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The strong maximum principle

Dedicated to Professor T. Nagai at the occasion of his 60th birthday, with my greatest admiration for his mathematical work

By

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Abstract

This paper reviews some of the most paradigmatic results on the minimum and the maximum principles for a class of second order linear elliptic operators, and establishes some new extremely sharp connections between them.

§ 1. Introduction

This paper considers a second order uniformly elliptic differential operator of the form

$$\mathfrak{L} := -\,\mathrm{div}\,(A\nabla\,\cdot\,) + \langle b, \nabla\,\cdot\,\rangle + c \tag{1.1}$$

in a bounded domain Ω of $\mathbb{R}^N$, $N \geq 1$, where 'div' stands for the divergence operator,

$$\mathrm{div}\,(u_1, \ldots, u_N) = \sum_{j=1}^{N} \frac{\partial u_j}{\partial x_j},$$

$\langle \cdot, \cdot \rangle$ is the Euclidean inner product of $\mathbb{R}^N$, and

$$\begin{cases} A = (a_{ij})_{1 \leq i,j \leq N} \in \mathcal{M}_N^{\mathrm{sym}}(W^{1,\infty}(\Omega)), \\ \\ b = (b_1, ..., b_N) \in (L^\infty(\Omega))^N, \quad c \in L^\infty(\Omega). \end{cases} \tag{1.2}$$

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Given a Banach space X, $\mathcal{M}_N^{\mathrm{sym}}(X)$ stands for the space of symmetric square matrices of order N with entries in X, and $W^{1,\infty}(\Omega)$ denotes the Sobolev space consisting of all functions of $L^\infty(\Omega)$ with weak derivatives of first order in $L^\infty(\Omega)$.

Also, throughout this paper, we are making the following general assumptions:

B1. Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, whose boundary, $\partial\Omega$, consists of two disjoint open and closed subsets, Γ_0 and Γ_1, of class $\mathcal{C}^1$,

$$\partial\Omega := \Gamma_0 \cup \Gamma_1,$$

some of which might be empty.

B2. $\beta \in \mathcal{C}(\Gamma_1)$, $\mathbf{n}$ denotes the *outward unit normal* vector field of Ω, and $\nu := A\mathbf{n}$ is the *conormal vector field*, i.e., for every $u \in \mathcal{C}^1(\Gamma_1)$,

$$\frac{\partial u}{\partial \nu} = \langle \nabla u, A\mathbf{n} \rangle = \langle A\nabla u, \mathbf{n} \rangle.$$

Under these assumptions, we denote by

$$\mathfrak{B} \ : \ \mathcal{C}(\Gamma_0) \otimes \mathcal{C}^1(\Gamma_1) \to C(\partial\Omega)$$

the boundary operator defined through

$$\mathfrak{B}\psi := \begin{cases} \psi & \text{on} \quad \Gamma_0, \\ \frac{\partial \psi}{\partial \nu} + \beta\psi & \text{on} \quad \Gamma_1, \end{cases} \qquad \psi \in \mathcal{C}(\Gamma_0) \otimes \mathcal{C}^1(\Gamma_1). \tag{1.3}$$

When $\Gamma_1 = \varnothing$, $\mathfrak{B}$ becomes the Dirichlet boundary operator on $\partial\Omega$; in such case, we will set $\mathfrak{D} := \mathfrak{B}$. If $\Gamma_0 = \varnothing$ and $\beta = 0$, then $\mathfrak{B}$ equals a Neumann boundary operator on $\partial\Omega$.

Essentially, this paper establishes some sharp connections between the classic minimum principles of Hopf [10], [11] and Protter and Weinberger [17] and the characterization of the strong maximum principle established by López-Gómez and Molina-Meyer [14]; further generalized by Amann and López-Gómez [3], López-Gómez [13], and Amann [2]. As a byproduct, the generalized minimum principle of Protter and Weinberger will be substantially generalized and considerably tidied up.

