$$
\text{(2.10)} \qquad \sup_{t>0} (1+t)^{n(1-1/q)/2+1} \|\nabla v(t) - M_0 \nabla G(1+t)\|_{L^q} < \infty,
$$

$$
\text{(2.11)} \qquad \sup_{t>0} (1+t)^{n(1-1/q)/2+n/2+1} \|(u\nabla v)(t) - M_0^2 (G\nabla G)(1+t)\|_{L^q} < \infty,
$$

where M_0 *is the one defined by* (1.2) *with* $\alpha = 0$.

Finally, to get the asymptotic behavior of v, we prepare the following lemma. The proof is obtained by the arguments similar to those in Kato [7] (see also Yamada [18]).

Lemma 2.7 ([7, 18]). *Let $n \geq 1$, $1 \leq q \leq \infty$ and (u, v) be the solution to* (KS) *with* (1.13). *Then*

$$\sup_{t>0} (1+t)^{n(1-1/q)/2+n/2+1} \|u(t) - v(t)\|_{L^q} < \infty. \tag{2.12}$$

§ 3. Moment estimate

This section is devoted to the moment estimate of u in order to show Theorem 1.2.

Proposition 3.1 (moment estimate). *Let $n \geq 2$, and let $l \geq 2$ be an integer. Then, under the condition $|x|^l u_0 \in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} |x|^l u(x,t)\,dx$ is finite for $t \geq 0$, and*

$$\int_{\mathbb{R}^n} |x|^l |u(x,t)|\,dx \leq C(1+t)^{l/2} \quad (t \geq 0). \tag{3.1}$$

For this purpose, we introduce the following cut-off function (see Sugiyama [17]).

Lemma 3.2 ([17]). *Define the function ψ by*

$$\psi(r) = \begin{cases} 1 & \text{for} \quad 0 \leq r \leq 1, \\ 1 - 2(r-1)^2 & \text{for} \quad 1 < r \leq 3/2, \\ 2(2-r)^2 & \text{for} \quad 3/2 < r \leq 2, \\ 0 & \text{for} \quad r > 2. \end{cases}$$

Then, the following assertions hold:

(i) $\psi \in C^1[0,\infty)$ and $\psi' \in W^{1,\infty}(0,\infty)$.

(ii) Put $\psi_m(x) := \psi(|x|/m)$ for $x \in \mathbb{R}^n$, $m \in \mathbb{N}$. Then, there exist positive constants c_i $(i = 1, 2)$ depending only on n such that

$$|\nabla \psi_m(x)| \leq c_1 m^{-1} \{\psi_m(x)\}^{1/2} \quad \text{for } x \in \mathbb{R}^n, \tag{3.2}$$

$$|\Delta \psi_m(x)| \leq c_2 m^{-2} \quad \text{for a.a. } x \in \mathbb{R}^n, \tag{3.3}$$

$$x \cdot \nabla \psi_m \leq 0 \quad \text{for } x \in \mathbb{R}^n. \tag{3.4}$$

Using the cut-off function given in Lemma 3.2, we show the following lemma before the proof of Proposition 3.1.

Lemma 3.3. *Let $k \geq 2$ and $m \in \mathbb{N}$. Then, under the condition $|x|^k u_0 \in L^1(\mathbb{R}^n)$,* *we have*

$$\int_{\mathbb{R}^n} |x|^k|u|\psi_m\,dx \le \||x|^k u_0\|_{L^1} + k(k+n-2)\int_0^t\int_{\mathbb{R}^n} |x|^{k-2}|u|\psi_m\,dxds \tag{3.5}$$
$$+C\int_0^t\int_{m\le|x|\le 2m} |x|^{k-2}|u|\,dxds + k\int_0^t\int_{\mathbb{R}^n}|x|^{k-1}|u||\nabla v|\psi_m\,dxds$$
$$+C\int_0^t\int_{m\le|x|\le 2m}|x|^{k-1}|u||\nabla v|\psi_m^{1/2}\,dxds \quad (t>0).$$

Proof. First, let ψ_m be the cut-off function defined in Lemma 3.2 and set

$$\varphi_\varepsilon(z) = \sqrt{z^2+\varepsilon} - \sqrt{\varepsilon} \quad \text{for} \quad z\in\mathbb{R},\, \varepsilon>0.$$

Multiplying the first equation of (KS) by $|x|^k\varphi_\varepsilon'(u)\psi_m$ and integrating by parts gives

$$\frac{d}{dt}\left(\int_{\mathbb{R}^n}|x|^k\varphi_\varepsilon(u)\psi_m\,dx\right) = -\int_{\mathbb{R}^n}\nabla(|x|^k\varphi_\varepsilon'(u)\psi_m)\cdot\nabla u\,dx \tag{3.6}$$
$$-\int_{\mathbb{R}^n}\nabla\cdot(u\nabla v)|x|^k\varphi_\varepsilon'(u)\psi_m\,dx$$

Here we have used the equality $\partial_t(\varphi_\varepsilon(u)) = \varphi_\varepsilon'(u)u_t$.

