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Nonlocal elliptic boundary value problems related to chemotactic movement of mobile species

Dedicated to Prof. Toshitaka Nagai, on the occasion of his sixtieth birthday

By

Dirk HORSTMANN * and Marcello LUCIA **

Abstract

The present paper focuses on the steady state problem of some chemotaxis models. Besides a summarization of known and new results for the so-called classical chemotaxis model additional results for some multi-species chemotaxis models are established. For example, an existence proof of nontrivial solutions to some nonlocal elliptic boundary value problems is given. Here the focus lies at conflict-free situations. Besides the presented steady state results some new tools and generalization for the analysis of multi-species chemotaxis models are presented, which include for example some more general Lyapunov functionals than those known up to now.

§ 1. Motivation and Introduction

"Movement is life!" Wilhelm Pfeffer (1845 - 1920)

This citation of the German botanist and plant physiologist Wilhelm Pfeffer might be motivation enough to study the mechanisms and underlying effects for the motion of mobile species. Therefore, studying the movement and selforganized pattern formation of mobile species is an interesting topic in mathematical biology. There are several aspects and mechanisms that influence the movement, in particular the direction of

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*Mathematisches Institut, Universität zu Köln, 50931 Köln, Germany.
e-mail: dhorst@math.uni-koeln.de
**Mathematics Department, The City University of New York, CSI, Staten Island NY 10314, USA
e-mail: mlucia@math.csi.cuny.edu

motion is in general the sum of several external impacts. For example, chemical substances in the environment may have a strong influence on the movement of mobile species. This can lead to strictly oriented movement or to partially oriented and partially tumbling movement of a single particle resulting in the corresponding movement of a whole population. One particular effect of chemical substances on the movement of mobile species is called chemotaxis. Here the movement towards a higher concentration of the chemical substance is termed positive chemotaxis, while the movement towards regions of a lower chemical concentration is called negative chemotactical movement. However, besides chemotactic effects also predator-prey interactions might cause the movement in some certain direction. All these aspects can lead to complex and strongly coupled systems of reaction-diffusion systems with cross-diffusion effects.

The present paper is organized as follows:

First we give the arguments and describe the underlying situations and mechanisms for modelling chemotactic movement of species that lead to multi-species chemotaxis systems that extend the classical Keller-Segel model to the situation of multi-species movement as an reaction to multi-agent attraction resp. repulsion. Then some known results from [13, 22] for these kind of models are recalled and a Lyapunov functional for some special systems is presented.

In section 2 we summarize the results available for the steady states of the classical single species Keller-Segel model. While section 3 is devoted to the analysis of some nonlocal elliptic boundary value problem that follows from some explicit examples of multi-species chemotaxis models introduced in the present paper and in[13]. We establish the mountain pass structure of the corresponding variational formulation of the nonlocal elliptic boundary value problem and prove the existence of nontrivial steady state solutions applying and generalizing some techniques that have been introduced in [19].

The paper finishes with some concluding remarks and comments on multi-species chemotaxis models.

§ 2. General multi-species chemotaxis models

To model chemotactic movement of several mobile species let us assume for a moment that we have three different populations of mobile species, that behave in a certain chemotactical manner to two chemical agents s_1 and s_2. Furthermore, we assume that the population p_1 flows from places where its density is high towards places where the density is low. At the same time the population p_2 and the substance s_1 has an attracting and the population p_3 and the substance s_2 has a repelling effect on p_1 and therefore influences the flow of p_1. Thus p_1 flows towards high of p_2 and s_1, resp. low density of p_3 and s_2. Accordingly the flow vector $\mathcal{J}_{p_1}(t,x)$ of the population density p_1 is a linear combination of the gradients of $p_1,\ p_2,\ p_3,\ s_1$ and s_2. Therefore, we get:

$$\mathcal{J}_{p_1} = -d_{11}\cdot \mathbf{grad}\ p_1 - d_{12}\cdot \mathbf{grad}\ p_2 - d_{13}\cdot \mathbf{grad}\ p_3 - d_{14}\cdot \mathbf{grad}\ s_1 - d_{15}\cdot \mathbf{grad}\ s_2,$$

where $d_{11}, d_{12}, d_{14} > 0$, and $d_{13}, d_{15} \leq 0$ are functions that depend on the p_i and s_j. According to Fick's law the flow contains a part that is proportional to the density gradients p_i and according to Fourier's law for the heat flow a part that is proportional to the chemoattractant resp. chemorepellent gradients s_i.

$$\frac{d}{dt}\int_D p_1(t,x)\ dx = Q^1_{(\mathbf{p},\mathbf{s})}(t,D) - \int_{\partial D} (\mathcal{J}_{p_1}(t,x)\cdot n(x))dS, \tag{2.1}$$

where $Q^i_{(\mathbf{p},\mathbf{s})}(t,D)$ denotes the growth of the population density $p_i(t,x)$ per domain and time volume.

Generalizing the situation to n mobile species that react on m different chemical substances and if we assume that the substances s_i simply diffusion according Fick's law, we can formulate the following generalized multi-species chemotaxis model:

$$\begin{cases} \frac{\partial}{\partial t}p_i = \nabla\left[\left(\sum\limits_{k=1}^{n} D_{ik}(\mathbf{p},\mathbf{s})\nabla p_k\right) + \left(\sum\limits_{j=1}^{m} C_{ij}(\mathbf{p},\mathbf{s})\nabla s_j\right)\right] + F_i(\mathbf{p},\mathbf{s}), \\ \qquad x\in\Omega\subset \mathbb{R}^N,\ t>0 \\ \frac{\partial}{\partial t}s_j = \nabla\cdot\left(\sum\limits_{l=1}^{m} A_{lj}(\mathbf{p},\mathbf{s})\nabla s_l\right) + H_j(\mathbf{p},\mathbf{s}), \\ \qquad x\in\Omega\subset \mathbb{R}^N,\ t>0 \end{cases} \tag{2.2}$$

together with either

$$\frac{\partial}{\partial n}p_i = 0\ (i=1,...,n),\ \frac{\partial}{\partial n}s_j \ = \ 0\ (j=1,...,m),\ (x,t)\in\partial\Omega\times(0,T) \tag{2.3}$$

or

$$(2.4)\qquad \begin{cases} \left(\sum\limits_{k=1}^{n} D_{ik}(\mathbf{p},\mathbf{s})\frac{\partial}{\partial n}p_k\right) + \left(\sum\limits_{j=1}^{m} C_{ij}(\mathbf{p},\mathbf{s})\frac{\partial}{\partial n}s_j\right) = 0\ (i=1,...,n), \\ (x,t)\in\partial\Omega\times(0,T) \\ s_j \ = \ 0\ (j=1,...,m), \\ (x,t)\in\partial\Omega\times(0,T) \end{cases}$$

as boundary conditions and given initial data

$$(2.5)\qquad \mathbf{p}(0,x) = \mathbf{p}_0(x),\ \mathbf{s}(0,x) \ = \ \mathbf{s}_0(x),\ x\in\Omega,$$

where the functions

$$F_i : \mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}\ (i=1,...,n) \text{ and } H_j : \mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}\ (j=1,...,m)$$

represent certain reaction terms that describe death and birth of the populations, resp. production and decay properties of the substances. For the "diffusion coefficients" $D_{ik}(\mathbf{p},\mathbf{s})$, $C_{ij}(\mathbf{p},\mathbf{s})$ and $A_{lj}(\mathbf{p},\mathbf{s})$ we assume that they are sufficiently smooth (at least once continuous differentiable) functions from $\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$.

§ 2.1. The variational structure for special models

In the presence of population growth the given generalized Keller-Segel models seem to have in general no variational structure, i.e. no Lyapunov functional seems to exist for these models except in those situations considered in [1]. However, in the absence of growth and under some additional assumptions these kind of models possess Lyapunov functionals. See for example [1, 13, 21] and [22] for more details.

To give an example for such cases we look at the following multi-species chemotaxis systems with mass conservation.

$$(2.6)\qquad \begin{cases} \frac{\partial}{\partial t}p_i = a_i\Delta p_i + \nabla\left[\left(\sum\limits_{l=1,l\neq i}^{n}\delta_{i,l}p_i\nabla p_l\right) - \left(\sum\limits_{j=1}^{m}\omega_{i,j}p_i\nabla s_j\right)\right], \\ \qquad x\in\Omega\subset\mathbb{R}^N,\ t>0 \\ \frac{\partial}{\partial t}s_j = \left(\sum\limits_{k=1}^{m} b_{k,j}\Delta s_k\right) - \sum\limits_{k=1}^{m}\gamma_{k,j}s_k + \sum\limits_{k=1}^{n}\alpha_{k,j}p_k, \\ \qquad x\in\Omega\subset\mathbb{R}^N,\ t>0 \end{cases}$$

where the a_i, $b_{k,j}$, $\alpha_{k,j}$, $\omega_{i,j}$, $\delta_{i,l}$ and $\gamma_{k,j}$ are given constants. G. Wolansky gave in [22] (for similar systems) the following definition to describe conflict-free systems, resp. systems in the presence of conflict:

Definition 2.1 (according to Wolansky [22]). We set

$$\lambda_{i,j} = \sum_{k=1}^{m} \omega_{i,k}\alpha_{j,k} = \omega_i\alpha_j.$$

1. A population i_1 is attracted (resp. repelled) to (resp. from) a population i_2 if $\lambda_{i_1,i_2} > 0$ (resp. $\lambda_{i_1,i_2} < 0$). In particular, a population is self-attracting (self-repelling) if $\lambda_{i,i} > 0$ (resp $\lambda_{i,i} < 0$).
2. A pair of populations $i_1, i_2 \in \{1, ..., n\}$ is said to be in a conflict, if

$$\lambda_{i_1,i_2} \times \lambda_{i_2,i_1} < 0.$$

Remark 2.2. A pair of populations $i_1, i_2 \in \{1, ..., n\}$ is said to be conflict-free, if $\lambda_{i_1,i_2} \times \lambda_{i_2,i_1} > 0$ and if there are n positive constants $\rho_1, ..., \rho_n$ such that $\rho_i\lambda_{i,l} = \rho_l\lambda_{l,j}$.

However, the system presented here covers more interactions than the system considered in [22]. Therefore, we have to define some additional kind of interaction between the mobile populations.

Definition 2.3 (according to Horstmann [13]). For $i \neq j$ we set

$$\kappa_{i,j} = \delta_{i,j}\delta_{j,i}.$$

1. A population i_1 has common objectives (has no common objectives) with a population i_2 if $\kappa_{i_1,i_2} > 0$ (resp. $\kappa_{i_1,i_2} < 0$).
2. If $\delta_{i_1,i_2} = \delta_{i_2,i_1}$ we say that the populations i_1 and i_2 have homogeneous common objectives.
3. We say that the system describes motion with common objectives, iff $\kappa_{i,j} > 0$ for all $i, j \in \{1, ..., n\}$ with $i \neq j$.

Similar to the Lyapunov functional for multi-species chemotaxis systems presented in [22] one can introduce Lyapunov functionals for the given multi-species systems where we have additionally to assume that

$$b_{i,j} = b_{j,i} \text{ for all } i, j \in \{1, ..., m\},\ \gamma_{i,k} = \gamma_{k,i} \text{ for all } i, k \in \{1, ..., n\} \text{ with } i \neq k,$$

$$\delta_{i,k} = \delta_{k,i} \text{ for all } i, k \in \{1, ..., n\} \text{ with } i \neq k$$

and

$$\omega_{i,j} = 0 \text{ iff } \alpha_{j,i} = 0 \text{ for } i \in \{1, .., n\} \text{ and } j \in \{1, .., m\}$$

holds true. For this kind of conflict-free systems in the presence of motion with common objectives, we have the following Lyapunov functional (compare [13]) at hand:

$$\begin{aligned}\mathcal{L}(\mathbf{p}, \mathbf{s}) = &\frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m}\beta_{l,j}\int_{\Omega}\left[b_{j,l}\nabla s_j \nabla s_l + \gamma_{j,l} s_j s_l\right] dx \\ &+\sum_{i=1}^{n}\rho_i\int_{\Omega} a_i p_i \ln(p_i) dx - \sum_{i=1}^{n}\sum_{j=1}^{m}\rho_i\omega_{i,j}\int_{\Omega} p_i s_j dx \\ &+\sum_{i=1}^{n}\sum_{k=1,k\neq i}^{n}\rho_i\delta_{i,k}\int_{\Omega} p_i p_k dx,\end{aligned} \tag{2.7}$$

where the matrix $\beta = (\beta_{l,j})_{m\times m}$ is such that

$$\beta\alpha_i = \rho_i\omega_i \text{ for all } 1 \leq i \leq n \text{ and } \sum_{l=1}^{m}\beta_{l,j} > 0 \text{ for all } j \in \{1, ..., m\}.$$

Remark 2.4. This Lyapunov functional that has been introduced in [13] generalizes the Lyapunov functional known from [22] since it takes also cross-diffusion effects between the different mobile species and the diffusible substances into account, which have not been considered before. For other multi-species chemotaxis models that process Lyapunov functionals we refer [1] and [13].