The distribution of the paper is as follows. Section 2 collects the minimum principles of Hopf, Section 3 collects the generalized minimum principle of Protter and Weinberger, Section 4 includes the characterization of the strong maximum principle, and Section 5 uses the characterization of Section 4 for sharpening the classical results of Sections 2 and 3.

§ 2. The minimum principle of E. Hopf

The next result goes back to Hopf [10]; it was the first minimum principle where the continuity assumptions on the coefficients of $\mathfrak{L}$ were removed away.

Theorem 2.1. *Suppose $c \geq 0$, and $u \in \mathcal{C}^2(\Omega)$ satisfies*

$$\mathfrak{L}u \geq 0 \quad in\ \Omega, \qquad m := \inf_{\Omega} u \in (-\infty, 0].$$

Then, either $u = m$ in Ω, or else $u(x) > m$ for all $x \in \Omega$. In other words, u cannot attain m in Ω, unless $u = m$ in Ω. Consequently,

$$\inf_{\bar{\Omega}} u = \inf_{\partial\Omega} u = m$$

if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$.

Usually, a function $u \in \mathcal{C}^2(\Omega)$ is said to be an *harmonic* function of $\mathfrak{L}$ in Ω if $\mathfrak{L}u = 0$ in Ω, while it is said to be *superharmonic* if $\mathfrak{L}u \geq 0$ in Ω, and *subharmonic* when $-u$ is superharmonic. According to this terminology, Theorem 2.1 establishes that no non-constant superharmonic function u can reach a non-positive absolute minimum in Ω. Consequently, by inter-exchanging u by $-u$, no non-constant subharmonic function can attain a non-negative absolute maximum in Ω. Therefore, no non-constant harmonic function can attain its absolute maximum neither its absolute minimum in Ω.

The next result improves Theorem 2.1 by establishing that any non-constant superharmonic function $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ of $\mathfrak{L}$ in Ω must decay linearly at any point $x_0 \in \partial\Omega$ where

$$u(x_0) = \inf_{\bar{\Omega}} u \leq 0$$

along any *outward pointing direction* for which u admits a directional derivative. Seemingly, it goes back to Giraud [8], [9], under some additional continuity properties on the coefficients of the operator. The version included here is attributable to Hopf [11] and Oleinik [16].

Theorem 2.2. *Suppose $c \geq 0$ and $u \in \mathcal{C}^2(\Omega)$ is a non-constant function satisfying*

$$\mathfrak{L}u \geq 0 \quad in\ \Omega, \qquad m := \inf_{\Omega} u \in (-\infty, 0].$$

Assume, in addition, that there exist $x_0 \in \partial\Omega$ and $R > 0$ such that

$$u(x_0) = m, \qquad u \in \mathcal{C}(B_R(x_0) \cap \bar{\Omega}),$$

and Ω satisfies an interior sphere property at x_0.

Then, for any outward pointing vector $\nu \in \mathbb{R}^N \setminus \{0\}$ at x_0 for which

$$\frac{\partial u}{\partial \nu}(x_0) := \lim_{\substack{x\in\Omega \\ x\to x_0}} \langle \nu, \nabla u(x)\rangle$$

exists, necessarily

$$\frac{\partial u}{\partial \nu}(x_0) < 0.$$

Thanks to Theorem 2.1, under the assumptions of Theorem 2.2,

$$u(x) > m \quad \text{for every} \quad x \in \Omega,$$

as we are assuming that u is non-constant. Therefore, Theorem 2.1 establishes that any non-constant superharmonic function $u(x)$ decays linearly towards its minimum, $m = u(x_0)$, as $x \in \Omega$ approximates $x_0 \in \partial\Omega$, if $m \leq 0$.

§ 3. The generalized minimum principle of Protter and Weinberger

The next result is a sharp generalization of Theorems 2.1 and 2.2 to cover the general case when the function $c(x)$ is not necessarily non-negative. It goes back to Theorem 10 of Protter and Weinberger [17, Chap. 2].