Now, we estimate the first term on the right-hand side of (3.6). Noting that $\varphi_\varepsilon''\ge 0$, $\varphi_\varepsilon'(u)\nabla u = \nabla\varphi_\varepsilon(u)$ and $\nabla|x|^k = k|x|^{k-2}x$, we have

$$-\int_{\mathbb{R}^n}\nabla(|x|^k\varphi_\varepsilon'(u)\psi_m)\cdot\nabla u\,dx \tag{3.7}$$
$$\le -k\int_{\mathbb{R}^n}|x|^{k-2}(x\cdot\nabla\varphi_\varepsilon(u))\psi_m\,dx - \int_{\mathbb{R}^n}|x|^k\nabla\varphi_\varepsilon(u)\cdot\nabla\psi_m\,dx.$$

Making use of $|\varphi_\varepsilon(z)|\le|z|$ $(z\in\mathbb{R})$ and (3.4) implies that

$$-\int_{\mathbb{R}^n}|x|^k\nabla\psi_m\cdot\nabla\varphi_\varepsilon(u)\,dx$$
$$=k\int_{\mathbb{R}^n}|x|^{k-2}\varphi_\varepsilon(u)(x\cdot\nabla\psi_m)\,dx + \int_{\mathbb{R}^n}|x|^k\varphi_\varepsilon(u)\Delta\psi_m\,dx$$
$$\le\int_{\mathbb{R}^n}|x|^k|u||\Delta\psi_m|\,dx.$$

Therefore, it follows from (3.3) that

$$-\int_{\mathbb{R}^n}|x|^k\nabla\psi_m\cdot\nabla\varphi_\varepsilon(u)\,dx \le \frac{c_2}{m^2}\int_{m\le|x|\le 2m}|x|^k|u|\,dx \tag{3.8}$$
$$\le C\int_{m\le|x|\le 2m}|x|^{k-2}|u|\,dx.$$

Similarly,

$$
\begin{aligned}
(3.9)\qquad & -k\int_{\mathbb{R}^n}|x|^{k-2}(x\cdot\nabla\varphi_\varepsilon(u))\psi_m\,dx\\
=&k\int_{\mathbb{R}^n}(\nabla(|x|^{k-2})\cdot x)\varphi_\varepsilon(u)\psi_m\,dx+kn\int_{\mathbb{R}^n}|x|^{k-2}\varphi_\varepsilon(u)\psi_m\,dx\\
&+k\int_{\mathbb{R}^n}|x|^{k-2}\varphi_\varepsilon(u)(x\cdot\nabla\psi_m)\,dx\\
\le&k(k+n-2)\int_{\mathbb{R}^n}|x|^{k-2}|u|\psi_m\,dx.
\end{aligned}
$$

Hence, from (3.7), (3.8) and (3.9), we obtain

$$
\begin{aligned}
(3.10)\qquad & -\int_{\mathbb{R}^n}\nabla(|x|^k\varphi_\varepsilon'(u)\psi_m)\cdot\nabla u\,dx\\
\le&k(k+n-2)\int_{\mathbb{R}^n}|x|^{k-2}|u|\psi_m\,dx+C\int_{m\le|x|\le 2m}|x|^{k-2}|u|\,dx.
\end{aligned}
$$

Next, we estimate the second term on the right hand-side of (3.6). For this purpose, we introduce the function $\zeta_\varepsilon(z)$ $(z\in\mathbb{R})$ as

$$
\zeta_\varepsilon(z)=\frac{\sqrt{\varepsilon}z^2}{\sqrt{z^2+\varepsilon}(\sqrt{z^2+\varepsilon}+\sqrt{\varepsilon})}.
$$

Since a direct calculation gives $z\varphi_\varepsilon'(z)=\varphi_\varepsilon(z)+\zeta_\varepsilon(z)$ $(z\in\mathbb{R})$, we have

$$
\begin{aligned}
&-\int_{\mathbb{R}^n}\nabla\cdot(u\nabla v)|x|^k\varphi_\varepsilon'(u)\psi_m\,dx\\
=&-\int_{\mathbb{R}^n}|x|^k(\nabla\varphi_\varepsilon(u)\cdot\nabla v)\psi_m\,dx-\int_{\mathbb{R}^n}|x|^k u\varphi_\varepsilon'(u)\Delta v\psi_m\,dx\\
=&-\int_{\mathbb{R}^n}|x|^k(\nabla\varphi_\varepsilon(u)\cdot\nabla v)\psi_m\,dx-\int_{\mathbb{R}^n}|x|^k(\varphi_\varepsilon(u)+\zeta_\varepsilon(u))\Delta v\psi_m\,dx\\
=&-\int_{\mathbb{R}^n}|x|^k\nabla\cdot(\varphi_\varepsilon(u)\nabla v)\psi_m\,dx-\int_{\mathbb{R}^n}|x|^k\zeta_\varepsilon(u)\Delta v\psi_m\,dx\\
=&k\int_{\mathbb{R}^n}|x|^{k-2}\varphi_\varepsilon(u)\,(x\cdot\nabla v)\,\psi_m\,dx+\int_{\mathbb{R}^n}|x|^k\varphi_\varepsilon(u)(\nabla v\cdot\nabla\psi_m)\,dx\\
&-\int_{\mathbb{R}^n}|x|^k\zeta_\varepsilon(u)\Delta v\psi_m\,dx\\
\le&k\int_{\mathbb{R}^n}|x|^{k-1}|u||\nabla v|\psi_m\,dx+\int_{\mathbb{R}^n}|x|^k|u||\nabla v||\nabla\psi_m|\,dx\\
&+\int_{\mathbb{R}^n}|x|^k\zeta_\varepsilon(u)|\Delta v|\psi_m\,dx.
\end{aligned}
$$

Using (3.2), we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |x|^k |u||\nabla v||\nabla \psi_m|\, dx \le & c_1 m^{-1} \int_{m\le |x|\le 2m} |x|^k |u||\nabla v|\psi_m^{1/2}\, dx \\ \le & C \int_{m\le |x|\le 2m} |x|^{k-1}|u||\nabla v|\psi_m^{1/2}\, dx.\end{aligned} \tag{3.11}$$

