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Lipschitz Semigroup Approach to Drift-diffusion Systems

Dedicated to Professor Toshitaka Nagai on the occasion of his sixtieth birthday

By

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Abstract

A characterization problem is discussed of semigroups of Lipschitz operators providing mild solutions to the Cauchy problem for the semilinear evolution equation of parabolic type $u'(t) = (A+B)u(t)$ for $t > 0$. By parabolic type is meant that the operator A is the infinitesimal generator of an analytic (C_0) semigroup on a general Banach space X. The operator B is assumed to be continuous from a closed subset of Y into X, where Y is a Banach space which is contained in X and has a stronger norm defined through a fractional power of $-A$. The abstract result is new in that a functional $V(t,s,x,y)$ depending on (t,s) can be taken as a metric-like functional used to show uniqueness in applications. This extension allows to make discussions based on L^p-L^q estimates as well as by fractional power $(-A)^\alpha$, so that the characterization is applied to the global solvability of the Cauchy problem for the drift-diffusion system. The existence and uniqueness, the continuous dependence on initial data, and the smoothing effect of C^1-solutions of the Cauchy problem for the drift-diffusion system can be obtained through the abstract result.

§ 1. Introduction

Let X be a general Banach space with norm $\|\cdot\|$ and D a closed subset of X. By a *semigroup on* D is meant a one-parameter family $\{S(t); t \geq 0\}$ of operators from D into itself satisfying the so-called *semigroup property* and the strong continuity in $t \geq 0$.

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In order to develop a general theory of nonlinear semigroups, it is necessary to consider the continuity of the operators $S(t)$ in an appropriate way. In this paper, we consider the continuity condition of the operators $S(t)$ in such a way that for each $\tau \geq 0$ there exists $L_\tau > 0$ satisfying

$$\|S(t)x - S(t)y\| \leq L_\tau \|x - y\| \qquad \text{for } x, y \in D \text{ and } t \in [0, \tau].$$

A semigroup on D satisfying the above-mentioned continuity condition is called a *semigroup of Lipschitz operators on D*. The generation of such semigroups has been recently studied in several settings. Among others, characterization theorems of nonlinearly perturbed (C_0) semigroups and analytic semigroups were given in [10, 18], respectively. In this paper we concern on studying a characterization of nonlinearly perturbed analytic (C_0) semigroups which is an extension of the previous result [18]. We apply this result to the global solvability of the Cauchy problem for the drift-diffusion system

$$\begin{cases} \partial_t u - \Delta u - \nabla \cdot (u \nabla \psi) = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ -\Delta \psi = \lambda u & \text{in } \mathbb{R}^N \times (0, \infty), \\ u(x, 0) = u_0(x) \geq 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 2$ and $\lambda = 1$. This system with $\lambda = 1$ is related to the mathematical model for semiconductor devices, and the system (1.1) with $\lambda = -1$ is a mathematical model for chemotaxis. For chemotaxis model we refer to Nagai [19, 20, 21]. Kurokiba-Nagai-Ogawa [12] studied the bipolar drift-diffusion system

$$\begin{cases} \partial_t n - \Delta n + \nabla \cdot (n \nabla \psi) = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ \partial_t p - \Delta p - \nabla \cdot (p \nabla \psi) = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ -\Delta \psi = \lambda (p - n) & \text{in } \mathbb{R}^N \times (0, \infty), \\ n(x, 0) = n_0(x), \quad p(x, 0) = p_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $N = 2$ and $\lambda = -1$. They showed the global existence and the uniform boundedness of solutions to (1.2) for initial values in the weighted space. Kurokiba-Ogawa [13] considered (1.2) with $\lambda = \pm 1$ and they verified the local existence of solutions to (1.2) and the global existence result for $\lambda = 1$ in $L^p(\mathbb{R}^N)$ under the restriction $N/2 < p < N$ or $N = p = 2$. Recently Ogawa-Shimizu [22] establish linear and bilinear estimates in the Hardy space $H^1(\mathbb{R}^2)$ and apply them to (1.2) to obtain the local existence of solutions for large data in $H^1(\mathbb{R}^2)$ and the global existence for small data. We shall apply our main results to show the global existence of solutions to (1.1) with $\lambda = 1$ in $L^p(\mathbb{R}^N)$ under the restriction $N/2 \leq p < N$. This fact will be proved in Section 7.