Theorem 3.1. *Suppose $(\mathfrak{L}, \Omega)$ admits a strictly positive superharmonic function $h \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$, in the sense that*

i) $h(x) > 0$ *for all* $x \in \bar{\Omega}$,

ii) $\mathfrak{L}h \geq 0$ *in* Ω.

Then, for any superharmonic function $u \in \mathcal{C}^2(\Omega)$ of $\mathfrak{L}$ in Ω such that

$$m := \inf_{\Omega} \frac{u}{h} \in (-\infty, 0], \tag{3.1}$$

either

$$u(x) > mh(x) \qquad \text{for all} \quad x \in \Omega, \tag{3.2}$$

or else

$$u = mh \qquad \text{in} \quad \Omega. \tag{3.3}$$

Further, suppose (3.2), *and the following three conditions:*

a) $h \in \mathcal{C}^1(\bar{\Omega})$,

b) *$u(x_0) = mh(x_0)$ for some $x_0 \in \partial\Omega$, and there exists $R > 0$ such that $u \in \mathcal{C}(B_R(x_0) \cap \bar{\Omega})$,*

c) *Ω satisfies the interior tangent sphere property at x_0 and there is an outward pointing vector $\nu \in \mathbb{R}^N \setminus \{0\}$ for which $\frac{\partial(u/h)}{\partial\nu}(x_0)$ exists.*

Then,

$$\frac{\partial(u/h)}{\partial\nu}(x_0) < 0.$$

It should be noted that, in case $c \geq 0$, the function $h := 1$ satisfies conditions (i) and (ii) and, hence, it provides us with a strict positive superharmonic function of $\mathfrak{L}$ in Ω. Consequently, in this special case, Theorem 3.1 provides us, simultaneously, with Theorems 2.1 and 2.2, for as $u/h = u$. Consequently, Theorem 3.1 seems to be substantially sharper than these results because it does not impose any restriction on the sign of $c \in L^\infty(\Omega)$. Its real strength will be revealed in Section 5.

§ 4. The characterization of the strong maximum principle

The following concept plays a pivotal role in the theory of elliptic partial differential equations. Subsequently, we will set

$$\mathcal{W}(\Omega) := \bigcap_{p>1} W^{2,p}(\Omega).$$

Definition 4.1. A function $h \in \mathcal{W}(\Omega)$ is said to be a supersolution of $(\mathfrak{L}, \mathfrak{B}, \Omega)$ if

$$\begin{cases} \mathfrak{L}h \geq 0 & \text{in } \Omega, \\ \mathfrak{B}h \geq 0 & \text{on } \partial\Omega. \end{cases}$$

The function h is said to be a *strict supersolution* of $(\mathfrak{L}, \mathfrak{B}, \Omega)$ if, in addition, some of these inequalities is strict (on a measurable set of positive measure). Also,

a) It is said that $(\mathfrak{L}, \mathfrak{B}, \Omega)$ satisfies the strong maximum principle if any nonzero supersolution $u \in \mathcal{W}(\Omega)$ of $(\mathfrak{L}, \mathfrak{B}, \Omega)$ (in particular, any strict supersolution) satisfies

$$u(x) > 0 \quad \forall\, x \in \Omega \cup \Gamma_1 \quad \text{and} \quad \frac{\partial u}{\partial \nu}(x) < 0 \quad \forall\, x \in u^{-1}(0) \cap \Gamma_0.$$

In such case, it will be simply said that $u \gg 0$.

b) It is said that $(\mathfrak{L}, \mathfrak{B}, \Omega)$ satisfies the maximum principle if any supersolution $u \in \mathcal{W}(\Omega)$ of $(\mathfrak{L}, \mathfrak{B}, \Omega)$ satisfies $u(x) \geq 0$ for all $x \in \bar{\Omega}$.