Similarly,

$$\int_{\mathbb{R}^n} |x|^k \zeta_\varepsilon(u)\psi_m |\Delta v|\, dx \le C_m\sqrt{\varepsilon}. \tag{3.12}$$

Here we have used the inequality $\zeta_\varepsilon(z) \le \sqrt{\varepsilon}$ $(z \in \mathbb{R})$. Hence putting together (3.11) and (3.12) implies that

$$\begin{aligned}& -\int_{\mathbb{R}^n} \nabla\cdot(u\nabla v)|x|^k \varphi_\varepsilon'(u)\psi_m\, dx \\ \le & k\int_{\mathbb{R}^n} |x|^{k-1}|u||\nabla v|\psi_m\, dx + C\int_{m\le|x|\le 2m} |x|^{k-1}|u||\nabla v|\psi_m^{1/2}\, dx + C_m\sqrt{\varepsilon}.\end{aligned} \tag{3.13}$$

Thus, substituting (3.10) and (3.13) into (3.6), we see that

$$\begin{aligned}\frac{d}{dt}\left(\int_{\mathbb{R}^n} |x|^k\varphi_\varepsilon(u)\psi_m\, dx\right) \le & k(k+n-2)\int_{\mathbb{R}^n} |x|^{k-2}|u|\psi_m\, dx \\ & + C\int_{m\le|x|\le 2m} |x|^{k-2}|u|\, dx + k\int_{\mathbb{R}^n} |x|^{k-1}|u||\nabla v|\psi_m\, dx \\ & + C\int_{m\le|x|\le 2m} |x|^{k-1}|u||\nabla v|\psi_m^{1/2}\, dx + C_m\sqrt{\varepsilon} \quad (t>0).\end{aligned} \tag{3.14}$$

Fix $m \in \mathbb{N}$ and $t > 0$. Integrating (3.14) from 0 to t and letting $\varepsilon \to 0$, by Lebesgue's dominated convergence theorem, we obtain the desired estimate (3.5). □

Proof of Proposition 3.1. The proof is given by induction for l. First of all, we shall show Proposition 3.1 for $l = 2$. Assume that $|x|^2u_0 \in L^1(\mathbb{R}^n)$. It follows from (3.5) with $k = 2$ that for $t > 0$,

$$\begin{aligned}& \int_{\mathbb{R}^n} |x|^2|u|\psi_m\, dx \\ \le & \||x|^2u_0\|_{L^1} + 2n\int_0^t\int_{\mathbb{R}^n} |u|\psi_m\, dxds + C\int_0^t\int_{m\le|x|\le 2m} |u|\, dxds \\ & + 2\int_0^t\int_{\mathbb{R}^n} |x||u||\nabla v|\psi_m\, dxds + C\int_0^t\int_{m\le|x|\le 2m} |x||u||\nabla v|\psi_m^{1/2}\, dxds.\end{aligned} \tag{3.15}$$

Making use of (2.7) with $q = 1$ and $0 \le \psi_m \le 1$ implies that

$$2n\int_0^t\int_{\mathbb{R}^n} |u|\psi_m\, dxds + C\int_0^t\int_{m\le|x|\le 2m} |u|\, dxds \le Ct. \tag{3.16}$$

Now we estimate the fourth term on the right-hand side of (3.15). Applying the Schwarz inequality and Lemma 2.5 gives

$$
\begin{aligned}
(3.17)\qquad & 2\int_0^t\int_{\mathbb{R}^n}|x||u||\nabla v|\psi_m\,dxds\\
&\le C\int_0^t\Big(\int_{\mathbb{R}^n}|x|^2|u|\psi_m\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^n}|u||\nabla v|^2\psi_m\,dx\Big)^{1/2}ds\\
&\le\int_0^t C(1+s)^{-n/2-1/2}\Big(\int_{\mathbb{R}^n}|x|^2|u|\psi_m\,dx\Big)^{1/2}ds.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(3.18)\qquad & C\int_0^t\int_{m\le|x|\le 2m}|x||u||\nabla v|\psi_m^{1/2}\,dxds\\
&\le C\int_0^t\int_{m\le|x|\le 2m}|x|^2|u|\psi_m\,dxds+C\int_0^t\int_{m\le|x|\le 2m}|u||\nabla v|^2\,dxds.
\end{aligned}
$$

Hence, substituting (3.16), (3.17) and (3.18) into (3.15), we see that

$$
\begin{aligned}
(3.19)\qquad & (1+t)^{-1}\int_{\mathbb{R}^n}|x|^2|u|\psi_m\,dx\\
&\le C+C(1+t)^{-1}\int_0^t(1+s)^{-n/2}\Big((1+s)^{-1}\int_{\mathbb{R}^n}|x|^2|u|\psi_m\,dx\Big)^{1/2}ds\\
&\quad+C\int_0^t(1+s)^{-1}\int_{m\le|x|\le 2m}|x|^2|u|\psi_m\,dxds\\
&\quad+C(1+t)^{-1}\int_0^t\int_{m\le|x|\le 2m}|u||\nabla v|^2\,dxds.
\end{aligned}
$$