In order to characterize nonlinearly perturbed analytic (C_0) semigroups, we interpret such a problem as a characterization problem of semigroups of Lipschitz operators providing mild solutions to the Cauchy problem for the semilinear evolution equation of parabolic type

$$u'(t) = (A + B)u(t) \quad \text{for } t > 0. \tag{SP}$$

By *parabolic type* we mean that the operator A is the infinitesimal generator of an analytic (C_0) semigroup $\{T(t); t \geq 0\}$ on X. The operator B is assumed to be continuous from $D \cap Y$ into X, where Y is a Banach space which is contained in X and has a stronger norm defined through a fractional power of $-A$.

The semilinear problem (SP) has been studied by many authors. If B is locally Lipschitz continuous from the set $D \cap Y$ into X, then the local solvability for (SP) can be shown in [1, 16] by the Banach-Picard fixed point theorem. In the setting that B is locally continuous from the set $D \cap Y$ into X, the construction of approximate solutions was done under various types of subtangential condition. ([15], [2] and [5].) Prüss proposed in [25] the following subtangential condition: There exists $\eta > 0$ such that to each $v \in D \cap Y$ and $\varepsilon > 0$ there correspond $h > 0$ and $w_h \in D \cap Y$, and z_h defined by

$$z_h = T(h)v + \int_0^h T(\xi)Bv \, d\xi - w_h$$

satisfies $\|z_h\| \leq \varepsilon h$ and $\|(-A)^\alpha z_h\| \leq \varepsilon h^\eta$. This condition is necessary for the existence of local mild solutions.

As is seen from the proof of Theorem 2.3 "(i) $\Rightarrow$ (ii)", a metric-like functional V_0 on $X \times X$ is necessary for the global existence of mild solutions depending Lipschitz continuously on their initial data. This fact implies that such a functional may be constructed for a given differential system which is well-posed.

The arguments in the above-mentioned papers studying (1.1) or (1.2) are based on L^p-L^q estimates. In the present paper, the abstract result in [18] is extended such that a functional $V(t, s, x, y)$ depending on (t, s) satisfying (V1) through (V3) can be taken as a metric-like functional in applications. This extension allows to make discussions based on L^p-L^q estimates as well as by fractional power $(-A)^\alpha$, as will be illustrated in Section 7.

In this paper we shall employ the subtangential condition in the sense of Prüss type and demonstrate that a sequence of approximate solutions converges to a mild solution to (SP) under the semilinear stability condition by means of a functional V on $\Delta \times Y \times Y$ satisfying (V1) through (V3).

This paper is organized as follows: In Section 2 we impose basic assumptions on A and B appearing in (SP) and characterize semigroups of Lipschitz operators providing mild solutions to semilinear evolution equations of parabolic type. The characterization is provided by Theorem 2.3. The uniqueness and the regularity results of mild solutions are given in Section 3. The proof of the existence of mild solutions is divided into two parts. Section 4 is devoted to construct approximate solutions to (SP). In Section 6 we discuss the convergence of a sequence of approximate solutions to a mild solution to the Cauchy problem for (SP) which forms a semigroup of Lipschitz operators. A key estimate to show the latter is given in Section 5. Section 7 deals with the drift-diffusion system. Our main theorem is applied to show the unique existence of solutions to the Cauchy problem for the drift-diffusion system.

§ 2. Main theorem

Let X be a general Banach space with norm $\|\cdot\|$ and D a closed subset of X. We begin by listing up basic assumptions on A and B appearing in (SP).