Subsequently, for a sufficiently large $\omega > 0$, e stands for the unique weak solution of

$$\begin{cases} (\mathfrak{L}+\omega)e = 1 & \text{in } \Omega, \\ \mathfrak{B}e = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

By elliptic regularity, $e \in \mathcal{W}(\Omega)$ and, owing to Amann and López-Gómez [3, Theorem 2.4], $e \gg 0$. Now, we can introduce the Banach space

$$\mathcal{C}_e(\bar{\Omega}) := \{ u \in \mathcal{C}(\bar{\Omega}) \; : \; \exists \lambda > 0 \text{ such that } -\lambda e \le u \le \lambda e \text{ in } \bar{\Omega} \} \tag{4.2}$$

equipped with the Minkowski norm

$$\|u\|_e := \inf \{ \lambda > 0 \; : \; -\lambda e \le u \le \lambda e \}, \qquad u \in \mathcal{C}_e(\bar{\Omega}).$$

According to Amann and López-Gómez [3, Theorem 2.4] and López-Gómez [13, Theorem 6.1], it readily follows the next characterization of the strong maximum principle, where $\sigma[\mathfrak{L}, \mathfrak{B}, \Omega]$ stands for the *principal eigenvalue* of the linear eigenvalue problem

$$\begin{cases} \mathfrak{L}\varphi = \sigma\varphi & \text{in } \Omega, \\ \mathfrak{B}\varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 4.2. *The following assertions are equivalent:*

i) $\sigma_0 := \sigma[\mathfrak{L}, \mathfrak{B}, \Omega] > 0$.

ii) $(\mathfrak{L}, \mathfrak{B}, \Omega)$ *possesses a positive strict supersolution* $h \in W^{2,p}(\Omega)$ *for some* $p > N$.

iii) $(\mathfrak{L}, \mathfrak{B}, \Omega)$ *satisfies the strong maximum principle.*

iv) $(\mathfrak{L}, \mathfrak{B}, \Omega)$ *satisfies the maximum principle.*

v) *The resolvent of the linear problem*

$$\begin{cases} \mathfrak{L}u = f \in \mathcal{C}_e(\bar{\Omega}) & \text{in } \Omega, \\ \mathfrak{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

$\mathfrak{R}_0 : \mathcal{C}_e(\bar{\Omega}) \to \mathcal{C}_e(\bar{\Omega})$, *is well defined and it is strongly positive.*

In case $\mathfrak{B} = \mathfrak{D}$, Theorem 4.2 goes back to López-Gómez and Molina-Meyer [14, Theorem 2.1]. Although in March 1994, when [14] appeared, there were already available a number of preliminary results trying to establish the hidden connections between the sign of σ_0, the validity of the maximum principle, the validity of the strong maximum principle, and the existence of a positive supersolution (e.g., Sweers [18], Figueiredo and Mitidieri [5], [6], López-Gómez and Pardo [15, Lemma 3.2], Fleckinger, Hernández and de Thélin [7]), the theorem establishing the equivalence between the following conditions goes back to [14, Theorem 2.1], not only for a single second order linear elliptic operator, but, more generally, for a rather general class of cooperative linear elliptic systems:

- $(\mathfrak{L}, \mathfrak{D}, \Omega)$ possesses a positive strict supersolution.
- The resolvent of $(\mathfrak{L}, \mathfrak{D}, \Omega)$ is well defined and it is strongly positive.
- $(\mathfrak{L}, \mathfrak{D}, \Omega)$ satisfies the strong maximum principle.
- $(\mathfrak{L}, \mathfrak{D}, \Omega)$ satisfies the maximum principle.
- $(\mathfrak{L}, \mathfrak{D}, \Omega)$ has a principal eigenvalue and

$$\sigma[\mathfrak{L}, \mathfrak{D}, \Omega] > 0. \tag{4.4}$$

Almost simultaneously, but in this case for the scalar operator (not for a cooperative system), Berestycki, Nirenberg and Varadhan [4, Theorem 1.1] established that $(\mathfrak{L}, \mathfrak{D}, \Omega)$ satisfies the maximum principle if and only if (4.4) holds; some precursors of this result had been already given by Agmon [1].