Putting $M_m(t)=(1+t)^{-1}\int_{\mathbb{R}^n}|x|^2|u|\psi_m\,dx$ and applying the Schwarz inequality and Lemma 2.5 to (3.19), we have

$$
M_m(t)\le C+C\int_0^t M_m(s)\,ds\quad(t>0).
$$

Therefore it follows from the Gronwall inequality and Fatou's lemma that the finiteness of integral $(1+t)^{-1}\int_{\mathbb{R}^n}|x|^2|u|\,dx$ is ensured. Letting $m\to\infty$, by applying Lebesgue's dominated convergence theorem to (3.19), we observe that for $t>0$,

$$
\begin{aligned}
(3.20)\qquad & (1+t)^{-1}\int_{\mathbb{R}^n}|x|^2|u|\,dx\\
&\le C+C(1+t)^{-1}\int_0^t(1+s)^{-n/2}\Big((1+s)^{-1}\int_{\mathbb{R}^n}|x|^2|u|\,dx\Big)^{1/2}ds.
\end{aligned}
$$

Fix $\eta > 0$. Since the inequality $ab \leq \eta a^2 + b^2/(4\eta)$ $(a, b \geq 0)$ holds, we obtain

$$\begin{aligned} &C(1+t)^{-1}\int_0^t (1+s)^{-n/2}\left((1+s)^{-1}\int_{\mathbb{R}^n}|x|^2|u|\,dx\right)^{1/2} ds \\ \leq& C\eta(1+t)^{-1}\int_0^t (1+s)^{-1}\int_{\mathbb{R}^n}|x|^2|u|\,dxds + C\eta^{-1}(1+t)^{-1}\int_0^t (1+s)^{-n}\,ds \\ \leq& C\eta^{-1} + C\eta \sup_{0\leq s\leq t}\left[(1+s)^{-1}\int_{\mathbb{R}^n}|x|^2|u|\,dx\right], \end{aligned}$$

which together with (3.20) implies that

$$\begin{aligned} &\sup_{0\leq s\leq t}\left[(1+s)^{-1}\int_{\mathbb{R}^n}|x|^2|u|\,dx\right] \\ \leq& C + C\eta^{-1} + C\eta \sup_{0\leq s\leq t}\left[(1+s)^{-1}\int_{\mathbb{R}^n}|x|^2|u|\,dx\right]. \end{aligned} \tag{3.21}$$

In (3.21) we take $\eta > 0$ such that $C\eta = 1/2$. Then

$$\sup_{0\leq s\leq t}\left[(1+s)^{-1}\int_{\mathbb{R}^n}|x|^2|u|\,dx\right] \leq C,$$

which establish the conclusion of Proposition 3.1 for $l = 2$.

Supposing that Proposition 3.1 is true for l, namely, if $|x|^l u_0 \in L^1(\mathbb{R}^n)$, then the integral $\int_{\mathbb{R}^n}|x|^l|u|\,dx$ is finite for all $t \geq 0$, and

$$\int_{\mathbb{R}^n}|x|^l|u|\,dx \leq C(1+t)^{l/2} \quad (t \geq 0), \tag{3.22}$$

we shall show that it is true for $l + 1$.

Assume that $|x|^{l+1}u_0 \in L^1(\mathbb{R}^n)$. Since u_0 and $|x|^{l+1}u_0$ are in $L^1(\mathbb{R}^n)$, $|x|^l u_0$ is so. Hence the inequality (3.22) holds. Applying (3.5) with $k = l + 1$, we have

$$\begin{aligned} &\int_{\mathbb{R}^n}|x|^{l+1}|u|\psi_m\,dx \\ \leq& \||x|^{l+1}u_0\|_{L^1} + (l+1)(l+n-1)\int_0^t\int_{\mathbb{R}^n}|x|^{l-1}|u|\psi_m\,dxds \\ &+ C\int_0^t\int_{m\leq|x|\leq 2m}|x|^{l-1}|u|\,dxds + (l+1)\int_0^t\int_{\mathbb{R}^n}|x|^l|u||\nabla v|\psi_m\,dxds \\ &+ C\int_0^t\int_{m\leq|x|\leq 2m}|x|^l|u||\nabla v|\psi_m^{1/2}\,dxds \quad (t > 0). \end{aligned} \tag{3.23}$$

Now we estimate the second term on the right-hand side of (3.23). By (2.7) with

$q = 1$, (3.22), and the Hölder inequality, we observe that

$$
\begin{aligned}
(3.24)\qquad & (l+1)(l+n-1)\int_0^t\int_{\mathbb{R}^n}|x|^{l-1}|u|\psi_m\,dxds \\
&\le C\int_0^t\left(\int_{\mathbb{R}^n}|x|^l|u|\,dx\right)^{(l-1)/l}\left(\int_{\mathbb{R}^n}|u|\,dx\right)^{1/l}ds \\
&\le C\int_0^t(1+s)^{(l-1)/2}\,ds \le C(1+t)^{(l+1)/2}.
\end{aligned}
$$

Similarly,

$$
(3.25)\qquad C\int_0^t\int_{m\le|x|\le 2m}|x|^{l-1}|u|\,dxds \le C(1+t)^{(l+1)/2}.
$$

Next we estimate the fourth and fifth terms on the right-hand side of (3.23). Using (2.8) with $q = \infty$ and (3.22) yields that

$$
\begin{aligned}
(3.26)\qquad & (l+1)\int_0^t\int_{\mathbb{R}^n}|x|^l|u||\nabla v|\psi_m\,dxds + C\int_0^t\int_{m\le|x|\le 2m}|x|^l|u||\nabla v|\psi_m^{1/2}\,dxds \\
&\le C\int_0^t(1+s)^{-n/2-1/2+l/2}\,ds \le C\int_0^t(1+s)^{(l-1)/2}\,ds \le C(1+t)^{(l+1)/2}.
\end{aligned}
$$

Therefore, substituting (3.24), (3.25) and (3.26) into (3.23) gives

$$
\int_{\mathbb{R}^n}|x|^{l+1}|u|\psi_m\,dx \le C(1+t)^{(l+1)/2} \quad (t>0),
$$

which together with Fatou's lemma implies that the integral $\int_{\mathbb{R}^n}|x|^{l+1}|u|\,dx$ is finite for all $t \ge 0$, and $\int_{\mathbb{R}^n}|x|^{l+1}|u|\,dx \le C(1+t)^{(l+1)/2}$ $(t \ge 0)$. As a consequence, Proposition 3.1 is true for $l+1$. □

§ 4. Proof of Theorem 1.2

The aim of this section is to prove Theorem 1.2. Now, we begin with the following proposition.

