The strong maximum principle

Dedicated to Professor T. Nagai at the occasion of his 60th birthday,
with my greatest admiration for his mathematical work

By

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Abstract

This paper reviews some of the most paradigmatic results on the minimum and the maximum principles for a class of second order linear elliptic operators, and establishes some new extremely sharp connections between them.

§1. Introduction

This paper considers a second order uniformly elliptic differential operator of the form

\[ \mathcal{L} := -\text{div} (A \nabla \cdot ) + \langle b, \nabla \cdot \rangle + c \]

in a bounded domain \( \Omega \) of \( \mathbb{R}^N \), \( N \geq 1 \), where ’div’ stands for the divergence operator,

\[ \text{div} (u_1, \ldots, u_N) = \sum_{j=1}^{N} \frac{\partial u_j}{\partial x_j}, \]

\( \langle \cdot, \cdot \rangle \) is the Euclidean inner product of \( \mathbb{R}^N \), and

\[ \begin{cases} A = (a_{ij})_{1 \leq i,j \leq N} \in \mathcal{M}_N^{\text{sym}}(W^{1,\infty}(\Omega)), \\ b = (b_1, \ldots, b_N) \in (L^\infty(\Omega))^N, \quad c \in L^\infty(\Omega). \end{cases} \]
Given a Banach space $X$, $\mathcal{M}_N^{\text{sym}}(X)$ stands for the space of symmetric square matrices of order $N$ with entries in $X$, and $W^{1,\infty}(\Omega)$ denotes the Sobolev space consisting of all functions of $L^\infty(\Omega)$ with weak derivatives of first order in $L^\infty(\Omega)$.

Also, throughout this paper, we are making the following general assumptions:

**B1.** $\Omega$ is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, whose boundary, $\partial \Omega$, consists of two disjoint open and closed subsets, $\Gamma_0$ and $\Gamma_1$, of class $\mathcal{C}^1$,

$$\partial \Omega := \Gamma_0 \cup \Gamma_1,$$

some of which might be empty.

**B2.** $\beta \in \mathcal{C}(\Gamma_1)$, $\mathbf{n}$ denotes the outward unit normal vector field of $\Omega$, and $\nu := A\mathbf{n}$ is the conormal vector field, i.e., for every $u \in \mathcal{C}^1(\Gamma_1)$,

$$\frac{\partial u}{\partial \nu} = \langle \nabla u, A\mathbf{n} \rangle = \langle A\nabla u, \mathbf{n} \rangle.$$

Under these assumptions, we denote by

$$\mathfrak{B} : \mathcal{C}(\Gamma_0) \otimes \mathcal{C}^1(\Gamma_1) \rightarrow \mathcal{C}(\partial \Omega)$$

the boundary operator defined through

$$(1.3) \quad \mathfrak{B} \psi := \begin{cases} 
\psi & \text{on } \Gamma_0, \\
\frac{\partial \psi}{\partial \nu} + \beta \psi & \text{on } \Gamma_1,
\end{cases} \quad \psi \in \mathcal{C}(\Gamma_0) \otimes \mathcal{C}^1(\Gamma_1).$$

When $\Gamma_1 = \emptyset$, $\mathfrak{B}$ becomes the Dirichlet boundary operator on $\partial \Omega$; in such case, we will set $\mathfrak{D} := \mathfrak{B}$. If $\Gamma_0 = \emptyset$ and $\beta = 0$, then $\mathfrak{B}$ equals a Neumann boundary operator on $\partial \Omega$.

Essentially, this paper establishes some sharp connections between the classic minimum principles of Hopf [10], [11] and Protter and Weinberger [17] and the characterization of the strong maximum principle established by López-Gómez and Molina-Meyer [14]; further generalized by Amann and López-Gómez [3], López-Gómez [13], and Amann [2]. As a byproduct, the generalized minimum principle of Protter and Weinberger will be substantially generalized and considerably tidied up.