Now, for classical solutions of (2.6) we see that

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}(\mathbf{p},\mathbf{s}) =& \frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m}\beta_{l,j}\int\limits_{\Omega}\left[b_{j,l}\nabla s_j \nabla\left(\frac{\partial}{\partial t}s_l\right)+\gamma_{j,l}s_j\left(\frac{\partial}{\partial t}s_l\right)\right]dx \\
&+\frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m}\beta_{l,j}\int\limits_{\Omega}\left[b_{j,l}\nabla\left(\frac{\partial}{\partial t}s_j\right)\nabla s_l+\gamma_{j,l}\left(\frac{\partial}{\partial t}s_j\right)s_l\right]dx \\
&+\sum_{i=1}^{n}\rho_i\int\limits_{\Omega}a_i\left(\frac{\partial}{\partial t}p_i\right)\ln(p_i)dx+\sum_{i=1}^{n}\rho_i\int\limits_{\Omega}a_i\left(\frac{\partial}{\partial t}p_i\right)dx \\
&-\sum_{i=1}^{n}\sum_{j=1}^{m}\rho_i\omega_{i,j}\int\limits_{\Omega}\left(\frac{\partial}{\partial t}p_i\right)s_j+p_i\left(\frac{\partial}{\partial t}s_j\right)dx \\
&+\sum_{i=1}^{n}\sum_{k=1,k\neq i}^{n}\rho_i\delta_{i,k}\int\limits_{\Omega}\left(\frac{\partial}{\partial t}p_i\right)p_k+p_i\left(\frac{\partial}{\partial t}p_k\right)dx \\
=&-\sum_{j=1}^{m}\int\limits_{\Omega}\left(\frac{\partial}{\partial t}s_j\right)\left[\sum_{l=1}^{m}\beta_{l,j}b_{j,l}\Delta s_l+\beta_{l,j}\gamma_{j,l}s_l++\sum_{i=1}^{n}\rho_i\omega_{i,j}p_i\right]dx \\
&+\sum_{i=1}^{n}\rho_i\int\limits_{\Omega}\left(\frac{\partial}{\partial t}p_i\right)\left(a_i\ln(p_i)-\sum_{j=1}^{m}\omega_{i,j}s_j+\sum_{k=1,k\neq i}^{n}\delta_{i,k}p_k dx\right) \\
=&-\sum_{j=1}^{m}\int\limits_{\Omega}\left(\frac{\partial}{\partial t}s_j\right)\left[\sum_{l=1}^{m}\beta_{l,j}\left(\frac{\partial}{\partial t}s_j\right)\right]dx \\
&+\sum_{i=1}^{n}\rho_i\int\limits_{\Omega}\left(\frac{\partial}{\partial t}p_i\right)\left(a_i\ln(p_i)-\sum_{j=1}^{m}\omega_{i,j}s_j+\sum_{k=1,k\neq i}^{n}\delta_{i,k}p_k dx\right) \\
=&-\int\limits_{\Omega}\left(\frac{\partial}{\partial t}\mathbf{s}\right)^T\beta\left(\frac{\partial}{\partial t}\mathbf{s}\right)dx \\
&-\sum_{i=1}^{n}\rho_i\int\limits_{\Omega}p_i\left|\nabla\left(\ln(p_i)-\sum_{j=1}^{m}\omega_{i,j}s_j+\sum_{k=1,k\neq i}^{n}\delta_{i,k}p_k\right)\right|^2dx
\end{aligned}
$$

holds. Thus, $\mathcal{L}(\mathbf{p},\mathbf{s})$ is monotone non-increasing if $\rho_i > 0$ for all $i \in \{1,...,n\}$, i.e. in the conflict-free situation.

Remark 2.5. We refer the interested reader to [7] for sufficient conditions that guarantee the existence of Lyapunov functionals for single species chemotaxis systems. furthermore, we refer to [1] for an alternative approach to derive Lyapunov functionals for chemotaxis systems including the situations in the presence of population growth.

§ 2.2. The steady state problem: reduction to a system of parameter-dependent elliptic partial differential equations for the agents

In [17] R. Schaaf focused on the properties of stationary solutions of the single species Keller-Segel model with homogeneous Neumann boundary data in a very general setting. She shows that the stationary problem of the single species Keller-Segel model can be reduced to a parameter-dependent single scalar equation. However, her result can easily be generalized to multi-species systems. Therefore, we consider the following steady state system:

$$\begin{cases} 0 = \nabla\left(k_{i,1}(p_i,\mathbf{s})\nabla p_i\right) + \nabla\left(k_{i,2}(p_i,\mathbf{s})\sum\limits_{j=1}^{m}\nabla s_j\right), \\ \\ 0 = \nabla\cdot\left(\sum\limits_{j=1}^{m} A_{lj}(\mathbf{p},\mathbf{s})\nabla s_l\right) + H_j(\mathbf{p},\mathbf{s}), \end{cases} \tag{2.8}$$

where both equations are considered in $x \in \Omega \subset \mathrm{I\!R}^N$, $t > 0$ together with either

$$\frac{\partial}{\partial n} p_i = 0 \ (i = 1, ..., n), \ \frac{\partial}{\partial n} s_j \ = \ 0 \ (j = 1, ..., m), \ (x, t) \in \partial\Omega \times (0, T) \tag{2.9}$$

or

$$\begin{cases} k_{i,1}(p_i,\mathbf{s})\frac{\partial}{\partial n}p_i + \left(k_{i,2}(p_i,\mathbf{s})\sum\limits_{j=1}^{m}\frac{\partial}{\partial n}s_j\right) = 0 \ (i = 1, ..., n), \\ \\ (x,t) \in \partial\Omega \times (0,T) \\ \\ s_j \ = \ 0 \ (j = 1, ..., m), \\ \\ (x,t) \in \partial\Omega \times (0,T) \end{cases} \tag{2.10}$$

as boundary conditions. Now it is easy to show that this coupled system can be reduced to a elliptic system of partial differential equations for the substances $\mathbf{s}$. However, this system of m elliptic partial differential equations for the agents depends on n parameters that depend on the different population densities. Therefore, the generalization of R. Schaaf's result reads as follows:

Theorem 2.6. A pair $(\tilde{\mathbf{p}}, \mathbf{s}) \in \{w \in \mathcal{X}_1 \mid w(\overline{\Omega}) \subset \mathrm{I\!R}^n_+\} \times \{w \in \mathcal{X}_2 \mid w(\overline{\Omega}) \subset \mathrm{I\!R}^m_+\}$ is a solution of (2.8) iff, for $\lambda_i \in \mathrm{I\!R}^+$ with $i \in \{1, ..., n\}$,

$$\tilde{p}_i(x) = \varphi_i(\mathbf{s}(x), \lambda_i) \text{ for all } x \in \overline{\Omega} \text{ and} \tag{2.11}$$

$$0 = \nabla\cdot\left(\sum_{l=1}^{m} A_{lj}(\varphi(\mathbf{s}(x),\lambda),\mathbf{s})\nabla s_l\right) + H_j(\varphi(\mathbf{s}(x),\lambda),\mathbf{s}), \tag{2.12}$$

where $\varphi(\mathbf{s}(x),\lambda)$ denotes $(\varphi_1(\mathbf{s}(x),\lambda_1),...,\varphi_n(\mathbf{s}(x),\lambda_n))^T \in \mathrm{IR}^n_+$.

Here the spaces $\mathcal{X}_i$ are defined as $\{w \in \mathcal{Z} \mid \partial w/\partial n = 0\}$ where $\mathcal{Z}$ is the space $C^{2,\beta}(\overline{\Omega},\mathrm{IR}^n)$ with $0<\beta<1$ for $N>1$ and $C^2(\overline{\Omega},\mathrm{IR}^n)$ for $N=1$ resp. the space $C^{2,\beta}(\overline{\Omega},\mathrm{IR}^m)$ with $0<\beta<1$ for $N>1$ and $C^2(\overline{\Omega},\mathrm{IR}^m)$ for $N=1$. The functions $\varphi_i(s,\lambda_i)$ are given by the functions $r_i(s)$ that solve the equations

$$\frac{d}{ds}r_i(s) = k_{i,2}(r_i,\mathbf{s})/k_{i,1}(r_i,\mathbf{s}),\ r_i(1) = \lambda_i.$$

For multi-species chemotaxis systems like system (2.6) with $\delta_{i,j} = 0$ for all $i,j \in \{1,..n\}, i \neq j$ the steady state problem therefore reduces to the following system of m elliptic equations:

$$0 = \left(\sum_{l=1}^{m} b_{l,j}\Delta s_l\right) - \sum_{l=1}^{m}\gamma_{j,l}s_l + \sum_{k=1}^{n}\alpha_{k,j}\lambda_k \exp\left(\sum_{l=1}^{m}\frac{\omega_{k,l}}{a_k}s_l\right),$$

completed with homogeneous Neumann boundary conditions for all $s_j,\ j \in \{1,..,m\}$. Some concrete results on the existence of steady states for conflict-free systems with homogeneous noflux for the first and Dirichlet boundary conditions for the second equation having a variational structure can be found for $N=2$ in [22], while some steady state analysis for a system with homogeneous Neumann conditions in the presence of conflict in one spatial dimension $N=1$ has been performed in [4].

§ 3. Results for the classical single species chemotaxis model

Before we concentrate on the steady state problem for some multi-species models we look at the results available for the classical single species chemotaxis model. The classical chemotaxis model, i.e. a simplified description of positive chemotactic movement is given by the equations:

$$\left\{\begin{array}{lll} u_t & = \nabla(\nabla u - \chi u \nabla v), & x\in\Omega,\ t>0 \\ v_t & = \Delta v - \tilde{\gamma} v + \alpha u, & x\in\Omega,\ t>0 \\ \partial u/\partial n = \partial v/\partial n & = \ 0, & x \in \partial\Omega,\ t>0 \\ u(0,x) = u_0(x),\ v(0,x) & = \ v_0(x), & x\in\Omega, \end{array}\right. \tag{3.1}$$

where χ, $\tilde{\gamma}$ and α are positive constants.