Proposition 4.1. *Let $n \ge 2$, $1 \le q \le \infty$ and (u, v) be the solution to* (KS) *with* (1.13)*. Then, under the condition $|x|^n u_0 \in L^1(\mathbb{R}^n)$, the following assertions hold:*

(i) The integral $F_{\alpha,p}$ given by (1.4) *converges for $|\alpha| + 2p \le n-1$.*

(ii) It holds that

$$
\begin{aligned}
(4.1)\qquad & \lim_{t\to\infty} t^{n(1-1/q)/2+n/2}\Bigg\|\int_0^t e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)\,ds \\
& - \sum_{|\alpha|+2p\le n-1}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot F_{\alpha,p} - W(t)\Bigg\|_{L^q} = 0,
\end{aligned}
$$

where $W(t)$ *is the one given by* (1.6).

Proof. First, we shall show the assertion (i) of Proposition 4.1. For this purpose, by applying the moment estimate (3.1) given in Proposition 3.1, it is sufficient to prove (1.16) introduced in Section 1, that is,

$$\begin{aligned}\int_{\mathbb{R}^n} & |y|^{|\alpha|}|(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)|\,dy \\ &\le C(1+s)^{-n/2-1+|\alpha|/2} \quad (s>0,\ |\alpha|\le n),\end{aligned}$$

where $M_0 = \int_{\mathbb{R}^n} u_0\,dy$. Indeed, we prove (1.16) only for $|\alpha|>0$ since (1.16) for $|\alpha|=0$ follows from (2.11). To show (1.16), we write $\int_{\mathbb{R}^n} |y|^{|\alpha|}|(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)|\,dy$ as follows:

$$\begin{aligned}\int_{\mathbb{R}^n} & |y|^{|\alpha|}|(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)|\,dy \\ &\le \int_{\mathbb{R}^n} |y|^{|\alpha|}|u(y,s)||\nabla v(y,s) - M_0\nabla G(y,1+s)|\,dy \\ &\quad + \int_{\mathbb{R}^n} |y|^{|\alpha|}|M_0\nabla G(y,1+s)||u(y,s) - M_0 G(y,1+s)|\,dy \\ &=: L_1(s) + L_2(s).\end{aligned}$$

Then, by (2.7) with $q=1$ and the moment estimate (3.1) with $l=n$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |y|^{|\alpha|}|u(y,s)|\,dy &= \int_{\mathbb{R}^n} \{|y|^n|u(y,s)|\}^{|\alpha|/n}|u(y,s)|^{1-|\alpha|/n}\,dy \\ &\le \||y|^n u(s)\|_{L^1}^{|\alpha|/n}\|u(s)\|_{L^1}^{1-|\alpha|/n} \le C(1+s)^{|\alpha|/2},\end{aligned}$$

which together with (2.10) implies that

$$\begin{aligned}L_1(s) &\le \|\nabla v(s) - M_0\nabla G(1+s)\|_{L^\infty}\int_{\mathbb{R}^n} |y|^{|\alpha|}|u(y,s)|\,dy \\ &\le C(1+s)^{-n/2-1+|\alpha|/2}.\end{aligned}$$

Similarly, we obtain

$$L_2(s) \le C(1+s)^{-n/2-1+|\alpha|/2}.$$

Here we have used the estimate $\sup_{s>0}(1+s)^{1/2-|\alpha|/2}\||y|^{|\alpha|}\nabla G(1+s)\|_{L^1} < \infty$. Hence the desired estimate (1.16) follows from these estimates.

Next, we are going to show (4.1). Let $t\ge 2$. We split $\int_0^t e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)\,ds$ as follows:

$$\begin{aligned}
(4.2) \qquad & \int_0^t e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)\,ds \\
&= \int_{t/2}^t e^{(t-s)\Delta}\nabla\cdot\{(u\nabla v)(s)-M_0^2(G\nabla G)(1+s)\}\,ds \\
&\quad + \int_0^{t/2} e^{(t-s)\Delta}\nabla\cdot\{(u\nabla v)(s)-M_0^2(G\nabla G)(1+s)\}\,ds + W(t),
\end{aligned}$$

where $M_0=\int_{\mathbb{R}^n} u_0\,dy$. Using

$$\nabla G(x-y,t-s)=\sum_{|\alpha|+2p\le n-1}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}y^\alpha(1+s)^p\nabla\partial_x^\alpha\partial_t^pG(x,1+t)+R_n,$$

where

$$\begin{aligned}
R_n &= \sum_{\substack{|\alpha|+2p=n,\\ |\alpha|\ge 1}}\frac{|\alpha|(-1)^{|\alpha|+p}}{\alpha!p!}\int_0^1(1-\theta)^{|\alpha|-1}y^\alpha(1+s)^p\nabla\partial_x^\alpha\partial_t^pG(x-\theta y,1+t)\,d\theta \\
&+\frac{(-1)^{[\frac{n-1}{2}]+1}}{([\frac{n-1}{2}])!}\int_0^1(1-\tau)^{[\frac{n-1}{2}]}(1+s)^{[\frac{n-1}{2}]+1}\nabla\partial_t^{[\frac{n-1}{2}]+1}G(x-y,1+t-\tau(1+s))\,d\tau,
\end{aligned}$$