The distribution of the paper is as follows. Section 2 collects the minimum principles of Hopf, Section 3 collects the generalized minimum principle of Protter and Weinberger, Section 4 includes the characterization of the strong maximum principle, and Section 5 uses the characterization of Section 4 for sharpening the classical results of Sections 2 and 3.
§ 2. The minimum principle of E. Hopf

The next result goes back to Hopf [10]; it was the first minimum principle where the continuity assumptions on the coefficients of $\mathcal{L}$ were removed away.

**Theorem 2.1.** Suppose $c \geq 0$, and $u \in C^2(\Omega)$ satisfies

$$\mathcal{L}u \geq 0 \quad \text{in} \quad \Omega, \quad m := \inf_{\Omega} u \in (\infty, 0].$$

Then, either $u = m$ in $\Omega$, or else $u(x) > m$ for all $x \in \Omega$. In other words, $u$ cannot attain $m$ in $\Omega$, unless $u = m$ in $\Omega$. Consequently,

$$\inf_{\Omega} u = \inf_{\partial \Omega} u = m$$

if $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Usually, a function $u \in C^2(\Omega)$ is said to be an *harmonic* function of $\mathcal{L}$ in $\Omega$ if $\mathcal{L}u = 0$ in $\Omega$, while it is said to be *superharmonic* if $\mathcal{L}u \geq 0$ in $\Omega$, and *subharmonic* when $-u$ is superharmonic. According to this terminology, Theorem 2.1 establishes that no non-constant superharmonic function $u$ can reach a non-positive absolute minimum in $\Omega$. Consequently, by inter-exchanging $u$ by $-u$, no non-constant subharmonic function can attain a non-negative absolute maximum in $\Omega$. Therefore, no non-constant harmonic function can attain its absolute maximum neither its absolute minimum in $\Omega$.

The next result improves Theorem 2.1 by establishing that any non-constant superharmonic function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of $\mathcal{L}$ in $\Omega$ must decay linearly at any point $x_0 \in \partial \Omega$ where

$$u(x_0) = \inf_{\Omega} u \leq 0$$

along any outward pointing direction for which $u$ admits a directional derivative. Seemingly, it goes back to Giraud [8], [9], under some additional continuity properties on the coefficients of the operator. The version included here is attributable to Hopf [11] and Oleinik [16].

**Theorem 2.2.** Suppose $c \geq 0$ and $u \in C^2(\Omega)$ is a non-constant function satisfying

$$\mathcal{L}u \geq 0 \quad \text{in} \quad \Omega, \quad m := \inf_{\Omega} u \in (\infty, 0].$$

Assume, in addition, that there exist $x_0 \in \partial \Omega$ and $R > 0$ such that

$$u(x_0) = m, \quad u \in C(B_R(x_0) \cap \overline{\Omega}),$$

and $\Omega$ satisfies an interior sphere property at $x_0$. 
Then, for any outward pointing vector \( v \in \mathbb{R}^N \setminus \{0\} \) at \( x_0 \) for which
\[
\frac{\partial u}{\partial \nu}(x_0) := \lim_{x \in \Omega, x \to x_0} \langle v, \nabla u(x) \rangle
\]
exists, necessarily
\[
\frac{\partial u}{\partial \nu}(x_0) < 0.
\]

Thanks to Theorem 2.1, under the assumptions of Theorem 2.2,
\[ u(x) > m \quad \text{for every} \quad x \in \Omega, \]
as we are assuming that \( u \) is non-constant. Therefore, Theorem 2.1 establishes that any non-constant superharmonic function \( u(x) \) decays linearly towards its minimum, \( m = u(x_0) \), as \( x \in \Omega \) approximates \( x_0 \in \partial \Omega \), if \( m \leq 0 \).

§ 3. The generalized minimum principle of Protter and Weinberger

The next result is a sharp generalization of Theorems 2.1 and 2.2 to cover the general case when the function \( c(x) \) is not necessarily non-negative. It goes back to Theorem 10 of Protter and Weinberger [17, Chap. 2].