In 1981 S. Childress and J. K. Percus [2] formulated the following conjecture concerning the time asymptotic behavior of solution for a linear chemotactic sensitivity function

> "*In particular, for the special model we have investigated, collapse cannot occur in a one-dimensional space; may or may not in two dimensions, depending upon the cell population; and must, we surmise, in three or more dimensions under a perturbation of sufficiently high symmetry.*"

This conjecture (and especially the statement about the two dimensional case) has motivated several scientists to analyze the classical chemotaxis model and to study the time-asymptotic behavior of the solution (see [11] and the references therein for several results). Before we briefly go more into details let us introduce the transformations

$$U(x,t) = \frac{u(x,t)}{\int\limits_\Omega u(x,t)dx},\ V(x,t) = \chi\left(v(x,t) - \frac{1}{|\Omega|}\int\limits_\Omega v(x,t)dx\right)$$

$$\gamma = \chi\tilde{\gamma} \text{ and } \lambda = \alpha\chi\int\limits_\Omega u(x,t)dx.$$

Thus we get the new system:

$$\begin{cases} U_t &= \nabla(\nabla U - U\nabla V), & x\in\Omega,\ t>0 \\ V_t &= \Delta V - \gamma V + \lambda\left(U - \frac{1}{|\Omega|}\right), & x\in\Omega,\ t>0 \\ \partial U/\partial n = \partial V/\partial n &= 0, & x\in\partial\Omega,\ t>0 \\ U(0,x) = U_0(x),\ V(0,x) &= V_0(x), & x\in\Omega. \end{cases} \tag{3.2}$$

Now this model fits into the previously considered class of systems that are of variational structure, therefore, we have the following tools at hand:

$$\mathcal{L}[U,V](t) = \frac{1}{2}\int\limits_\Omega |\nabla V|^2 + \gamma V^2\ dx + \lambda\int\limits_\Omega U(\log(U)-1) - 1 - UV dx$$

$$\mathcal{F}[V](t) = \frac{1}{2}\int\limits_\Omega |\nabla V|^2 + \gamma V^2\ dx - \lambda\log\left(\frac{1}{|\Omega|}\int\limits_\Omega e^V dx\right).$$

and the corresponding parameter dependent nonlocal elliptic boundary value problem

$$-\Delta V + \gamma V = \lambda\left(\frac{e^V}{\int\limits_\Omega e^V dx} - \frac{1}{|\Omega|}\right) \text{ in } \Omega,\ \frac{\partial V}{\partial n} = 0 \text{ on } \partial\Omega.$$

Applying Young's inequality we see that one can bound the Lyapunov functional $\mathcal{L}$ from below by $\mathcal{F}$. We see that:

$$U\left(V - \log\left(\frac{1}{|\Omega|}\int_\Omega e^V dx\right)\right) \le U(\log U - 1) + e^{V - \log\left(\frac{1}{|\Omega|}\int_\Omega e^V dx\right)}.$$

holds. Since $\int_\Omega U = 1$, we have:

$$\mathcal{L}\big[U,V\big](t) \ge \mathcal{F}[V](t) := \frac{1}{2}\int\limits_\Omega |\nabla V|^2 + \gamma V^2 dx - \lambda\log\left(\frac{1}{|\Omega|}\int\limits_\Omega e^V dx\right).$$

A typical blow-up result that can be achieved in connection with the conjecture of Childress and Percus for $N = 2$ is given by:

Theorem 3.1 (Horstmann & Wang [9]).
Let $\Omega \subset \mathbb{R}^2$ be a smooth simply connected domain and suppose that $\gamma > 0$. Furthermore, we assume that:

$$4\pi < \lambda \text{ and } \lambda \neq 4\pi m \text{ for } m \in \mathbb{N}.$$

Then there exist a constant $-\infty < \hat{K} \leq 0$ and initial data (U_0, V_0), such that

$$\hat{K} > \mathcal{L}(U_0, V_0).$$

The corresponding solution to this initial data has to blow up in finite or in infinite time.

For similar results and further blow-up results see [10, 11] and the references therein.

§ 3.1. Existence of nontrivial steady states

This result shows how important the steady state analysis is also in connection with the time asymptotic behavior of the solution. Therefore, an important question is whether nontrivial steady states exist or not. Thus, we focus on this question in the following of the present section.

For a simply connected domain $\Omega \subset \mathbb{R}^2$ we consider the nonlocal elliptic boundary value problem

$$0 = \Delta V - \gamma V + \lambda \left(\frac{e^V}{\int\limits_\Omega e^V dx} - \frac{1}{|\Omega|} \right) \text{ in } \Omega, \ \frac{\partial V}{\partial n} = 0 \text{ on } \partial\Omega,$$

where $\gamma \geq 0$ and $\lambda > 0$. This problem is the Euler-Lagrange equation of the variational problem

$$\min_{V \in \mathcal{D}} \frac{1}{2} \int\limits_\Omega |\nabla V|^2 + \gamma V^2 \, dx - \lambda \log \left(\frac{1}{|\Omega|} \int_\Omega e^V dx \right) \equiv \min_{V \in \mathcal{D}} \mathcal{F}(V),$$

where

$$\mathcal{D} := \left\{ V \in H^1(\Omega) \mid \int\limits_\Omega V dx = 0 \right\}.$$

Critical points of this variational problem are therefore solutions of our nonlocal boundary value problem. To show the existence of a minimizer for the variational problem we can use the following exponential Sobolev inequalities that are also known as Moser-Trudinger, resp. Onofri, resp. Chang-Yang type inequalities.

Lemma 3.2. Suppose that $\Omega \subset \mathbb{R}^2$ is a simply connected domain with C^2 boundary. Then for all $V \in \mathcal{D}$ we have that

$$\int_{\Omega} e^V dx \leq k_\Omega \cdot e^{\frac{1}{8\pi} \int_{\Omega} |\nabla V|^2 \ dx}.$$

For radially symmetric $V \in \mathcal{D}$ and $\Omega = B(0,R)$ (a disk in $\mathbb{R}^2$ with radius R) we have:

$$\int_{\Omega} e^V dx \leq k_\Omega \cdot e^{\frac{1}{16\pi} \int_{\Omega} |\nabla V|^2 \ dx + V(R)}.$$

Applying this lemma we see that the variational problem is coercive in both cases, in some parameter range of λ. Therefore, one can use the direct method of the calculus of variations to establish the existence of a minimizer and thus the existence of a possibly nontrivial solution of our nonlocal elliptic boundary value problem.

Theorem 3.3. Suppose $\Omega \subset \mathbb{R}^2$ is simply connected with C^2 boundary. If $\lambda < 4\pi$ holds, then there exists a possibly nontrivial minimizer of the given variational problem. If Ω denotes a disk in $\mathbb{R}^2$ and if one restricts oneself to the radial function belonging to $\mathcal{D}$, then the existence of a minimizer is guaranteed for any $\lambda \leq 8\pi$. This minimizer is given by $V \equiv 0$.

Using the mountain pass structure of the functional $\mathcal{F}$ one can prove the existence of nontrivial steady state solutions for the given Keller-Segel equations. These results can be summarized in the following two statements.

Theorem 3.4 (Horstmann [8], Wang & Wei [20]). Let us suppose that $\Omega \subset \mathbb{R}^2$ is a simply connected domain of class C^2. If $\gamma \geq \lambda$, then there exists a nontrivial solution of the nonlocal elliptic boundary value problem for almost every $\lambda > 4\pi$.

Corollary 3.5 (Horstmann [8]). Let $\Omega \subset \mathbb{R}^2$ denote a simply connected domain of class C^2 and let μ_1 denote the first nonzero eigenvalue of the Laplace-operator with Neumann boundary conditions on $\partial\Omega$. Suppose that

$$4\pi < \mu_1$$

is satisfied. Then there exists at least one nontrivial solution of the nonlocal elliptic boundary value problem for almost every $\lambda \in (4\pi, \mu_1)$ with $0 \leq \gamma < \lambda$.

§ 3.2. Some symmetry properties of steady state solutions

Now we want to analyze some symmetry properties of the steady state solutions. For the rest of our analysis in the present section we consider the following nonlocal elliptic boundary value problem:

$$\text{(3.3)} \qquad \begin{cases} -\Delta V = \lambda \left(\frac{e^V}{\int\limits_\Omega e^V\,dx} - \frac{1}{|\Omega|} \right), \text{ in } \Omega, \\ \partial V/\partial n = 0, \qquad\qquad\qquad \text{on } \partial\Omega, \\ \int_\Omega V = 0. \end{cases}$$

Given a group of isometry G of $\mathbb{R}^2$ we consider the space H^G defined as the class of functions of mean zero which are invariant under the action of G:

$$H^G := \left\{ V \in H^1(B) \,:\, \int_B V = 0 \text{ and } V = V \circ g \;\forall g \in G \right\}.$$

Mostly G will be:

- the group generated by a rotation R_θ of angle θ denoted henceforth $\langle R_\theta \rangle$,
- or the Dihedral group D_n generated by n reflections and a rotation (a group with $2n$ elements).

First one observes the following property of steady states belonging to some $H^{\langle R_\theta \rangle}$, with $\theta = \frac{2\pi}{n}$ $(n \in \mathbb{N})$.

Theorem 3.6 (Horstmann & Lucia [12]). Let $V \in C^2(\overline{B})$ be a solution our nonlocal boundary value problem satisfying:

$$V \in H^{\langle R_\theta \rangle}, \quad \theta = \frac{2\pi}{n} \; (n \in \mathbb{N}),$$

and consider a critical point C of V in $\overline{B} \setminus \{0\}$. Then $V \in H^{D_n}$ where D_n is the Dihedral group with generators R_θ, σ_C, where we use the notations

$$R_\theta := \text{ Rotation of angle } \theta \text{ and } \sigma_C := \text{ Reflection respectively to } OC, \text{ for } C \neq 0.$$

We conclude from the maximum principle that $V \equiv 0$ is the unique solution of the nonlocal problem if $\lambda \leq 0$. For positive λ the uniqueness is not obvious. Let us first consider the following relaxed problem:

$$\text{(3.4)} \qquad \begin{cases} -\Delta V = \lambda \left(\frac{e^V}{\int\limits_\Omega e^V\,dx} - \frac{1}{|\Omega|} \right), \\ V \in C^2(\Omega) \cap C^0(\overline{\Omega}), \quad \mathrm{osc}_{\partial\Omega}(V) \equiv 0. \end{cases}$$

Senba and Suzuki [18] proved that radially symmetric solutions of this problem have to be constant if $\lambda < 8\pi$.

Theorem 3.7 (Horstmann & Lucia [12]). For $\lambda \leq 8\pi$ the constant functions are the unique solution of Problem (3.4).

When the domain Ω is a disk and under the assumption (9) we immediately deduce from the last Theorem the following:

- Any radial solutions of (3.4) must be constant whenever $\lambda \leq 8\pi$;
- $V \equiv 0$ is the unique radial solution of Problem (3.3) whenever $\lambda \leq 8\pi$.

The uniqueness result about radial solutions raises the following question:

In a ball B are there any non-trivial solutions to the nonlocal elliptic boundary value problem that are invariant under a group of isometries G of B?

Of course a natural attempt to get such solutions is to consider the minimizing problem in the space H^G. In the simplest case when $G = \{id, R_\pi\}$ (the group generated by a rotation of angle π), this approach has been used by Senba and Suzuki [18] who proved:

$$\mathcal{F}|_{H^G} \text{ admits a minimizer for each } \lambda < 8\pi. \tag{3.5}$$

However, one can generalize such a result as follows [12]: Consider the group G generated by a rotation of angle $2\pi/n$ ($n \in \mathbb{N}$), then conclusion (3.5) holds.

Theorem 3.8 (Horstmann & Lucia [12]). If $V \in H^{\langle R_{2\pi/n}\rangle}\backslash\{0\}$ is a solution of problem (3.3), then

$$\lambda > \Lambda_n := \begin{cases} \frac{64}{\pi} \text{ if } n = 2, \\ 8\pi \text{ if } n \geq 3. \end{cases} \tag{3.6}$$

Our above theorem gives several optimal exponential Sobolev inequalities. Indeed one has for each $\lambda < \Lambda_n$ the following optimal inequality:

$$\frac{1}{|B|}\int_B e^V \leq e^{\frac{\pi}{128}\int_B |\nabla V|^2} \quad \forall V \in H^{\langle R_\pi\rangle}$$

and for $n \geq 3$:

$$\frac{1}{|B|}\int_B e^V \leq e^{\frac{1}{16\pi}\int_B |\nabla V|^2} \quad \forall V \in H^{\langle R_{2\pi/n}\rangle}.$$

Summarizing the results for the steady state problem of the single species model we have the following results:

1. Suppose $\Omega \subset \mathbb{R}^2$ is simply connected and its boundary is piecewise of class C^2. For $\lambda < 4\theta$ there exists a possibly nontrivial minimizer of the functional $\mathcal{F}(V)$ over the set of functions from $H^1(\Omega)$ having mean value equal to zero. Here θ denotes the smallest interior angle of $\partial\Omega$ [compare Gajewski & Zacharias [6]].

2. Suppose $\Omega \subset \mathbb{R}^2$ is a simply connected domain and its boundary is piecewise of class C^2, $\gamma \geq 1$ and $\lambda > 4\theta$. Then there exists a nontrivial solution of the nonlocal elliptic boundary value problem [compare Horstmann [8] and Wang & Wei [20]].