The fact that the characterization of the strong maximum principle in terms of the existence of a strict positive supersolution had been left outside the general scope of Berestycki, Nirenberg and Varadhan [4], prompted López-Gómez to include all technical details of the proof of Theorem 4.2, in the special case when $\mathfrak{B} = \mathfrak{D}$, in [12, Theorem 2.5], for as he realized that even the simplest version of [14, Theorem 2.1], for the scalar operator, was unknown for the most recognized specialists in the field. All the materials covered by [12] had been already delivered by J. López-Gómez in his PhD course on *Bifurcation Theory* in the Department of Mathematics of the University of Zürich during the summer semester of 1994 (see the Acknowledgements of [12]).

From the point of view of the applications, the most crucial feature from Theorem 4.2 is the fact that *the existence of a positive strict supersolution characterizes the strong maximum principle*, for as this is the usual strategy adopted in the applications to make sure that (4.4), or, equivalently, the strong maximum principle, holds. This provides to [14, Theorem 2.1] with its greatest significance when it is weighted versus Berestycki, Nirenberg and Varadhan [4, Theorem 1.1].

Three years later, in March 1997, Amann and López-Gómez [3, Theorem 2.4] generalized López-Gómez [12, Theorem 2.5] up to cover general boundary operators of the type considered in this paper. Some very very weak versions of this theorem have been recently given by Amann [2].

§ 5. The classical minimum principles revisited

Throughout this section we suppose that $\mathfrak{B} = \mathfrak{D}$ is the Dirichlet boundary operator. Then,

$$\sigma_0 := \sigma[\mathfrak{L}, \mathfrak{D}, \Omega].$$

The next consequence from Theorem 4.2 shows that the assumption that $(\mathfrak{L}, \Omega)$ admits a superharmonic function h such that

$$h(x) > 0 \qquad \text{for all} \quad x \in \bar{\Omega}$$

in Theorem 3.1 is nothing more than the positivity of σ_0.

Corollary 5.1. *Suppose $\mathfrak{B} = \mathfrak{D}$. Then, conditions* i)–v) *of Theorem 4.2 are equivalent to*

vi) $(\mathfrak{L}, \mathfrak{D}, \Omega)$ *admits a supersolution $h \in \mathcal{W}(\Omega)$ such that $h(x) > 0$ for all $x \in \bar{\Omega}$.*

vii) $(\mathfrak{L}, \mathfrak{D}, \Omega)$ *admits a positive strict supersolution $h \in \mathcal{W}(\Omega)$ such that $h = 0$ on $\partial\Omega$.*

Proof: Suppose $\sigma_0 > 0$. Then, by Theorem 4.2, the unique solution of

$$\begin{cases} \mathfrak{L}h = 0 & \text{in } \Omega, \\ h = 1 & \text{on } \partial\Omega, \end{cases}$$

provides us with a strict supersolution satisfying vi). Note that

$$h = 1 - \mathfrak{R}_0\, c,$$

where c is the zero order term of $\mathfrak{L}$, and $\mathfrak{R}_0$ is the resolvent of (4.3). Also, any principal eigenfunction φ_0 provides us with a positive strict supersolution satisfying vii).

Conversely, under any of the conditions vi) or vii), h provides us with a positive strict supersolution of $(\mathfrak{L}, \mathfrak{D}, \Omega)$ and, hence, thanks to Theorem 4.2, we find that $\sigma_0 > 0$. The proof is complete. $\square$

Note that if $c \geq 0$, then, the constant function $h := 1$ provides us with a supersolution satisfying condition vi), and, hence, $\sigma_0 > 0$. Consequently, the next result provides us with a substancial generalization of the theory of E. Hopf (Theorems 2.1 and 2.2) and of M. H. Protter and H. F. Weinberger (Theorem 3.1).