we have

$$\begin{aligned}
\int_0^{t/2} e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)\,ds &= \sum_{|\alpha|+2p\le n-1}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^pG(1+t)\cdot \\
&\cdot\int_0^{t/2}\int_{\mathbb{R}^n} y^\alpha(1+s)^p\{(u\nabla v)(y,s)-M_0^2(G\nabla G)(y,1+s)\}\,dyds \\
&+\int_0^{t/2}\int_{\mathbb{R}^n} R_n\{(u\nabla v)(y,s)-M_0^2(G\nabla G)(y,1+s)\}\,dyds.
\end{aligned}$$

Thus, substituting this equality into (4.2) and using the fact that the integral $F_{\alpha,p}$ converges for $|\alpha|+2p\le n-1$, we find that

$$\begin{aligned}
(4.3) \quad & \int_0^t e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)\,ds-\sum_{|\alpha|+2p\le n-1}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^pG(1+t)\cdot F_{\alpha,p}-W(t) \\
&=: I_1(t)+I_2(t)+I_3(t),
\end{aligned}$$

where

$$
\begin{aligned}
I_1(t) &= \int_{t/2}^{t} e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)\,ds,\\
I_2(t) &= -\sum_{|\alpha|+2p\le n-1}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot\\
&\quad\cdot\int_{t/2}^{\infty}\int_{\mathbb{R}^n} y^\alpha(1+s)^p\{(u\nabla v)(y,s)-M_0^2(G\nabla G)(y,1+s)\}\,dyds,\\
I_3(t) &= \int_0^{t/2}\int_{\mathbb{R}^n} R_n\{(u\nabla v)(y,s)-M_0^2(G\nabla G)(y,1+s)\}\,dyds.
\end{aligned}
$$

We now estimate $\|I_i(t)\|_{L^q}$ $(i=1,2,3)$ appearing in (4.3). By using Lemma 2.4 and (2.11), we have

$$
\begin{aligned}
\|I_1(t)\|_{L^q} &\le C\int_{t/2}^{t}(t-s)^{-1/2}\|(u\nabla v)(s)-M_0^2(G\nabla G)(1+s)\|_{L^q}\,ds\\
&\le C\int_{t/2}^{t}(t-s)^{-1/2}s^{-n(1-1/q)/2-n/2-1}\,ds\\
&\le Ct^{-n(1-1/q)/2-n/2-1/2},
\end{aligned}
$$

which implies that

$$
\lim_{t\to\infty} t^{n(1-1/q)/2+n/2}\|I_1(t)\|_{L^q}=0. \tag{4.4}
$$

For $\|I_2(t)\|_{L^q}$, it holds from the Minkowski's inequality, (1.16) and (2.3) that

$$
\begin{aligned}
\|I_2(t)\|_{L^q} \le &\sum_{|\alpha|+2p\le n-1}\frac{\|\nabla\partial_x^\alpha\partial_t^p G(1+t)\|_{L^q}}{\alpha!p!}\int_{t/2}^{\infty}(1+s)^p\,ds\\
&\times\int_{\mathbb{R}^n}|y|^{|\alpha|}|(u\nabla v)(y,s)-M_0^2(G\nabla G)(y,1+s)|\,dy\\
\le &\sum_{|\alpha|+2p\le n-1} C(1+t)^{-n(1-1/q)/2-|\alpha|/2-p-1/2}\int_{t/2}^{\infty}(1+s)^{-n/2-1+|\alpha|/2+p}\,ds\\
\le &Ct^{-n(1-1/q)/2-n/2-1/2},
\end{aligned}
$$

which yields that

$$
\lim_{t\to\infty} t^{n(1-1/q)/2+n/2}\|I_2(t)\|_{L^q}=0. \tag{4.5}
$$

As for $\|I_3(t)\|_{L^q}$, using the Minkowski's inequality and (2.3), we see that

$$
\begin{aligned}
(4.6) \qquad \|R_n\|_{L^q} \le& C \sum_{\substack{|\alpha|+2p=n,\\ |\alpha|\ge 1}} \int_0^1 |y|^{|\alpha|}(1+s)^p \|\nabla\partial_x^\alpha\partial_t^p G(\cdot-\theta y, 1+t)\|_{L^q}\, d\theta \\
&+ C\int_0^1 (1+s)^{[\frac{n-1}{2}]+1}\|\nabla\partial_t^{[\frac{n-1}{2}]+1}G(\cdot-y,1+t-\tau(1+s))\|_{L^q}\, d\tau \\
\le& C(1+t)^{-n(1-1/q)/2-n/2-1/2} \sum_{\substack{|\alpha|+2p=n,\\ |\alpha|\ge 1}} |y|^{|\alpha|}(1+s)^p \\
&+ C(t-s)^{-n(1-1/q)/2-[\frac{n-1}{2}]-3/2}(1+s)^{[\frac{n-1}{2}]+1}.
\end{aligned}
$$

Therefore, it follows from (1.16) and (4.6) that

$$
\begin{aligned}
\|I_3(t)\|_{L^q} \le& C(1+t)^{-n(1-1/q)/2-n/2-1/2} \sum_{\substack{|\alpha|+2p=n,\\ |\alpha|\ge 1}} \int_0^{t/2} (1+s)^{-n/2-1+|\alpha|/2+p}\, ds \\
&+ Ct^{-n(1-1/q)/2-[\frac{n-1}{2}]-3/2}\int_0^t (1+s)^{-n/2+[\frac{n-1}{2}]}\, ds \\
\le& Ct^{-n(1-1/q)/2-n/2-1/2}\log t,
\end{aligned}
$$

which yields that

$$
(4.7) \qquad \lim_{t\to\infty} t^{n(1-1/q)/2+n/2}\|I_3(t)\|_{L^q} = 0.
$$

As a consequence, (4.1) follows from (4.3) through (4.7). □

The following proposition is a key one to obtain Theorem 1.2.