3. Let Ω denote a disk in $\mathbb{R}^2$, $\gamma = 0$ and $\lambda \leq 8\pi$, then there exists no nontrivial solution of the nonlocal elliptic boundary value problem in the class of all radial symmetric functions [compare Horstmann & Lucia [12]].

4. Suppose Ω is a disk in $\mathbb{R}^2$, $\gamma = 0$ and $\lambda \leq 8\pi$. Furthermore let R_θ denote a rotation of angle θ and $H^{R_\theta} := \{u \in H^1(\Omega) \, : \, u \circ g = u, \quad \forall g \in R_\theta\}$. Then there exists a global minimizer of the functional $\mathcal{F}(V)$ in the class $H^{R_{2\pi/n}}$. For $n \geq 3$ this minimizer is given by the trivial solution $V \equiv 0$. For $n = 2$ the minimizer is also the trivial solution $V \equiv 0$, provided $\lambda\pi \leq 64$ [compare Horstmann & Lucia [12]].

§ 3.3. Consequences for the time asymptotic behavior

For the time dependent problem these results allow us to draw the following conclusions. Suppose $\gamma = 0$. Let $(U, V) \in H^{\langle R_{2\pi/n} \rangle}$ denote a solution of the chemotaxis equations. For $\lambda < 8\pi$ we have that:

$$\mathcal{L}\big[U, V\big](t) \geq \frac{1}{2}\left(1 - \frac{\lambda}{8\pi}\right) \int_\Omega |\nabla V|^2 dx - C, \quad \forall V \in H^{\langle R_{2\pi/n} \rangle}.$$

According to Gajewski & Zacharias in [6] one can prove the global existence and the uniform boundedness of all terms of the Lyapunov functional for any $t \geq 0$. In fact $\mathcal{L}[U, V](t) \geq -C$ implies that for a solution that exists global in time

$$\frac{d}{dt}\mathcal{L}[U, V](t) \to 0 \text{ as } t \to \infty \text{ has to hold.}$$

From this we conclude:

$$\lim_{t\to\infty} \left\| \frac{\partial V}{\partial t}(\cdot, t) \right\|^2_{L^2(\Omega)} = 0 \text{ and } \lim_{t\to\infty} \int_\Omega U(t) \big|\nabla(\log U(t) - V(t))\big|^2 dx = 0.$$

Standard arguments show that $(U, V) \in L^\infty\big(0, \infty; H^{D_n} \times H^{D_n}\big)$. Applying recent results by Feireisl, Laurençot & Petzeltová [5] it follows that classical solutions have to converge to steady state solutions in the C^1-sense as $t \to \infty$.

For weak solutions it has been shown in [Gajewski & Zacharias [6]] that this convergence holds true for a subsequence $(t_k)_{k\in\mathbb{N}}$ such that

$$U(t_k) \to |\Omega| e^{V^*} / \int_\Omega e^{V^*} dx$$

in $L^2(\Omega)$ and $V(t_k) \to V^*$ in $H^1(\Omega)$ as $t_k \to \infty$. Here V^* denotes a solution of the nonlocal elliptic boundary value problem.

Therefore we see that for $\lambda < \Lambda_n$ (Λ_n as given in (3.6) the solution $(U, V) \in H^{D_n} \times H^{D_n}$ of the chemotaxis system has to converge to the uniform distribution as $t \to \infty$, since $V \equiv 0$ is the unique solution of the nonlocal elliptic boundary value problem in this case. Since any radial function belongs to each H^{D_n}, this observation also implies that:

Corollary 3.9 (Horstmann & Lucia [12]). Consider the Keller-Segel model in a disk and let (U, V) be a radial solution. If $\lambda < 8\pi$, then the solution is globally defined and $\lim_{t\to\infty}(U, V)(\cdot, t) = (1, 0)$ in $C^1(\overline{\Omega})$.

Furthermore, we see that:

Theorem 3.10 (Horstmann & Lucia [12]). Consider the Keller-Segel model in a disk and let (U, V) be a C^2-solution belonging to the space $H^{D_n} \times H^{D_n}$. If $\lambda < \Lambda_n$ with Λ_n defined in (3.6), then the solution is globally defined and $\lim_{t\to\infty}(U, V)(\cdot, t) = (1, 0)$ in $C^1(\overline{\Omega})$.

§ 4. Results for the steady state problem of some multi-species system

In this section we want to pay our attention to some simplified multi-species chemotaxis models of the following type:

$$
\begin{cases}
u_t = \nabla(\nabla u - \chi_1 u \nabla v), & x \in \Omega,\ t > 0 \\
w_t = \nabla(\nabla w \pm \chi_2 w \nabla v), & x \in \Omega,\ t > 0 \\
v_t = \Delta v - \gamma v + \alpha_1 u \mp \alpha_2 w, & x \in \Omega,\ t > 0 \\
\frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = \frac{\partial v}{\partial n} = 0 & x \in \partial\Omega,\ t > 0 \\
u(0, x) = u_0(x),\ w(0, x) = w_0(x), & x \in \Omega, \\
v(0, x) = v_0(x), & x \in \Omega.
\end{cases}
\tag{4.1}
$$

We remark that the first two equations imply together with the boundary conditions a conservation of the initial mass for all times. Furthermore, four different situations are given. For

$$
\begin{cases}
u_t = \nabla(\nabla u - \chi_1 u \nabla v), & x \in \Omega,\ t > 0 \\
w_t = \nabla(\nabla w - \chi_2 w \nabla v), & x \in \Omega,\ t > 0 \\
v_t = \Delta v - \gamma v + \alpha_1 u - \alpha_2 w, & x \in \Omega,\ t > 0
\end{cases}
\tag{4.2}
$$

and

$$
\begin{cases}
u_t = \nabla(\nabla u - \chi_1 u \nabla v), & x \in \Omega,\ t > 0 \\
w_t = \nabla(\nabla w + \chi_2 w \nabla v), & x \in \Omega,\ t > 0 \\
v_t = \Delta v - \gamma v + \alpha_1 u + \alpha_2 w, & x \in \Omega,\ t > 0
\end{cases}
\tag{4.3}
$$

the equations are in the presence of a conflict of interests, while the system is conflict-free for

$$\begin{cases} u_t = \nabla(\nabla u - \chi_1 u \nabla v), & x \in \Omega,\ t > 0 \\ w_t = \nabla(\nabla w + \chi_2 w \nabla v), & x \in \Omega,\ t > 0 \\ v_t = \Delta v - \gamma v + \alpha_1 u - \alpha_2 w, & x \in \Omega,\ t > 0 \end{cases} \tag{4.4}$$

and

$$\begin{cases} u_t = \nabla(\nabla u - \chi_1 u \nabla v), & x \in \Omega,\ t > 0 \\ w_t = \nabla(\nabla w - \chi_2 w \nabla v), & x \in \Omega,\ t > 0 \\ v_t = \Delta v - \gamma v + \alpha_1 u + \alpha_2 w, & x \in \Omega,\ t > 0 \end{cases} \tag{4.5}$$

Processing in the same way as in the single species case we now introduce the new variables

$$U(x,t) = \frac{u(x,t)}{\int\limits_\Omega u(x,t)dx}, W(x,t) = \frac{w(x,t)}{\int\limits_\Omega w(x,t)dx}$$

and

$$V(x,t) = v(x,t) - \frac{1}{|\Omega|}\int\limits_\Omega v(x,t)dx$$

as well as the new parameters

$$\nu^\pm = \alpha_1 \int\limits_\Omega u(x,t)dx \pm \alpha_2 \int\limits_\Omega w(x,t)dx,\ \lambda = \frac{\alpha_1}{\nu^\pm}\int\limits_\Omega u(x,t)dx$$

and

$$\mu = \frac{\alpha_2}{\nu^\pm}\int\limits_\Omega w(x,t)dx \text{ with } \lambda \pm \mu = 1.$$

Before we go on, let us make the following important convention for our analysis:

Hypothesis 4.1. Throughout the present paper we assume that ν^+ resp. ν^- are positive.

For our later observations we remark here that according to our hypothesis the following limiting behavior is possible:

1. $\nu^\pm \to \infty$ with $\lambda \to 1$ and $\mu \to 0$
2. $\nu^+ \to \infty$ with $\lambda \to 0$ and $\mu \to 1$
3. $\nu^\pm \to \infty$ with $\lambda \to \frac{\alpha_1}{\alpha_1 \pm \alpha_2}$ and $\mu \to \frac{\alpha_2}{\alpha_1 \pm \alpha_2}$

Therefore, the given transformations and new notations lead us to the new systems:

$$(4.6)\qquad \begin{cases} U_t &= \nabla(\nabla U - \chi_1 U \nabla V), & x \in \Omega,\ t>0\\ W_t &= \nabla(\nabla W \pm \chi_2 W \nabla V), & x \in \Omega,\ t>0\\ V_t &= \Delta V - \gamma V + \nu^{\pm}\left(\lambda U \pm \mu W - \frac{1}{|\Omega|}\right), & x \in \Omega,\ t>0\\ \frac{\partial U}{\partial n} &= \frac{\partial W}{\partial n} = \frac{\partial V}{\partial n} = 0 & x \in \partial\Omega,\ t>0\\ U(0,x) &= U_0(x),\ W(0,x) = W_0(x), & x \in \Omega\\ V(0,x) &= V_0(x), & x \in \Omega. \end{cases}$$

Thus the corresponding steady state problems with homogeneous Neumann boundary conditions read as follows:

$$(4.7)\qquad 0 = \Delta V - \gamma V + \nu^{\pm}\left(\lambda \frac{e^{\chi_1 V}}{\int\limits_\Omega e^{\chi_1 V}dx} \mp \mu \frac{e^{\mp\chi_2 V}}{\int\limits_\Omega e^{\mp\chi_2 V}dx} - \frac{1}{|\Omega|}\right).$$

Consequently, the corresponding four variational problems are now given by the two situations in the presence of conflict:

$$\min_{V\in\mathcal{D}} \frac{1}{2}\int\limits_\Omega |\nabla V|^2 + \gamma V^2\ dx - \nu^-\lambda \log\left(\frac{1}{|\Omega|}\int_\Omega e^{\chi_1 V}dx\right) + \nu^-\mu\log\left(\frac{1}{|\Omega|}\int_\Omega e^{\chi_2 V}dx\right)$$

$$\equiv \min_{V\in\mathcal{D}} \mathcal{F}^{\nu^-,\chi_2^+}_{\lambda,\mu}(V),$$

$$\min_{V\in\mathcal{D}} \frac{1}{2}\int\limits_\Omega |\nabla V|^2 + \gamma V^2\ dx - \nu^+\lambda \log\left(\frac{1}{|\Omega|}\int_\Omega e^{\chi_1 V}dx\right) + \nu^+\mu\log\left(\frac{1}{|\Omega|}\int_\Omega e^{-\chi_2 V}dx\right)$$

$$\equiv \min_{V\in\mathcal{D}} \mathcal{F}^{\nu^+,\chi_2^-}_{\lambda,\mu}(V),$$

and the two conflict-free problems:

$$\min_{V\in\mathcal{D}} \frac{1}{2}\int\limits_\Omega |\nabla V|^2 + \gamma V^2\ dx - \nu^+\lambda \log\left(\frac{1}{|\Omega|}\int_\Omega e^{\chi_1 V}dx\right) - \nu^+\mu\log\left(\frac{1}{|\Omega|}\int_\Omega e^{\chi_2 V}dx\right)$$

$$\equiv \min_{V\in\mathcal{D}} \mathcal{F}^{\nu^+,\chi_2^+}_{\lambda,\mu}(V),$$

and

$$\min_{V\in\mathcal{D}} \frac{1}{2}\int\limits_\Omega |\nabla V|^2 + \gamma V^2\ dx - \nu^-\lambda \log\left(\frac{1}{|\Omega|}\int_\Omega e^{\chi_1 V}dx\right) - \nu^-\mu\log\left(\frac{1}{|\Omega|}\int_\Omega e^{-\chi_2 V}dx\right)$$

$$\equiv \min_{V\in\mathcal{D}} \mathcal{F}^{\nu^-,\chi_2^-}_{\lambda,\mu}(V),$$

where in all four cases

$$\mathcal{D} := \left\{ V \in H^1(\Omega) \mid \int\limits_{\Omega} V dx = 0 \right\}.$$

Applying the exponential Sobolev inequalities and remembering that according to Jensen's inequality

$$\log\left(\frac{1}{|\Omega|}\int_\Omega e^{\mp\chi_2 V} dx\right) \geq 0$$

holds for all $V \in \mathcal{D}$, we can state the following results:

Theorem 4.1. Suppose $\Omega \subset \mathbb{R}^2$ is simply connected with C^2 boundary. If $\nu^{\mp}\lambda\chi_1 < 4\pi$ holds, then there exists a possibly nontrivial minimizer of the given variational problem with $\mathcal{F}_{\lambda,\mu}^{\nu^-,\chi_2^+}(V)$ or $\mathcal{F}_{\lambda,\mu}^{\nu^+,\chi_2^-}(V)$. If Ω denotes a disk in $\mathbb{R}^2$ and if one restricts oneself to the radial function belonging to $\mathcal{D}$, then the existence of a minimizer is guaranteed for any $\nu^{\pm}\lambda\chi_1 \leq 8\pi$. This minimizer is given by $V \equiv 0$.