(A) The operator A is the infinitesimal generator of an analytic (C_0) semigroup $\{T(t); t \geq 0\}$ on X whose type is negative.

Let $\alpha \in (0,1)$ and let Y be the Banach space $D((-A)^\alpha)$ equipped with the norm $\|x\|_Y = \|(-A)^\alpha x\|$ for $x \in D((-A)^\alpha)$. We consider the set $C := D \cap Y$ in Y and assume that C is dense in D. Then we introduce the relative continuity on the perturbing operator B from C into X and the linear growth condition for B in the following sense:

(B1) The operator B is continuous from C into X.

(B2) There exists $M_B > 0$ such that $\|Bx\| \leq M_B(1 + \|x\|_Y)$ for $x \in C$.

The Cauchy problem for the semilinear evolution equation (SP) with initial condition $u(0) = u_0$ is denoted by (SP;u_0). In oder to characterize semigroups of Lipschitz operators associated with semilinear evolution equations of parabolic type, we need the following notion of solutions that may not be differentiable in general.

Definition 2.1. Let $u_0 \in D$ and $\bar\tau > 0$. A function $u \in C([0,\bar\tau]; X) \cap C((0,\bar\tau]; Y)$ is called a *mild solution to* (SP;u_0) *on* $[0,\bar\tau]$ if $u(t) \in C$ for $t \in (0,\bar\tau]$, $Bu \in C((0,\bar\tau]; X) \cap L^1(0,\bar\tau; X)$ and u satisfies the integral equation

$$u(t) = T(t)u_0 + \int_0^t T(t-s)Bu(s)\,ds \quad \text{for } t \in [0,\bar\tau]. \tag{2.1}$$

A function $u \in C([0,\infty); X) \cap C((0,\infty); Y)$ is called a *global mild solution to* (SP;u_0) if for each $\bar\tau > 0$ the restriction u to $[0,\bar\tau]$ is a mild solution to (SP;u_0) on $[0,\bar\tau]$.

We start with the definition of semigroups of Lipschitz operators.

Definition 2.2. A one-parameter family $\{S(t); t \geq 0\}$ of Lipschitz operators from D into itself is called a *semigroup of Lipschitz operators on D* if the following three conditions are satisfied:

(S1) $S(0)x = x$ for $x \in D$, and $S(t+s)x = S(t)S(s)x$ for s, $t \geq 0$ and $x \in D$.

(S2) For each $x \in D$, $S(\cdot)x : [0,\infty) \to X$ is continuous.

(S3) For each $\tau \geq 0$ there exists $L_\tau > 0$ such that

$$\|S(t)x - S(t)y\| \leq L_\tau \|x - y\| \quad \text{for } x,\, y \in D \text{ and } t \in [0,\tau].$$

The main theorem in this paper is given by

Theorem 2.3. *Assume that conditions* (A) *and* (B) *are satisfied. Then, the following two statements are equivalent:*

(i) *There exists a semigroup $\{S(t); t \geq 0\}$ of Lipschitz operators on D such that for each $x \in D$, $S(\cdot)x$ is a global mild solution to* (SP;x).

(ii) *There exist α_0, $\beta_0 \in [0,1)$ such that the following three conditions are satisfied:*

(ii-1) *There exist $\tau > 0$ and a nonnegative functional V on $\Delta \times Y \times Y$, where $\Delta = \{(t,s); 0 \leq s \leq t \leq \tau\}$, such that*