Proposition 4.2. *Under the assumptions of Proposition 4.1, the following assertions hold:*

(i) *The integral $E_{\alpha,p}$ given by (1.3) converges for $|\alpha|+2p\le n-2$ and the one $F_{\alpha,p}$ given by (1.4) is well-defined for $|\alpha|+2p=n-1$.*

(ii) *The asymptotic behavior of $u(t)$ is given by*

$$
\begin{aligned}
(4.8) \qquad t^{n(1-1/q)/2+n/2}\Bigg\| & u(t) - \sum_{|\alpha|+2p\le n}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\partial_x^\alpha\partial_t^p G(1+t)M_\alpha \\
&+ \sum_{|\alpha|+2p\le n-2}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot E_{\alpha,p} \\
&+ \sum_{|\alpha|+2p=n-1}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot F_{\alpha,p} + N_n(t;t) + R(t)\Bigg\|_{L^q} \to 0
\end{aligned}
$$

as $t \to \infty$, where M_α, $N_n(t;t)$ and $R(t)$ are the ones given by (1.2), (2.4) *with $l = n$, $z = t > 0$ and* (1.5), *respectively.*

Proof. First, we recall from Proposition 4.1 that the integral $F_{\alpha,p}$ given by (1.4) is well-defined for $|\alpha| + 2p \le n - 1$. Moreover, we see that the integral $E_{\alpha,p}$ given by (1.3) converges for $|\alpha| + 2p \le n - 2$ because using arguments similar to those in the proof of (1.16) gives (1.15). Therefore we can get the assertion (i) of Proposition 4.1.

Next we are going to prove the assertion (ii) of Proposition 4.2. For this purpose, we claim that the following equality holds:

$$
\begin{aligned}
(4.9)\quad & u(t) - \sum_{|\alpha|+2p\le n} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \partial_x^\alpha \partial_t^p G(1+t) M_\alpha \\
& + \sum_{|\alpha|+2p\le n-2} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot E_{\alpha,p} \\
& + \sum_{|\alpha|+2p= n-1} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot F_{\alpha,p} + N_n(t;t) + R(t) \\
=& \left\{ e^{t\Delta} u_0 - \sum_{|\alpha|+2p\le n} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \partial_x^\alpha \partial_t^p G(1+t) M_\alpha \right\} - \left\{ \int_0^t e^{(t-s)\Delta} \nabla \cdot (u \nabla v)(s)\, ds \right. \\
& \left. - \sum_{|\alpha|+2p\le n-1} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot F_{\alpha,p} - W(t) \right\} \\
=:& \, M_1(t) + M_2(t),
\end{aligned}
$$

where M_α, $N_n(t;t)$, $R(t)$ and $W(t)$ are the ones given by (1.2), (2.4) with $l = n$, $z = t > 0$, (1.5) and (1.6), respectively. Indeed, by making use of the convergence of integral $\int_0^\infty \int_{\mathbb{R}^n} y^\alpha (1+s)^p (G\nabla G)(y, 1+s)\, dyds$ for $|\alpha| + 2p \le n - 2$ and Lemma 2.3, the correction term $R(t)$ given in (1.5) has the following relation:

$$
\begin{aligned}
(4.10)\qquad N_n(t;t) + R(t) =& W(t) - M_0^2 \sum_{|\alpha|+2p\le n-2} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \\
& \cdot \int_0^\infty \int_{\mathbb{R}^n} y^\alpha (1+s)^p (G\nabla G)(y, 1+s)\, dyds.
\end{aligned}
$$

Hence, it follows from (2.1) and (4.10) that

$$
\begin{aligned}
&u(t) - \sum_{|\alpha|+2p\le n} \frac{(-1)^{|\alpha|+p}}{\alpha!p!}\partial_x^\alpha \partial_t^p G(1+t) M_\alpha \\
&+ \sum_{|\alpha|+2p\le n-2} \frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha \partial_t^p G(1+t)\cdot E_{\alpha,p} \\
&+ \sum_{|\alpha|+2p= n-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha \partial_t^p G(1+t)\cdot F_{\alpha,p} + N_n(t;t) + R(t) \\
=&\left\{ e^{t\Delta}u_0 - \sum_{|\alpha|+2p\le n} \frac{(-1)^{|\alpha|+p}}{\alpha!p!}\partial_x^\alpha \partial_t^p G(1+t) M_\alpha \right\} - \left\{ \int_0^t e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)\,ds \right. \\
&- \sum_{|\alpha|+2p\le n-2} \frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha \partial_t^p G(1+t)\cdot E_{\alpha,p} \\
&+ M_0^2 \sum_{|\alpha|+2p\le n-2} \frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha \partial_t^p G(1+t)\cdot \\
&\qquad \cdot \int_0^\infty \int_{\mathbb{R}^n} y^\alpha (1+s)^p (G\nabla G)(y,1+s)\,dyds \\
&\left. - \sum_{|\alpha|+2p= n-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha \partial_t^p G(1+t)\cdot F_{\alpha,p} - W(t) \right\} \\
=&M_1(t) + M_2(t).
\end{aligned}
$$

This implies the desired equality (4.9).