Theorem 4.2. Suppose $\Omega \subset \mathbb{R}^2$ is simply connected with C^2 boundary. If $\nu^{\pm}(\chi_1\lambda + \chi_2\mu) < 4\pi$ holds, then there exists a possibly nontrivial minimizer of the given variational problem with $\mathcal{F}_{\lambda,\mu}^{\nu^+,\chi_2^+}(V)$ resp. $\mathcal{F}_{\lambda,\mu}^{\nu^-,\chi_2^-}(V)$. If Ω denotes a disk in $\mathbb{R}^2$ and if one restricts oneself to the radial function belonging to $\mathcal{D}$, then the existence of a minimizer is guaranteed for any $\nu^{\pm}(\chi_1\lambda + \chi_2\mu) \leq 8\pi$. This minimizer is given by $V \equiv 0$.

Remark 4.3. The given conditions on the existence of global minimizers for the present variational problems correspond to condition (iii) in [22, Theroem 4] for the boundedness of the Lyapunov functionals considered there and to the one given in [3, Theorem 2] to guarantee the global existence of the solution to the explicit conflict-free multi-species model studied therein. To simplify the comparison we remark that the inequalities

$$\nu^{\pm}(\chi_1\lambda + \chi_2\mu) < 4\pi \text{ resp. } \nu^{\pm}(\chi_1\lambda + \chi_2\mu) < 8\pi$$

can be rewritten as:

$$\nu^{\pm}(\chi_1\lambda + \chi_2\mu)(\chi_2\lambda + \chi_1\mu) < 4\pi(\chi_2\lambda + \chi_1\mu)$$

resp.

$$\nu^{\pm}(\chi_1\lambda + \chi_2\mu)(\chi_2\lambda + \chi_1\mu) < 8\pi(\chi_2\lambda + \chi_1\mu)$$

Remembering the definition of our used notation this is transformed into

$$\left(\alpha_1 \int\limits_{\Omega} u_0(x)dx + \alpha_2 \int\limits_{\Omega} w_0(x)dx\right)^2 < 4\pi\left(\frac{\alpha_1}{\chi_1}\int\limits_{\Omega} u_0(x)dx + \frac{\alpha_2}{\chi_2}\int\limits_{\Omega} w_0(x)dx\right)$$

resp.

$$\left(\alpha_1 \int\limits_\Omega u_0(x)dx + \alpha_2 \int\limits_\Omega w_0(x)dx\right)^2 < 8\pi\left(\frac{\alpha_1}{\chi_1}\int\limits_\Omega u_0(x)dx + \frac{\alpha_2}{\chi_2}\int\limits_\Omega w_0(x)dx\right)$$

Let us now look a little bit closer at the structure of the four given functionals. Calculating the first and the second variation of the functionals we get

$$\begin{aligned}\frac{d}{d\delta}\mathcal{F}^{\nu^\mp,\chi_2^+}_{\lambda,\mu}(V+\delta\varphi)|_{\delta=0} = &\int\limits_\Omega \nabla V\nabla\varphi + \gamma V\varphi\ dx\\ &-\nu^\mp\lambda\chi_1\frac{\int\limits_\Omega \varphi e^{\chi_1 V}\ dx}{\frac{1}{|\Omega|}\int_\Omega e^{\chi_1 V}dx} \pm \nu^\mp\mu\chi_2\frac{\int\limits_\Omega \varphi e^{\chi_2 V}\ dx}{\frac{1}{|\Omega|}\int_\Omega e^{\chi_2 V}dx}\end{aligned}$$

$$\begin{aligned}\frac{d}{d\delta}\mathcal{F}^{\nu^\mp,\chi_2^-}_{\lambda,\mu}(V+\delta\varphi)|_{\delta=0} = &\int\limits_\Omega \nabla V\nabla\varphi + \gamma V\varphi\ dx\\ &-\nu^\mp\lambda\chi_1\frac{\int\limits_\Omega \varphi e^{\chi_1 V}\ dx}{\frac{1}{|\Omega|}\int_\Omega e^{\chi_1 V}dx} \pm \nu^\mp\mu\chi_2\frac{\int\limits_\Omega \varphi e^{-\chi_2 V}\ dx}{\frac{1}{|\Omega|}\int_\Omega e^{-\chi_2 V}dx}\end{aligned}$$

$$\begin{aligned}\frac{d^2}{d\delta^2}\mathcal{F}^{\nu^\mp,\chi_2^+}_{\lambda,\mu}(V+\delta\varphi)|_{\delta=0} = &\int\limits_\Omega |\nabla\varphi|^2 + \gamma\varphi^2\ dx\\ &-\nu^\mp\lambda\chi_1^2\frac{\int_\Omega e^{\chi_1 V}dx\int\limits_\Omega \varphi^2 e^{\chi_1 V}\ dx - \left(\int\limits_\Omega \varphi e^{\chi_1 V}\ dx\right)^2}{|\Omega|\left(\frac{1}{|\Omega|}\int_\Omega e^{\chi_1 V}dx\right)^2}\\ &\pm\nu^\mp\mu\chi_2^2\frac{\int_\Omega e^{\chi_2 V}dx\int\limits_\Omega \varphi^2 e^{\chi_2 V}\ dx - \left(\int\limits_\Omega \varphi e^{\chi_2 V}\ dx\right)^2}{|\Omega|\left(\frac{1}{|\Omega|}\int_\Omega e^{\chi_2 V}dx\right)^2}\end{aligned}$$

$$\begin{aligned}\frac{d^2}{d\delta^2}\mathcal{F}^{\nu^\mp,\chi_2^-}_{\lambda,\mu}(V+\delta\varphi)|_{\delta=0} = &\int\limits_\Omega |\nabla\varphi|^2 + \gamma\varphi^2\ dx\\ &-\nu^\mp\lambda\chi_1^2\frac{\int_\Omega e^{\chi_1 V}dx\int\limits_\Omega \varphi^2 e^{\chi_1 V}\ dx - \left(\int\limits_\Omega \varphi e^{\chi_1 V}\ dx\right)^2}{|\Omega|\left(\frac{1}{|\Omega|}\int_\Omega e^{\chi_1 V}dx\right)^2}\\ &\mp\nu^\mp\mu\chi_2^2\frac{\int_\Omega e^{-\chi_2 V}dx\int\limits_\Omega \varphi^2 e^{-\chi_2 V}\ dx - \left(\int\limits_\Omega \varphi e^{-\chi_2 V}\ dx\right)^2}{|\Omega|\left(\frac{1}{|\Omega|}\int_\Omega e^{-\chi_2 V}dx\right)^2}\end{aligned}$$

for all $\varphi \in \mathcal{D}$. Therefore, we conclude that

$$V \equiv 0 \text{ is a strict local minima of } \mathcal{F}_{\lambda,\mu}^{\nu^{\mp},\chi_2^{+}} \text{ for all } \nu^{\mp}(\lambda\chi_1^2 \mp \mu\chi_2^2) \leq \mu_1 + \gamma$$

and

$$V \equiv 0 \text{ is a strict local minima of } \mathcal{F}_{\lambda,\mu}^{\nu^{\mp},\chi_2^{-}} \text{ for all } \nu^{\mp}(\lambda\chi_1^2 \pm \mu\chi_2^2) \leq \mu_1 + \gamma.$$

Remark 4.4. We want to point out that this is completely different from the situation given and considered in [16]. There a similar problem with homogeneous Dirichlet conditions is analyzed, where the given conditions exclude $V \equiv 0$ as a solution.

§ 4.1. Unboundedness of the functionals from below under the condition that $\nu^{\pm}(\chi_1\lambda + \chi_2\mu) > 4\pi$

For the rest of the present paper we focus at the conflict-free situations, i.e. we look at the functionals $\mathcal{F}_{\lambda,\mu}^{\nu^{+},\chi_2^{+}}$ and $\mathcal{F}_{\lambda,\mu}^{\nu^{-},\chi_2^{-}}$, where we will use the notation $\mathcal{F}_{\lambda,\mu}^{\nu^{\pm},\chi_2^{\pm}}$ for these two functionals for briefness.

Now in the same way as it has been done in [8] for the single species case we can show that for

$$\nu^{\pm}(\chi_1\lambda + \chi_2\mu) > 4\pi \text{ and } \nu^{+}(\lambda\chi_1^2 + \mu\chi_2^2) \leq \mu_1 + \gamma$$

there exists a $1 > \varepsilon_0 > 0$ and a function $v_{\varepsilon_0} \in \mathcal{D}$ such that

$$\mathcal{F}_{\lambda,\mu}^{\nu^{+},\chi_2^{+}}(v_{\varepsilon_0}) < 0 \text{ and } ||\nabla v_{\varepsilon_0}||^2_{L^2(\Omega)} \geq 1 \text{ holds.}$$

The functional $\mathcal{F}_{\lambda,\mu}^{\nu^{-},\chi_2^{-}}$ has a mountain pass structure and is unbounded from below, if

$$\nu^{-} \min\{\chi_1\lambda, \chi_2\mu\} > 4\pi \text{ and } \nu^{-}(\lambda\chi_1^2 + \mu\chi_2^2) \leq \mu_1 + \gamma,$$

i. e. there exists a $1 > \varepsilon_0 > 0$ and a function $v_{\varepsilon_0} \in \mathcal{D}$ such that

$$\mathcal{F}_{\lambda,\mu}^{\nu^{-},\chi_2^{-}}(v_{\varepsilon_0}) < 0 \text{ and } ||\nabla v_{\varepsilon_0}||^2_{L^2(\Omega)} \geq 1 \text{ holds.}$$

To show that the functionals $\mathcal{F}_{\lambda,\mu}^{\nu^{\pm},\chi_2^{\pm}}$ are no longer bounded from below in this cases we restrict ourselves for simplicity to the case that $\Omega = B(y_0, 1)$ is a disk in $\mathbb{R}^2$ and $x_0 = 0 \in \partial\Omega$.

Now we look for a sequence $(v_\varepsilon)_{\varepsilon \geq 0} \subset H^1(\Omega)$, with

$$\lim_{\varepsilon \to 0} \mathcal{F}_{\lambda,\mu}^{\nu^{\pm},\chi_2^{\pm}}(v_\varepsilon) = -\infty.$$

Exactly as in the single species case that has been considered in [8] we choose

$$
\begin{aligned}
v_\varepsilon(x) &= \log\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x-x_0|^2)^2}\right) - \frac{1}{|\Omega|}\int_\Omega \log\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x-x_0|^2)^2}\right) \\
&= u_\varepsilon(x) - \frac{1}{|\Omega|}\int_\Omega u_\varepsilon(x).
\end{aligned}
$$

Remark 4.5. Since each

$$
\psi_{\varepsilon,x_0}(x) \equiv u_\varepsilon(x) + \log(8\pi) = \log\left(\frac{8\pi\varepsilon^2}{(\varepsilon^2+\pi|x-x_0|^2)^2}\right),
$$

$\varepsilon > 0,\ x_0 \in \mathbf{R}^2$, is a solution of

$$
\begin{aligned}
-\Delta\psi(x) &= e^{\psi(x)},\ x \in \mathbf{R}^2 \\
\int_{\mathbf{R}^2} e^{\psi(x)}dx &< \infty,
\end{aligned}
$$

the sequence $(v_\varepsilon)_{\varepsilon\geq 0}$ with

$$
v_\varepsilon = \psi_{\varepsilon,x_0}(x) - \frac{1}{|\Omega|}\int_\Omega \psi_{\varepsilon,x_0}(x)\ dx \in \mathcal{D}
$$

and $x_0 \in \overline{\Omega}$ seems to be a good choice for a sequence of functions with the desired properties in the limiting situation $\varepsilon \to 0$.