Theorem 5.2. *Suppose $\sigma_0 > 0$ and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$ satisfies*

$$\mathfrak{L}u \geq 0 \quad \textit{in} \quad \Omega \quad \textit{and} \quad \inf_{\bar{\Omega}} u \geq 0. \tag{5.1}$$

Then, $u \gg 0$, unless $u = 0$.

If, instead of (5.1), *u satisfies*

$$\mathfrak{L}u \geq 0 \quad \textit{in} \quad \Omega \quad \textit{and} \quad \inf_{\bar{\Omega}} u < 0, \tag{5.2}$$

then, for every $h \in \mathcal{W}(\Omega)$ such that

$$\mathfrak{L}h \geq 0 \quad \textit{in} \quad \Omega \quad \textit{and} \quad \inf_{\bar{\Omega}} h > 0, \tag{5.3}$$

the quotient function

$$v(x) := \frac{u(x)}{h(x)}, \qquad x \in \bar{\Omega}, \tag{5.4}$$

satisfies

$$m := \inf_{\bar{\Omega}} v < 0$$

and

$$v(x) > m \ \ \forall \ \ x \in \Omega \quad \wedge \quad \frac{\partial v}{\partial \nu}(x) < 0 \ \ \forall \ \ x \in v^{-1}(m) \cap \partial\Omega, \tag{5.5}$$

unless $v = m$ *in* $\bar{\Omega}$.

Therefore, if, for any given $f > 0$, h *is chosen as the unique solution of*

$$\begin{cases} \mathfrak{L} h = f & \text{in } \Omega, \\ h = 1 & \text{on } \partial\Omega, \end{cases} \tag{5.6}$$

then, we necessarily have that

$$\inf_{\bar{\Omega}} \frac{u}{h} = \inf_{\partial\Omega} \frac{u}{h} = \inf_{\partial\Omega} u < 0, \tag{5.7}$$

$$u(x) > \left(\inf_{\partial\Omega} u \right) h(x) \qquad \text{for all} \quad x \in \Omega, \tag{5.8}$$

and

$$\frac{\partial}{\partial \nu} \frac{u}{h}(x_0) < 0 \qquad \text{for all} \quad x_0 \in u^{-1}\left(\inf_{\partial\Omega} u \right) \cap \partial\Omega. \tag{5.9}$$

Proof: Since $\sigma_0 > 0$, according to Theorem 4.2, $(\mathfrak{L}, \mathfrak{B}, \Omega)$ satisfies the strong maximum principle. Suppose $u \neq 0$ satisfies (5.1). Then, u provides us with a nonzero supersolution of $(\mathfrak{L}, \mathfrak{B}, \Omega)$ and, therefore, u satisfies the requested properties.

Subsequently, we suppose that u and h satisfy (5.2) and (5.3), respectively. By Theorem 4.2, the solution of (5.6) provides us with one of those functions h for every $f > 0$. Note that

$$h = 1 + \mathfrak{R}_0(f - c).$$

Now, consider the quotient function v defined by (5.4). As $\inf_{\bar{\Omega}} u < 0$ and $\inf_{\bar{\Omega}} h > 0$, we have that

$$m := \inf_{\bar{\Omega}} v < 0.$$

Moreover, owing to Theorem 3.1, (5.5) holds, unless $v = m$ in $\bar{\Omega}$. In any of these circumstances, we have that

$$\inf_{\bar{\Omega}} v = \inf_{\partial\Omega} v. \tag{5.10}$$

Subsequently, we fix $f > 0$ and suppose that h is the unique solution of (5.6). Then, since $h = 1$ on $\partial\Omega$, we have that $v = u$ on $\partial\Omega$ and, hence, (5.10) implies (5.7).

Suppose $v = m$ in $\bar{\Omega}$. Then, $u = mh$ and, hence,

$$0 \leq \mathfrak{L}u = m\mathfrak{L}h = mf < 0,$$

which is impossible. Therefore, (5.8) and (5.9) follow from (5.5). This completes the proof of the theorem. $\square$

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