**Theorem 3.1.** Suppose \( (\mathcal{L}, \Omega) \) admits a strictly positive superharmonic function \( h \in C^2(\Omega) \cap C(\overline{\Omega}) \), in the sense that
i) \( h(x) > 0 \) for all \( x \in \overline{\Omega} \),
ii) \( \mathcal{L}h \geq 0 \) in \( \Omega \).

Then, for any superharmonic function \( u \in C^2(\Omega) \) of \( \mathcal{L} \) in \( \Omega \) such that
\[
(3.1) \quad m := \inf_{\Omega} \frac{u}{h} \in (-\infty, 0],
\]
either
\[
(3.2) \quad u(x) > mh(x) \quad \text{for all} \quad x \in \Omega,
\]
or else
\[
(3.3) \quad u = mh \quad \text{in} \quad \Omega.
\]
Further, suppose (3.2), and the following three conditions:
\[ a) \quad h \in C^1(\overline{\Omega}), \]
b) $u(x_0) = mh(x_0)$ for some $x_0 \in \partial \Omega$, and there exists $R > 0$ such that $u \in C(B_R(x_0) \cap \Omega)$,

c) $\Omega$ satisfies the interior tangent sphere property at $x_0$ and there is an outward pointing vector $\nu \in \mathbb{R}^N \setminus \{0\}$ for which $\frac{\partial (u/h)}{\partial \nu}(x_0)$ exists.

Then,

$$\frac{\partial (u/h)}{\partial \nu}(x_0) < 0.$$ 

It should be noted that, in case $c \geq 0$, the function $h := 1$ satisfies conditions (i) and (ii) and, hence, it provides us with a strict positive superharmonic function of $\mathcal{L}$ in $\Omega$. Consequently, in this special case, Theorem 3.1 provides us, simultaneously, with Theorems 2.1 and 2.2, for as $u/h = u$. Consequently, Theorem 3.1 seems to be substantially sharper than these results because it does not impose any restriction on the sign of $c \in L^\infty(\Omega)$. Its real strength will be revealed in Section 5.

§ 4. The characterization of the strong maximum principle

The following concept plays a pivotal role in the theory of elliptic partial differential equations. Subsequently, we will set

$$\mathcal{W}(\Omega) := \bigcap_{p>1} W^{2,p}(\Omega).$$

**Definition 4.1.** A function $h \in \mathcal{W}(\Omega)$ is said to be a supersolution of $(\mathcal{L}, \mathcal{B}, \Omega)$ if

$$\begin{cases} 
\mathcal{L}h \geq 0 & \text{in } \Omega, \\
\mathcal{B}h \geq 0 & \text{on } \partial \Omega.
\end{cases}$$

The function $h$ is said to be a strict supersolution of $(\mathcal{L}, \mathcal{B}, \Omega)$ if, in addition, some of these inequalities is strict (on a measurable set of positive measure). Also,

a) It is said that $(\mathcal{L}, \mathcal{B}, \Omega)$ satisfies the strong maximum principle if any nonzero supersolution $u \in \mathcal{W}(\Omega)$ of $(\mathcal{L}, \mathcal{B}, \Omega)$ (in particular, any strict supersolution) satisfies

$$u(x) > 0 \quad \forall x \in \Omega \cup \Gamma_1 \quad \text{and} \quad \frac{\partial u}{\partial \nu}(x) < 0 \quad \forall x \in u^{-1}(0) \cap \Gamma_0.$$ 

In such case, it will be simply said that $u \gg 0$.

b) It is said that $(\mathcal{L}, \mathcal{B}, \Omega)$ satisfies the maximum principle if any supersolution $u \in \mathcal{W}(\Omega)$ of $(\mathcal{L}, \mathcal{B}, \Omega)$ satisfies $u(x) \geq 0$ for all $x \in \Omega$. 