To complete the proof of Proposition 4.2, we estimate $\|M_i(t)\|_{L^q}$ $(i=1,2)$. For $\|M_1(t)\|_{L^q}$,

$$\lim_{t\to\infty} t^{n(1-1/q)/2+n/2}\|M_1(t)\|_{L^q} = 0$$

follows from the method similar to that in Proposition 3.1 of Fujigaki-Miyakawa [2]. Also, for $\|M_2(t)\|_{L^q}$, applying (4.1) yields that

$$\lim_{t\to\infty} t^{n(1-1/q)/2+n/2}\|M_2(t)\|_{L^q} = 0.$$

Thus, these estimates and (4.9) yield the desired estimate (4.8). □

Proof of Theorem 1.2. Let $t\ge 2$ and $1\le q\le\infty$. Once the asymptotic expansion of u is shown, that of v is obtained by Lemma 2.7. Hence we prove only the asymptotic expansion of u.

In the odd dimensional case, we observe from the assertion (i) of Proposition 4.2 that the integral $E_{\alpha,p}$ given by (1.3) converges for $|\alpha|+2p\le n-2$. Moreover this integral is well-defined for $|\alpha| + 2p = n - 1$ because the convergence of integral $F_{\alpha,p}$ given by (1.4) is assured due to the assertion (i) of Proposition 4.1 and the integral $\int_{\mathbb{R}^n} y^\alpha (1+s)^p (G\nabla G)(y, 1+s)\, dy$ is zero for $|\alpha| + 2p = n - 1$. Therefore, we see that the integral $E_{\alpha,p}$ given by (1.3) is well-defined for $|\alpha| + 2p \le n - 1$, and the asymptotic expansion (1.10) follows from (2.5) and (4.8).

On the other hand, in the even dimensional case, it follows from the assertion (i) of Proposition 4.2 that the convergence of integral $E_{\alpha,p}$ given by (1.3) is well-defined for $|\alpha| + 2p \le n - 2$, and the integral $F_{\alpha,p}$ given by (1.4) converges for $|\alpha| + 2p = n - 1$. Furthermore, from (2.5) with $l = n$, $z = t > 0$, we have

$$N_n(t; t) = P_{(n-2)/2}(t) \log(1 + t),$$

which together with (4.8) gives the asymptotic expansion (1.11). $\square$

References

[1] S. Childress, J. K. Percus, *Nonlinear aspects of chemotaxis*, Math. Biosci., **56** (1981), 271–237.

[2] Y. Fujigaki, T. Miyakawa, *Asymptotic profiles of the nonstationary incompressible Navier-Stokes flow in the whole space*, SIAM J. Math. Anal., **33** (2001), 523–544.

[3] Y. Giga, M.-H. Giga, *Self-similar solutions and asymptotic behavior*(in Japanese). In: Nonlinear Partial Differential Equation, Kyoritsu, 1999.

[4] M. A. Herrero, J. J. L. Velázquez, *A blow-up mechanism for a chemotactic model*, Ann. Scuola Normale Sup. Pisa, **XXIV** (1997), 633–683.

[5] D. Horstmann, *From 1970 until present: The Keller-Segel model in chemotaxis and its consequences. I*, Jahresber. Deutsch. Math.-Verein., **105** (2003), 103–165.

[6] W. Jäger, S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc., **329** (1992), 819–824.

[7] M. Kato, *Sharp asymptotics for parabolic system of chemotaxis in one space dimension*, Differential and Integral Equations., **22** (2009) 35–51.

[8] E. F. Keller, L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theoret. Biol., **26** (1970), 399–415.

[9] R. Kobayashi, S. Kawashima, *Decay estimates and large time behavior of solutions to the drift-diffusion system*, Funcialaj Ekvasioj, **51** (2008) 371–394.

[10] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Uralt'seva, *Linear and Quasi-linear Equations of Parabolic Type*, Nauka, Moscow, 1967: English translation: Amer. Math. Soc., Providence, R. I., 1968.

[11] T. Miyakawa, *Hardy spaces of solenoidal vector fields, with application to the Navier-Stokes equations*, Kyushu J. Math., **50** (1996), 1-64.

[12] T. Nagai, T. Senba, K. Yoshida, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcialaj Ekvacioj, **40** (1997), 411–433.

[13] T. Nagai, R. Syukuinn, M. Umesako, *Decay properties and asymptotic profiles of bounded solutions to a parabolic system of chemotaxis in R^N*, Funkcialaj Ekvacioj, **46** (2003), 383–407.

[14] T. Nagai, T. Yamada, *Large time behavior of bounded solutions to a parabolic system of system of chemotaxis in the whole space*, J. Math. Anal. Appl., **336** (2007), 704–726.

[15] K. Nishihara, *Asymptotic profile of solutions to a parabolic system of chemotaxis in one dimensional space*, preprint.

[16] T. Ogawa, M. Yamamoto, *Asymptotic behavior of solutions to drift-diffusion system with generalized dissipation*, Math. Models Methods Appl. Sci., **19** (2009), 939–967.

[17] Y. Sugiyama, *Global existence in sub-critical cases and finite time blow-up in super-critical cases to degenerate Keller-Segel systems*, Diff. Integral Equations., **19** (2006), 841–876.

[18] T. Yamada, *Higher-order asymptotic expansions for a parabolic system modeling chemotaxis in the whole space*, Hiroshima Math. J., **39** (2009), 363–420.