Lemma 4.6. Let Ω be as stated above. Suppose $\nu^+(\chi_1\lambda+\chi_2\mu) > 4\pi$ and $\varepsilon \to 0$ then

$$
\mathcal{F}_{\lambda,\mu}^{\nu^+,\chi_2^+}(v_\varepsilon) \to -\infty \text{ and } \int_\Omega |\nabla v_\varepsilon|^2 dx \to \infty.
$$

For $\nu^- \min\{\chi_1\lambda, \chi_2\mu\} > 4\pi$ and $\varepsilon \to 0$ either

$$
\mathcal{F}_{\lambda,\mu}^{\nu^-,\chi_2^-}(v_\varepsilon) \to -\infty \text{ and } \int_\Omega |\nabla v_\varepsilon|^2 dx \to \infty
$$

or

$$
\mathcal{F}_{\lambda,\mu}^{\nu^-,\chi_2^-}(-v_\varepsilon) \to -\infty \text{ and } \int_\Omega |\nabla v_\varepsilon|^2 dx \to \infty
$$

Remark 4.7. For the analogous result in the radially symmetric case one has to consider the given sequence with x_0 as the center of the disk $\Omega = B(0,1)$.

Proof:
We see that

$$\frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2\ dx = \frac{16\pi^2}{2}\int_\Omega \frac{|x|^2}{(\varepsilon^2+\pi|x|^2)^2}\ dx$$

$$\frac{\gamma}{2}\int_\Omega v_\varepsilon^2\ dx = \frac{\gamma}{2}\int_\Omega (\log(\varepsilon^2+\pi|x|^2)^2)^2 dx - \frac{\gamma}{2}\int_\Omega \log(\varepsilon^2+\pi|x|^2)^{2^2}$$

$$\log\frac{1}{\pi}\int_\Omega e^{\chi_1 v_\varepsilon} = \log\frac{1}{\pi}\int_\Omega e^{\chi_1 \log\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right)} - \frac{\chi_1}{\pi}\int_\Omega \log\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right)$$

$$\log\frac{1}{\pi}\int_\Omega e^{\pm\chi_2 v_\varepsilon} = \log\frac{1}{\pi}\int_\Omega e^{\pm\chi_2 \log\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right)} - \pm\frac{\chi_2}{\pi}\int_\Omega \log\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right).$$

Now we look a bit closer at

$$\frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2\ dx.$$

Using the substitution $y = x/\varepsilon$ and polar coordinates with respect to x_0 we can calculate after an integration

$$\begin{aligned}\frac{1}{2}||\nabla v_\varepsilon||^2_{L^2(\Omega)} &= 8\pi^3\int_0^{2\varepsilon^{-1}} \frac{r^3}{(1+\pi r^2)^2}\ dr - Error\\ &= 4\pi^3\left(\frac{\varepsilon^2}{\pi^2(\varepsilon^2+4\pi)} + \frac{\log\left(\frac{1}{\varepsilon^2}(\varepsilon^2+4\pi)\right)}{\pi^2} - \frac{1}{\pi^2}\right) - Error\\ &= 8\pi\log\left(\frac{1}{\varepsilon}\right) + O(1) - Error.\end{aligned}$$

Now we remark that the $Error$-term in this calculation is positive. Therefore, we see that

$$\frac{1}{2}||\nabla v_\varepsilon||^2_{L^2(\Omega)} \le 8\pi\log\left(\frac{1}{\varepsilon}\right) + O(1).$$

In the same way we get after an integration of v_ε^2:

$$\begin{aligned}\frac{\gamma}{2}\int_\Omega v_\varepsilon^2\ dx &= 2\pi\gamma\int_0^2 r(\log(\varepsilon^2+\pi r^2))^2\ dr - 2\gamma\int_\Omega \log(\varepsilon^2+\pi|x|^2)^2 - Error\\ &\le 2\pi\gamma\int_0^2 r(\log(\varepsilon^2+\pi r^2))^2\ dr - 2\gamma\int_\Omega \log(\varepsilon^2+\pi|x|^2)^2 \ = \ O(1)\end{aligned}$$

Now we have to determine the last two expressions. Here, we have to be more careful. This time we make no use of the substitution applied for the other two integrals. We see that for positive χ_i:

$$\log \frac{1}{\pi} \int\limits_{\Omega} \left(\frac{\varepsilon^2}{(\varepsilon^2 + \pi|x|^2)^2} \right)^{\chi_i} = \log \frac{1}{\pi} \int\limits_{\Omega} \frac{\varepsilon^{2\chi_i}}{(\varepsilon^2 + \pi|x|^2)^{2\chi_i}} dx$$
$$= O(1)$$

and

$$\begin{aligned} \frac{\chi_i}{\pi} \int\limits_{\Omega} \log \left(\frac{\varepsilon^2}{(\varepsilon^2 + \pi|x|^2)^2} \right) &= \frac{\chi_i}{\pi} \int\limits_{\Omega} u_\varepsilon \; dx \\ &= 2\chi_i \log(\varepsilon) - 2\frac{\chi_i}{\pi} \int\limits_{\Omega} \log(\varepsilon^2 + \pi|x|^2) \; dx \\ &= 2\chi_i \log(\varepsilon) + O(1). \end{aligned}$$

for $i \in \{1, 2\}$. This all together implies our claim that

$$\mathcal{F}_{\lambda,\mu}^{\nu^+,\chi_2^+}(v_\varepsilon) \leq \big(8\pi - 2\nu^+(\chi_1\lambda + \chi_2\mu)\big) \log \left(\frac{1}{\varepsilon} \right) + O(1)$$

and therefore

$$\mathcal{F}_{\lambda,\mu}^{\nu^+,\chi_2^+}(v_\varepsilon) \to -\infty \text{ as } \epsilon \to 0.$$

The claim for $\mathcal{F}_{\lambda,\mu}^{\nu^-,\chi_2^-}(v_\varepsilon)$ follows from the inequalities

$$\mathcal{F}_{\lambda,\mu}^{\nu^-,\chi_2^-}(v) \leq \frac{1}{2} \int\limits_{\Omega} |\nabla v|^2 + \gamma v^2 \; dx - \nu^- \lambda \log \left(\frac{1}{|\Omega|} \int_{\Omega} e^{\chi_1 v} dx \right) \tag{4.8}$$

and

$$\mathcal{F}_{\lambda,\mu}^{\nu^-,\chi_2^-}(v) \leq \frac{1}{2} \int\limits_{\Omega} |\nabla v|^2 + \gamma v^2 \; dx - \nu^- \mu \log \left(\frac{1}{|\Omega|} \int_{\Omega} e^{-\chi_2 v} dx \right) \tag{4.9}$$

that hold for all $v \in \mathcal{D}$. □

Remark 4.8. The calculations presented above give us some insights to the structure of the functional $\mathcal{F}_{\lambda,\mu}^{\nu^-,\chi_2^+}$ for a situation in the presence of conflicts. Let Ω be as stated above. Suppose $\nu^-(\chi_1\lambda - \chi_2\mu) > 4\pi$ and $\varepsilon \to 0$ then

$$\mathcal{F}_{\lambda,\mu}^{\nu^-,\chi_2^+}(v_\varepsilon) \to -\infty \text{ and } \int\limits_{\Omega} |\nabla v_\varepsilon|^2 dx \to \infty.$$

Remembering that the functionals are bounded from below and coercive for $\nu^\pm \chi_1 \lambda < 4\pi$, the following questions are obvious (using again the simplified notation $\mathcal{F}_{\lambda,\mu}^{\nu^\mp,\chi_2^\pm}$):

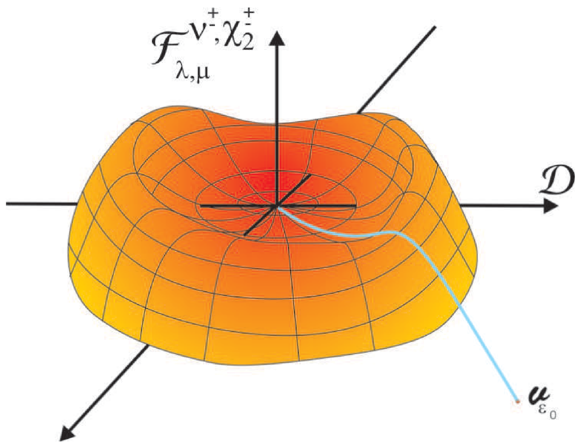

Figure 1. Sketch of the mountain pass structure of the functionals $\mathcal{F}^{\nu^{\pm},\chi_2^{\pm}}_{\lambda,\mu}$.

1. Are the functionals $\mathcal{F}^{\nu^{\mp},\chi_2^{\pm}}_{\lambda,\mu}$ bounded from below for $4\pi \leq \nu^{\pm}\chi_1\lambda \leq 4\pi + \nu^{\pm}\chi_2\mu$?
2. If the functionals $\mathcal{F}^{\nu^{\mp},\chi_2^{\pm}}_{\lambda,\mu}$ are bounded from below for $4\pi \leq \nu^{\pm}\chi_1\lambda \leq 4\pi + \nu^{\pm}\chi_2\mu$ are there any nontrivial minimizers in $\mathcal{D}$?

Remark 4.9. The statement of the lemma remains also true for more general domains (with piecewise smooth boundary) where one might have to replace π by the smallest interior angle of the boundary $\partial\Omega$ of the considered domain Ω.

Now it seems as if the famous mountain pass theorem by Ambrosetti and Rabinowitz might be applicable to establish the existence of a nontrivial solution in the given conflict-free cases, but unfortunately the functionals do not satisfy the Palais-Smale condition. Therefore, we have to proceed in a slightly different way.

§ 4.2. Existence of nontrivial steady state solutions in the conflict-free cases

According to our observations concerning the limiting behavior of $\lambda(\nu)$ and $\mu(\nu)$ as $\nu^{\pm} \to \infty$ we see that

$$\mathcal{F}^{s,\chi_2^{\pm}}_{\lambda(s),\mu(s)}(v_{\epsilon_0}) < \mathcal{F}^{\nu^{\pm},\chi_2^{\pm}}_{\lambda(\nu^{\pm}),\mu(\nu^{\pm})}(v_{\epsilon_0}) < 0$$

for every $s \geq \nu^{\pm}$, with $\nu^{+}(\chi_1\lambda + \chi_2\mu) > 4\pi$ resp. $\nu^{-}\min\{\chi_1\lambda, \chi_2\mu\} > 4\pi$.

However, since λ and μ depend both on $\nu^{\pm}$ the case $\lambda \neq \mu$ is a little bit more subtle than it seems at a first glance.

Remark 4.10. We remark that for $\lambda = \mu$ the functionals only depend on the parameter $\lambda\nu^{\pm}$. Therefore we can simplify the notation of the functionals in this case in the following way:

$$\mathcal{F}^{\chi_2^{\pm}}_{\nu^{\pm}\lambda} := \mathcal{F}^{\nu^{\pm},\chi_2^{\pm}}_{\lambda,\lambda}$$

Therefore, we remark that according to our observations above we have the inequalities:

$$\mathcal{F}^{\chi_2^{\pm}}_{s}(v_{\epsilon_0}) < \mathcal{F}^{\chi_2^{\pm}}_{\nu^{\pm}\lambda}(v_{\epsilon_0}) < 0$$

for every $s \geq \lambda\nu^{\pm}$ with $\lambda\nu^{+}(\chi_1 + \chi_2) > 4\pi$ resp. $\lambda\nu^{-}\min\{\chi_1, \chi_2\} > 4\pi$.