Subsequently, for a sufficiently large $\omega > 0$, $e$ stands for the unique weak solution of
\begin{equation}
(L + \omega)e = 1 \quad \text{in } \Omega, \\
\mathfrak{B}e = 0 \quad \text{on } \partial \Omega.
\end{equation}

By elliptic regularity, $e \in \mathcal{W}(\Omega)$ and, owing to Amann and López-Gómez [3, Theorem 2.4], $e \gg 0$. Now, we can introduce the Banach space
\begin{equation}
C_{e}(\overline{\Omega}) := \{ u \in C(\overline{\Omega}) : \exists \lambda > 0 \text{ such that } -\lambda e \leq u \leq \lambda e \text{ in } \overline{\Omega} \}
\end{equation}
equipped with the Minkowski norm
\[ \| u \|_{e} := \inf \{ \lambda > 0 : -\lambda e \leq u \leq \lambda e \}, \quad u \in C_{e}(\overline{\Omega}) \).

According to Amann and López-Gómez [3, Theorem 2.4] and López-Gómez [13, Theorem 6.1], it readily follows the next characterization of the strong maximum principle, where $\sigma[L, \mathfrak{B}, \Omega]$ stands for the principal eigenvalue of the linear eigenvalue problem
\begin{equation}
\begin{align*}
L\varphi &= \sigma \varphi \quad \text{in } \Omega, \\
\mathfrak{B}\varphi &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\end{equation}

**Theorem 4.2.** The following assertions are equivalent:

i) $\sigma_{0} := \sigma[L, \mathfrak{B}, \Omega] > 0$.

ii) $(L, \mathfrak{B}, \Omega)$ possesses a positive strict supersolution $h \in W^{2,p}(\Omega)$ for some $p > N$.

iii) $(L, \mathfrak{B}, \Omega)$ satisfies the strong maximum principle.

iv) $(L, \mathfrak{B}, \Omega)$ satisfies the maximum principle.

v) The resolvent of the linear problem
\begin{equation}
\begin{align*}
Lu &= f \in C_{e}(\overline{\Omega}) \quad \text{in } \Omega, \\
\mathfrak{B}u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\end{equation}
$\mathfrak{R}_{0} : C_{e}(\overline{\Omega}) \rightarrow C_{e}(\overline{\Omega})$, is well defined and it is strongly positive.

In case $\mathfrak{B} = \emptyset$, Theorem 4.2 goes back to López-Gómez and Molina-Meyer [14, Theorem 2.1]. Although in March 1994, when [14] appeared, there were already available a number of preliminary results trying to establish the hidden connections between the sign of $\sigma_{0}$, the validity of the maximum principle, the validity of the strong maximum principle, and the existence of a positive supersolution (e.g., Sweers [18], Figueiredo and Mitidieri [5], [6], López-Gómez and Pardo [15, Lemma 3.2], Fleckinger, Hernández and de Thélin [7]), the theorem establishing the equivalence between the following conditions goes back to [14, Theorem 2.1], not only for a single second order linear elliptic operator, but, more generally, for a rather general class of cooperative linear elliptic systems:
• \((\mathcal{L}, \mathfrak{D}, \Omega)\) possesses a positive strict supersolution.

• The resolvent of \((\mathcal{L}, \mathfrak{D}, \Omega)\) is well defined and it is strongly positive.

• \((\mathcal{L}, \mathfrak{D}, \Omega)\) satisfies the strong maximum principle.

• \((\mathcal{L}, \mathfrak{D}, \Omega)\) satisfies the maximum principle.

• \((\mathcal{L}, \mathfrak{D}, \Omega)\) has a principal eigenvalue and

\[
\sigma_{0} := \sigma[\mathcal{L}, \mathfrak{D}, \Omega] > 0.
\]

Almost simultaneously, but in this case for the scalar operator (not for a cooperative system), Berestycki, Nirenberg and Varadhan [4, Theorem 1.1] established that \((\mathcal{L}, \mathfrak{D}, \Omega)\) satisfies the maximum principle if and only if (4.4) holds; some precursors of this result had been already given by Agmon [1].