We define $\mathcal{P} := \{p : [0,1] \to \mathcal{D} \mid p$ is continous and $p(0) = 0,\ p(1) = v_{\varepsilon_0}\}$ and set

$$k_s \equiv \inf_{p\in\mathcal{P}} \max_{t\in[0,1]} \mathcal{F}^{\chi_2^{\pm}}_{s}(p(t))$$

for all $s \geq \nu^{\pm}\lambda$. Since $\nu^{\pm}\lambda \mapsto \mathcal{F}^{\chi_2^{\pm}}_{\nu^{\pm}\lambda}(v)$ is monotone decreasing for all $v \in \mathcal{D}$, we see that $s \mapsto k_s$ is also monotone decreasing for all $s \geq \nu^{\pm}\lambda$ and, therefore, this mapping is differentiable for almost any $s \geq \nu^{\pm}\lambda$. Furthermore, we know that there is a constant k_0 not depending on the parameter $\nu^{\pm}\lambda$ such that

$$k_s \geq k_0 > 0$$

(since for all $\nu^{\pm}(\chi_1^2\lambda + \chi_2^2\mu) \leq \mu_1 + \gamma$, i.e. $\nu^{\pm}\lambda(\chi_1^2 + \chi_2^2) \leq \mu_1 + \gamma$, a strict local minima is given by the trivial solution $V \equiv 0$).

To prove the existence of nontrivial solutions to our nonlocal elliptic boundary value problems we have to establish the following statement.

Lemma 4.11. We assume that $\lambda = \mu$ and $\lambda\nu^{+}(\chi_1 + \chi_2) > 4\pi$ resp. that $\lambda = \mu$ and $\lambda\nu^{-}\min\{\chi_1, \chi_2\} > 4\pi$. Let us suppose that the mapping $s \mapsto k_s$ is differentiable at $s > \nu^{\pm}\lambda$. Then k_s denotes a critical value of the functional $\mathcal{F}^{\chi_2^{\pm}}_{s}$, i.e. there exists a nontrivial stationary solution for almost every $s > \nu^{\pm}\lambda$.

As in [19] we need two lemmata to prove this result.

Lemma 4.12.

1. For each pair of functions $u,\ v \in \mathcal{D}$ and every $s > 0$ we have:

$$\begin{aligned}\mathcal{F}^{s,\chi_2^{\pm}}_{\lambda,\mu}(u+v) \leq{}& \mathcal{F}^{s,\chi_2^{\pm}}_{\lambda,\mu}(u) + \int_\Omega \nabla v \nabla u + \gamma v u \; dx \\ & - s\lambda\chi_1 \frac{\int_\Omega v e^{\chi_1 u}\; dx}{\int_\Omega e^{\chi_1 u}} - \pm s\mu\chi_2 \frac{\int_\Omega v e^{\pm\chi_2 u}\; dx}{\int_\Omega e^{\pm\chi_2 u}} \\ & + \frac{1}{2}\int_\Omega |\nabla v|^2 + \gamma v^2 \; dx \end{aligned} \tag{4.10}$$

2. For any $C_1 \geq 0$ there exists a constant K such that for any $s,\ q \in \mathbb{R}$ the estimate

$$\left\| \left(\mathcal{F}^{s,\chi_2^\pm}_{\lambda,\mu}\right)'(v) - \left(\mathcal{F}^{q,\chi_2^\pm}_{\lambda,\mu}\right)'(v) \right\|_{H^{-1}} \leq K|s-q| \tag{4.11}$$

holds uniformly in $v \in \mathcal{D}$, with $||\nabla v||^2_{L^2(\Omega)} \leq C_1$.

Remark 4.13. For $\lambda = \mu$ the two statements of Lemma 4.12 read as follows:

1. For each pair of functions $u,\ v \in \mathcal{D}$ and every $s > 0$ we have:

$$\begin{aligned} \mathcal{F}^{\chi_2^\pm}_s(u+v) \leq \mathcal{F}^{\chi_2^\pm}_s(u) &+ \int_\Omega \nabla v \nabla u + \gamma v u \; dx \\ &- s\chi_1 \frac{\int_\Omega v e^{\chi_1 u} \; dx}{\int_\Omega e^{\chi_1 u}} - \pm s\chi_2 \frac{\int_\Omega v e^{\pm\chi_2 u} \; dx}{\int_\Omega e^{\pm\chi_2 u}} \\ &+ \frac{1}{2}\int_\Omega |\nabla v|^2 + \gamma v^2 \; dx \end{aligned} \tag{4.12}$$

2. For any $C_1 \geq 0$ there exists a constant K such that for any $s,\ q \in \mathbb{R}$ the estimate

$$\left\| \left(\mathcal{F}^{\chi_2^\pm}_s\right)'(v) - \left(\mathcal{F}^{\chi_2^\pm}_q\right)'(v) \right\|_{H^{-1}} \leq K|s-q| \tag{4.13}$$

holds uniformly in $v \in \mathcal{D}$, with $||\nabla v||^2_{L^2(\Omega)} \leq C_1$.

Proof of Lemma 4.12:
First we consider the first statement and remark that:

$$\begin{aligned} \mathcal{F}^{s,\chi_2^\pm}_{\lambda,\mu}(u+v) - \mathcal{F}^{s,\chi_2^\pm}_{\lambda,\mu}(u) - \int_\Omega \nabla v \nabla u + \gamma v u \; dx - \frac{1}{2}\int_\Omega |\nabla v|^2 + \gamma v^2 \; dx \\ + s\lambda\chi_1 \frac{\int_\Omega v e^{\chi_1 u} \; dx}{\int_\Omega e^{\chi_1 u}} + \pm s\mu\chi_2 \frac{\int_\Omega v e^{\pm\chi_2 u} \; dx}{\int_\Omega e^{\pm\chi_2 u}} \end{aligned}$$

$$=$$

$$\begin{aligned} &-s\lambda \left[\log\left(\frac{\int_\Omega e^{\chi_1(u+v)} \; dx}{\int_\Omega e^{\chi_1 u} dx} \right) - \frac{\chi_1 \int_\Omega v e^{\chi_1 u} \; dx}{\int_\Omega e^{\chi_1 u} dx} \right] \\ &-s\mu \left[\log\left(\frac{\int_\Omega e^{\pm\chi_2(u+v)} \; dx}{\int_\Omega e^{\pm\chi_2 u} dx} \right) - \frac{\pm\chi_2 \int_\Omega v e^{\pm\chi_2 u} \; dx}{\int_\Omega e^{\pm\chi_2 u} dx} \right] \end{aligned}$$

$$= -\int_0^1 \int_0^{\delta'} \frac{d^2 f}{d\delta^2}(\delta'')\, d\delta'' d\delta',$$

with

$$f(\delta) = \lambda \log \left(\frac{\int_\Omega e^{\chi_1(u+\delta v)}\, dx}{\int_\Omega e^{\chi_1 u}\, dx} \right) + \mu \log \left(\frac{\int_\Omega e^{\pm\chi_2(u+\delta v)}\, dx}{\int_\Omega e^{\pm\chi_2 u}\, dx} \right).$$

Now since

$$\begin{aligned} f''(\delta) &= \frac{\lambda \left[\int_\Omega e^{\chi_1(u+\delta v)} \chi_1^2 v^2\, dx \cdot \int_\Omega e^{\chi_1(u+\delta v)}\, dx - \left(\int_\Omega e^{\chi_1(u+\delta v)} \chi_1 v\, dx \right)^2 \right]}{\left(\int_\Omega e^{\chi_1(u+\delta v)}\, dx \right)^2} \\ &\quad + \frac{\mu \left[\int_\Omega e^{\pm\chi_2(u+\delta v)} \chi_2^2 v^2\, dx \cdot \int_\Omega e^{\pm\chi_2(u+\delta v)}\, dx - \left(\int_\Omega e^{\pm\chi_2(u+\delta v)} (\pm\chi_2) v\, dx \right)^2 \right]}{\left(\int_\Omega e^{\pm\chi_2(u+\delta v)}\, dx \right)^2} \\ &\geq 0 \end{aligned}$$

is true according to Hölder's inequality we see that the first statement of our Lemma is valid.

Let us now turn to the second claim:
For all $u \in \mathcal{D}$ with $||\nabla u||_{L^2(\Omega)} \leq 1$ we see that:

$$\begin{aligned} \left\langle \left(\mathcal{F}^{s,\chi_2^\pm}_{\lambda,\mu} \right)'(v), u \right\rangle - \left\langle \left(\mathcal{F}^{q,\chi_2^\pm}_{\lambda,\mu} \right)'(v), u \right\rangle &= (q-s) \left(\frac{\chi_1 \lambda \int_\Omega u e^{\chi_1 v} dx}{\int_\Omega e^{\chi_1 v} dx} + \frac{\pm\chi_2 \mu \int_\Omega u e^{\pm\chi_2 v} dx}{\int_\Omega e^{\pm\chi_2 v} dx} \right) \\ &\leq |s-q| \max\{\chi_1, \chi_2\} ||\lambda| + |\mu|| \tilde{K}(C_1) \\ &\leq 2 \max\{|\lambda|, |\mu|\} \max\{\chi_1, \chi_2\} \tilde{K}(C_1) |s-q| \\ &\leq K|s-q| \end{aligned}$$

where $\tilde{K}(C_1)$ is a positive constant that can be determined explicitly by using Hölder's inequality and the Moser-Trudinger type inequality in the present paper. (For $\lambda = \mu$ and the functionals $\mathcal{F}_s^{\chi_2^\pm}$ the proof reduces to a simple application of Hölder's inequality and the Moser-Trudinger type inequality.) □

Proof of Lemma 4.11:

Let us fix a point s where k_s is differentiable. Now we choose a monotone decreasing sequence $(s_n)_{n\in\mathbf{N}}$ with $s_n \to s$ as $n \to \infty$. For $n \in \mathbf{N}$ and each $p_n \in \mathcal{P}$ with

$$\max_{t\in[0,1]} \mathcal{F}_s^{\chi_2^{\pm}}(p_n(t)) \leq k_s + (s_n - s) \tag{4.14}$$

we look at a point $v = p_n(t_n)$, such that

$$\mathcal{F}_s^{\chi_2^{\pm}}(v) \geq k_{s_n} - 2(s_n - s) \tag{4.15}$$

holds. Now we set $\alpha = -k_s' + 3$ and choose $n_0 \in \mathbf{N}$ sufficiently large. For all $n \geq n_0$ we have the following:

$$\begin{aligned} k_s - \alpha(s_n - s) \leq k_{s_n} - 2(s_n - s) \;&\leq\; \mathcal{F}_{s_n}^{\chi_2^{\pm}}(v) \;\leq\; \mathcal{F}_s^{\chi_2^{\pm}}(v) \\ &\leq \max_{0\leq t\leq 1} \mathcal{F}_s^{\chi_2^{\pm}}(p_n(t)) \;\leq\; k_s + (s_n - s) \end{aligned} \tag{4.16}$$

n_0 is independent of p_n. The inequality (4.16) implies

$$0 \leq \mathcal{F}_s^{\chi_2^{\pm}}(v) - \mathcal{F}_{s_n}^{\chi_2^{\pm}}(v) \leq k_s - \mathcal{F}_{s_n}^{\chi_2^{\pm}}(v) + (s_n - s)$$

and thus, according to (4.15),

$$\begin{aligned} 0 &\leq \frac{\mathcal{F}_s^{\chi_2^{\pm}}(v) - \mathcal{F}_{s_n}^{\chi_2^{\pm}}(v)}{s_n - s} \\ &= \log\left(\frac{1}{|\Omega|}\int_\Omega e^{\chi_1 V} dx\right) + \log\left(\frac{1}{|\Omega|}\int_\Omega e^{\pm\chi_2 V} dx\right) \\ &\leq \alpha + 1. \end{aligned} \tag{4.17}$$

However, this inequality also guarantees that

$$||\nabla v||^2_{L^2(\Omega)} \leq C_1 \tag{4.18}$$

for all $v = p_n(t_n)$, with $n \geq n_0$, since

$$\begin{aligned} ||\nabla v||^2_{L^2(\Omega)} &= 2\mathcal{F}_s^{\chi_2^{\pm}}(v) + 2s\left[\log\left(\frac{1}{|\Omega|}\int_\Omega e^{\chi_1 V} dx\right) + \log\left(\frac{1}{|\Omega|}\int_\Omega e^{\pm\chi_2 V} dx\right)\right] \\ &\leq 2k_s + 2(s_n - s) + 2s(\alpha + 1) \\ &\leq C_1. \end{aligned}$$

Now we make use of the next lemma, which we will prove at the end of this section.