The fact that the characterization of the strong maximum principle in terms of the existence of a strict positive supersolution had been left outside the general scope of Berestycki, Nirenberg and Varadhan [4], prompted López-Gómez to include all technical details of the proof of Theorem 4.2, in the special case when \(\mathfrak{B} = \mathfrak{D}\), in [12, Theorem 2.5], for as he realized that even the simplest version of [14, Theorem 2.1], for the scalar operator, was unknown for the most recognized specialists in the field. All the materials covered by [12] had been already delivered by J. López-Gómez in his PhD course on Bifurcation Theory in the Department of Mathematics of the University of Zürich during the summer semester of 1994 (see the Acknowledgements of [12]).

From the point of view of the applications, the most crucial feature from Theorem 4.2 is the fact that the existence of a positive strict supersolution characterizes the strong maximum principle, for as this is the usual strategy adopted in the applications to make sure that (4.4), or, equivalently, the strong maximum principle, holds. This provides to [14, Theorem 2.1] with its greatest significance when it is weighted versus Berestycki, Nirenberg and Varadhan [4, Theorem 1.1].

Three years later, in March 1997, Amann and López-Gómez [3, Theorem 2.4] generalized López-Gómez [12, Theorem 2.5] up to cover general boundary operators of the type considered in this paper. Some very very weak versions of this theorem have been recently given by Amann [2].

§ 5. The classical minimum principles revisited

Throughout this section we suppose that \(\mathfrak{B} = \mathfrak{D}\) is the Dirichlet boundary operator. Then,

\[
\sigma_{0} := \sigma[\mathcal{L}, \mathfrak{D}, \Omega].
\]
The next consequence from Theorem 4.2 shows that the assumption that \((\mathcal{L}, \Omega)\) admits a superharmonic function \(h\) such that
\[
h(x) > 0 \quad \text{for all} \quad x \in \overline{\Omega}
\]
in Theorem 3.1 is nothing more than the positivity of \(\sigma_0\).

**Corollary 5.1.** Suppose \(\mathfrak{B} = \mathfrak{D}\). Then, conditions i)–v) of Theorem 4.2 are equivalent to

vi) \((\mathcal{L}, \mathfrak{D}, \Omega)\) admits a supersolution \(h \in \mathcal{W}(\Omega)\) such that \(h(x) > 0\) for all \(x \in \overline{\Omega}\).

vii) \((\mathcal{L}, \mathfrak{D}, \Omega)\) admits a positive strict supersolution \(h \in \mathcal{W}(\Omega)\) such that \(h = 0\) on \(\partial \Omega\).

**Proof:** Suppose \(\sigma_0 > 0\). Then, by Theorem 4.2, the unique solution of
\[
\begin{cases}
\mathcal{L}h = 0 & \text{in } \Omega, \\
h = 1 & \text{on } \partial \Omega,
\end{cases}
\]
provides us with a strict supersolution satisfying vi). Note that
\[
h = 1 - \mathfrak{R}_0 c,
\]
where \(c\) is the zero order term of \(\mathcal{L}\), and \(\mathfrak{R}_0\) is the resolvent of (4.3). Also, any principal eigenfunction \(\varphi_0\) provides us with a positive strict supersolution satisfying vii).

Conversely, under any of the conditions vi) or vii), \(h\) provides us with a positive strict supersolution of \((\mathcal{L}, \mathfrak{D}, \Omega)\) and, hence, thanks to Theorem 4.2, we find that \(\sigma_0 > 0\). The proof is complete. \(\square\)

Note that if \(c \geq 0\), then, the constant function \(h := 1\) provides us with a supersolution satisfying condition vi), and, hence, \(\sigma_0 > 0\). Consequently, the next result provides us with a substantial generalization of the theory of E. Hopf (Theorems 2.1 and 2.2) and of M. H. Protter and H. F. Weinberger (Theorem 3.1).