Lemma 4.14. There exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$, such that $||\nabla v_n||^2_{L^2(\Omega)} \leq C_1$ and $\mathcal{F}_s^{\chi_2^{\pm}}(v_n) \to k_s$ as well as $\left(\mathcal{F}_s^{\chi_2^{\pm}}\right)'(v_n) \to 0$ is true as $n \to \infty$.

Let $(v_n)_{n\in\mathbf{N}}$ denote a sequence mentioned in Lemma 4.14. We assume that $v_n \rightharpoonup v$ for $n \to \infty$ in $\mathcal{D}$ and according to the compact imbedding of $H^1(\Omega)$ into the Orlicz space $L_{e^v-v-1}(\Omega)$ that $e^{v_n} \to e^v$ in $L^2(\Omega)$. Thus we have

$$\begin{aligned} o(1) &= \int_\Omega \nabla v_n \nabla(v_n - v) + \gamma v_n(v_n - v)\ dx \\ &\quad - s\chi_1 \frac{\int_\Omega e^{\chi_1 v_n}(v_n - v)\ dx}{\int_\Omega e^{\chi_1 v_n}} - \pm s\chi_2 \frac{\int_\Omega e^{\pm\chi_2 v_n}(v_n - v)\ dx}{\int_\Omega e^{\pm\chi_2 v_n}} \\ &= \|\nabla(v_n - v)\|^2_{L^2(\Omega)} + \gamma\|v_n - v\|^2_{L^2(\Omega)} - o(1), \end{aligned}$$

where $o(1) \to 0$ for $n \to \infty$. Therefore we have the strong convergence of $(v_n)_{n\in\mathbf{N}}$ in $H^1(\Omega)$. Using now the lower semi-continuity of the functionals $\mathcal{F}_s^{\chi_2^\pm}$ the proof is finished. □

Proof of Lemma 4.14 (compare [19]):
We prove the claim of Lemma 4.14 via contradiction. Therefore, we assume that no such sequence exists. The assumptions imply that there is a $\epsilon > 0$ such that

$$\left\|\left(\mathcal{F}_s^{\chi_2^\pm}\right)'(v)\right\|_{H^{-1}} \geq 2\epsilon \text{ for all } v \in \mathcal{D}, \text{ with } \|\nabla v\|^2_{L^2(\Omega)} \leq K_1$$

and

$$|\mathcal{F}_s^{\chi_2^\pm}(v) - k_s| < 2\epsilon.$$

Furthermore, we can assume that for $n \geq n_0$ the inequalities

$$\alpha(s_n - s) < \frac{\epsilon}{4}\sqrt{s_n - s} < \epsilon \text{ and } \frac{\mu_1 + \gamma}{2\mu_1}|s_n - s| < \frac{\epsilon}{4}\sqrt{s_n - s}$$

hold true. First we choose a function $\psi \in C^\infty(\mathbf{R})$ with $0 \leq \psi \leq 1$ defined by

$$\psi(r) = 1, \text{ if } r \geq -1, \ \psi(r) = 0, \text{ if } r \leq -2,$$

$$\psi_n(v) = \psi\left(\frac{\mathcal{F}_{s_n}(v) - k_{s_n}}{s_n - s}\right), \text{ if } n \in \mathbf{N}, \text{ and } \ v \in \mathcal{D}.$$

Now we choose $p_n \in \mathcal{P}$ satisfying inequality (4.14) and define

$$\tilde{p}_n(t) \equiv p_n(t) - \sqrt{s_n - s} \cdot \psi_n(p_n(t)) \frac{\left(\mathcal{F}_s^{\chi_2^\pm}\right)'(p_n(t))}{\left\|\left(\mathcal{F}_s^{\chi_2^\pm}\right)'(p_n(t))\right\|_{H^{-1}}}.$$

Inequality (4.16) was satisfied for all $v = p_n(t_n)$ with $\mathcal{F}_{s_n}^{\chi_2^\pm}(v) \geq k_{s_n} - 2(s_n - s)$. Thus (4.17) is true for such v, if $n \geq n_0$. (4.16) implies that

$$|\mathcal{F}_s^{\chi_2^\pm}(v) - k_s| < 2\epsilon \text{ and } \|\left(\mathcal{F}_s^{\chi_2^\pm}\right)'(v)\|_{H^{-1}} \geq 2\epsilon.$$

According to (4.18) and the second statement of Lemma 4.12 we have for such v and sufficiently large $n \geq n_0$ the following inequalities:

$$
\begin{aligned}
\left\langle \left(\mathcal{F}_{s_n}^{\chi_2^{\pm}}\right)'(v), \left(\mathcal{F}_{s}^{\chi_2^{\pm}}\right)'(v) \right\rangle &= \left\| \left(\mathcal{F}_{s}^{\chi_2^{\pm}}\right)'(v) \right\|_{H^{-1}}^2 - \left\langle \left(\mathcal{F}_{s}^{\chi_2^{\pm}}\right)'(v), \left(\mathcal{F}_{s}^{\chi_2^{\pm}}\right)'(v) \right\rangle \\
&\quad + \left\langle \left(\mathcal{F}_{s_n}^{\chi_2^{\pm}}\right)'(v), \left(\mathcal{F}_{s}^{\chi_2^{\pm}}\right)'(v) \right\rangle \\
&\geq \frac{\left\| \left(\mathcal{F}_{s}^{\chi_2^{\pm}}\right)'(v) \right\|_{H^{-1}}^2 - \left\| \left(\mathcal{F}_{s}^{\chi_2^{\pm}}\right)'(v) - \left(\mathcal{F}_{s_n}^{\chi_2^{\pm}}\right)'(v) \right\|_{H^{-1}}^2}{2} \\
&\geq \frac{1}{2} \left\| \left(\mathcal{F}_{s}^{\chi_2^{\pm}}\right)'(v) \right\|_{H^{-1}}^2 - K|s - s_n|^2 \\
&\geq \frac{1}{4} \left\| \left(\mathcal{F}_{s}^{\chi_2^{\pm}}\right)'(v) \right\|_{H^{-1}}^2 \geq \epsilon^2
\end{aligned}
$$

But for such v we have from the point of view of the first statement in Lemma 4.12, with $\tilde{v} \equiv \tilde{p}_n(t_n)$,

$$
\begin{aligned}
\mathcal{F}_{s_n}^{\chi_2^{\pm}}(\tilde{v}) &\leq \mathcal{F}_{s_n}^{\chi_2^{\pm}}(v) - \frac{1}{4}\sqrt{s_n - s} \cdot \psi_n(v) \left\| \left(\mathcal{F}_{s}^{\chi_2^{\pm}}\right)'(v) \right\|_{H^{-1}} + \frac{\mu_1 + \gamma}{2\mu_1} |s_n - s| \psi_n^2(v) \\
&\leq \mathcal{F}_{s_n}^{\chi_2^{\pm}}(v) - \frac{\epsilon}{2}\sqrt{s_n - s} \cdot \psi_n(v) + \frac{\epsilon}{4}\sqrt{s_n - s} \cdot \psi_n(v) \\
&\leq \mathcal{F}_{s_n}^{\chi_2^{\pm}}(v) - \frac{\epsilon}{4}\sqrt{s_n - s} \cdot \psi_n(v) \\
&\leq \mathcal{F}_{s_n}^{\chi_2^{\pm}}(v)
\end{aligned}
$$

for $n \geq n_0$. Thus we have for $n \geq n_0$:

$$
\begin{aligned}
k_{s_n} \leq \max_{0 \leq t \leq 1} \mathcal{F}_{s_n}^{\chi_2^{\pm}}(\tilde{p}_n(t)) &= \max_{\left\{ t \,\middle|\, \left(\mathcal{F}_{s_n}^{\chi_2^{\pm}}\right)'(p_n(t)) \geq k_{s_n} - (s_n - s) \right\}} \mathcal{F}_{s_n}^{\chi_2^{\pm}}(\tilde{p}_n(t)) \\
&\leq \max_{0 \leq t \leq 1} \mathcal{F}_{s_n}^{\chi_2^{\pm}}(p_n(t)) - \frac{\epsilon}{4}\sqrt{s_n - s} \leq \max_{0 \leq t \leq 1} \mathcal{F}_{s}^{\chi_2^{\pm}}(p_n(t)) - \frac{\epsilon}{4}\sqrt{s_n - s} \\
&\leq k_s + (s_n - s) - \frac{\epsilon}{4}\sqrt{s_n - s} \leq k_{s_n} + \alpha(s_n - s) - \frac{\epsilon}{4}\sqrt{s_n - s} \\
&< k_{s_n}
\end{aligned}
$$

But this is a contradiction. □

No we have everything at hand to formulate the main Theorem of this subsection.

Theorem 4.15. Let $\Omega \subset \mathbb{R}^2$ be a simply connected smooth domain. Then there exists a nontrivial solution for problem

$$0 = \Delta V - \gamma V + \nu^{\pm}\lambda\left(\frac{e^{\chi_1 V}}{\int\limits_\Omega e^{\chi_1 V}dx} + \frac{e^{\chi_2 V}}{\int\limits_\Omega e^{\chi_2 V}dx} - 1\right) \text{ in } \Omega,\ \frac{\partial V}{\partial n} = 0 \text{ on } \partial\Omega$$

for almost every $\lambda\nu^{+}(\chi_1+\chi_2) > 4\pi$ that satisfy $\nu^{+}\lambda(\chi_1^2+\chi_2^2) \le \mu_1+\gamma$ and there exists a nontrivial solution for problem

$$0 = \Delta V - \gamma V + \nu^{\pm}\lambda\left(\frac{e^{\chi_1 V}}{\int\limits_\Omega e^{\chi_1 V}dx} - \frac{e^{-\chi_2 V}}{\int\limits_\Omega e^{-\chi_2 V}dx} - 1\right) \text{ in } \Omega,\ \frac{\partial V}{\partial n} = 0 \text{ on } \partial\Omega$$

for almost every $\lambda\nu^{-}\min\{\chi_1,\chi_2\} > 4\pi$ satisfying $\nu^{-}\lambda(\chi_1^2+\chi_2^2) \le \mu_1+\gamma$.

Remark 4.16. Unfortunately the given existence proof of nontrivial steady state solutions in the conflict-free cases needs essentially the assumption that $\lambda = \mu$. Therefore, it would be interesting to know whether one can overcome this assumption or not.

§ 5. Concluding remarks

The analysis of multi-species chemotaxis systems is a challenging research topic. Especially the effects caused by possible interactions between the mobile species seem to be quite interesting and have not been that well studied so far. Most results available for multi-species chemotaxis systems apply for models with only one mobile species but a various number of substances, that attracted or repelled the mobile species. Models in the presence of conflicts seem to be quite complicated and several techniques that work in the conflict-free situation fail in the presence of conflicts. For example the techniques used in the present paper to establish the existence of nontrivial steady state solutions for some conflict-free systems seem to be not applicable in the presence of conflicts.

Although we have not discussed the "presence of conflict"-case in this paper, the present results bring some new insights in studying the multi-species chemotaxis models. More alternative aspects and approaches will be discussed in [13] where we will also focus at models in the presence of conflict.

While the present paper deals with the steady state analysis and, therefore, leaves the time asymptotic behavior of the multi-species chemotaxis models relatively untouched, it would be quite interesting to get also more insights into this aspect. Once again the situation in the presence of conflict seems to be (at a first glance) very challenging. In conflict-free systems one finds generalizations of the well-known threshold phenomena for the single species chemotaxis model in two spatial dimensions. It would not be surprising if one could use similar arguments to those for single species chemotaxis systems to prove blow-up results for conflict-free models. For example one can think of different blow-up times for each mobile species. However, first finite time blow-up results for conflict-free two species systems that have been achieved in [3] show simultaneous blow-up of the solution in finite time. Of course there are still open questions in this context. For example: Is there blow-up in multi-species chemotaxis models that do not describe motion with grounds of the mobile species? If there is a blow-up behavior for such kind of systems, is the blow-up simultaneous?

However, also the asymptotic behavior of the solution for models in the presence of conflict seem to allow much more features than in the conflict-free situations. Therefore, there might be different mechanisms that influence the time asymptotic behavior of the solution that have not been discovered yet.

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