**Theorem 5.2.** Suppose \(\sigma_0 > 0\) and \(u \in C^2(\Omega) \cap C^1(\overline{\Omega})\) satisfies
\[
(5.1) \quad \mathcal{L}u \geq 0 \quad \text{in } \Omega \quad \text{and} \quad \inf_{\overline{\Omega}} u \geq 0.
\]
Then, \(u \gg 0\), unless \(u = 0\).

If, instead of (5.1), \(u\) satisfies
\[
(5.2) \quad \mathcal{L}u \geq 0 \quad \text{in } \Omega \quad \text{and} \quad \inf_{\overline{\Omega}} u < 0,
\]
then, for every \(h \in \mathcal{W}(\Omega)\) such that
\[
(5.3) \quad \mathcal{L}h \geq 0 \quad \text{in } \Omega \quad \text{and} \quad \inf_{\overline{\Omega}} h > 0,
\]
the quotient function

\[ v(x) := \frac{u(x)}{h(x)}, \quad x \in \Omega, \]

satisfies

\[ m := \inf_{\Omega} v < 0 \]

and

\[ v(x) > m \quad \forall \ x \in \Omega \quad \land \quad \frac{\partial v}{\partial v}(x) < 0 \quad \forall \ x \in v^{-1}(m) \cap \partial \Omega, \]

unless \( v = m \) in \( \bar{\Omega} \).

Therefore, if, for any given \( f > 0 \), \( h \) is chosen as the unique solution of

\[ \begin{cases} \mathcal{L}h = f & \text{in } \Omega, \\ h = 1 & \text{on } \partial \Omega, \end{cases} \]

then, we necessarily have that

\[ \inf_{\bar{\Omega}} \frac{u}{h} = \inf_{\partial \Omega} \frac{u}{h} = \inf_{\partial \Omega} u < 0, \]

\[ u(x) > \left( \inf_{\partial \Omega} u \right) h(x) \quad \text{for all } \ x \in \Omega, \]

and

\[ \frac{\partial}{\partial v} \frac{u}{h}(x_0) < 0 \quad \text{for all } \ x_0 \in u^{-1} \left( \inf_{\partial \Omega} u \right) \cap \partial \Omega. \]

**Proof:** Since \( \sigma_0 > 0 \), according to Theorem 4.2, \( (\mathcal{L}, B, \Omega) \) satisfies the strong maximum principle. Suppose \( u \neq 0 \) satisfies (5.1). Then, \( u \) provides us with a nonzero supersolution of \( (\mathcal{L}, B, \Omega) \) and, therefore, \( u \) satisfies the requested properties.

Subsequently, we suppose that \( u \) and \( h \) satisfy (5.2) and (5.3), respectively. By Theorem 4.2, the solution of (5.6) provides us with one of those functions \( h \) for every \( f > 0 \). Note that

\[ h = 1 + \Re_0(f - c). \]

Now, consider the quotient function \( v \) defined by (5.4). As \( \inf_\Omega u < 0 \) and \( \inf_\Omega h > 0 \), we have that

\[ m := \inf_{\Omega} v < 0. \]

Moreover, owing to Theorem 3.1, (5.5) holds, unless \( v = m \) in \( \bar{\Omega} \). In any of these circumstances, we have that

\[ \inf_{\Omega} v = \inf_{\partial \Omega} v. \]
Subsequently, we fix \( f > 0 \) and suppose that \( h \) is the unique solution of (5.6). Then, since \( h = 1 \) on \( \partial \Omega \), we have that \( v = u \) on \( \partial \Omega \) and, hence, (5.10) implies (5.7).

Suppose \( v = m \) in \( \Omega \). Then, \( u = mh \) and, hence,

\[
0 \leq \mathcal{L}u = m\mathcal{L}h = mf < 0,
\]

which is impossible. Therefore, (5.8) and (5.9) follow from (5.5). This completes the proof of the theorem. \( \square \)

References