(V1) *there exists $L > 0$ such that*

$$\begin{aligned}&|V(t,s,x,y) - V(t,s,\hat{x},\hat{y})| \\ &\quad \leq L(\|x-\hat{x}\| + \|y-\hat{y}\| + t^{\beta_0}\|T(t-s)(x-\hat{x})\|_Y + t^{\beta_0}\|T(t-s)(y-\hat{y})\|_Y)\end{aligned}$$

for (t,s,x,y), $(t,s,\hat{x},\hat{y}) \in \Delta \times Y \times Y$,

(V2) *there exist $M \geq m > 0$ such that*

$$V(t,s,x,y) \leq Mt^{\beta_0}(t-s)^{-\beta_0}\|x-y\| \quad \text{for } (t,s) \in \Delta \text{ with } t \neq s \text{ and } x,y \in C,$$
$$m\|x-y\| \leq V(t,t,x,y) \quad \text{for } t \in [0,\tau] \text{ and } x,y \in C,$$

(V3) *there is a nondecreasing function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{r\downarrow 0}\theta(r) = 0$ and*

$$|V(t,s,x,y) - V(\hat{t},\hat{s},x,y)| \leq \theta(|t-\hat{t}| + |s-\hat{s}|)(1 + \|x\|_Y + \|y\|_Y)$$

for $(t,s,x,y), (\hat{t},\hat{s},x,y) \in \Delta \times C \times C$.

(ii-2) *There exists $\omega \geq 0$ such that to each $\varepsilon > 0$, $(t,s) \in \Delta$ with $t \neq s$, and $x,y \in C$ there corresponds $h \in (0,\varepsilon]$ such that $s+h \leq t$ and*

$$\begin{aligned}&(V(t,s+h,J(h)x,J(h)y) - V(t,s,x,y))/h \\ &\quad \leq t^{\beta_0}(t-s)^{-\alpha_0}(s+h)^{-\beta_0}(\omega V(s,s,x,y) + \varepsilon),\end{aligned}$$

where $J(\sigma)w = T(\sigma)w + \int_0^\sigma T(\xi)Bw\,d\xi$ for $(\sigma,w) \in [0,\infty) \times C$.

(ii-3) *There exists* $\beta \in (0,1)$ *such that to each* $x \in C$ *and* $\varepsilon > 0$ *there correspond* $h \in (0,\varepsilon]$, $x_h \in C$ *and* $z_h \in Y$ *satisfying*

$$x_h = T(h)x + \int_0^h T(s)Bx\,ds + z_h, \quad \|z_h\| \leq \varepsilon h \quad \text{and} \quad \|z_h\|_Y \leq \varepsilon h^{\beta}.$$

Remark. (a) The proof of the implication "(ii) $\Rightarrow$ (i)" of Theorem 2.3 will be shown by defining a semigroup $\{S(t); t \geq 0\}$ of Lipschitz operators on D by $S(t)u_0 = u(t)$ for $t \geq 0$, where $u(t)$ is the unique global mild solution to (SP;u_0) whose existence is ensured by Section 4 through Section 6. For this reason, we need to demonstrate the existence and uniqueness of global mild solutions under condition (ii) of Theorem 2.3. The important point for our arguments is that we may assume, without loss of generality, that there exists $\omega_A < 0$ such that the analytic (C_0) semigroup $\{T(t); t \geq 0\}$ on X satisfies $\|T(t)\| \leq e^{\omega_A t}$ for $t \geq 0$. This is ensured by [23, Proposition 2.5] and the renorming technique ([6]).
(b) In proving that (ii) implies (i), the following conditions on the functional V derived from (V1) and (V2) will be used. (b-i) By (V2) we have $V(t,s,x,x) = 0$ for $(t,s) \in \Delta$ and $x \in C$. (b-ii) By setting $\hat{x} = \hat{y} = y$ in condition (V1), we have $V(t,s,x,y) \leq L(\|x-y\| + t^{\beta_0}\|x-y\|_Y)$ for $(t,s,x,y) \in \Delta \times C \times C$. (b-iii) By (V1) there exists $L_Y > 0$ such that $|V(t,s,x,y) - V(t,s,\hat{x},\hat{y})| \leq L_Y(\|x-\hat{x}\|_Y + \|y-\hat{y}\|_Y)$ for $(t,s,x,y), (t,s,\hat{x},\hat{y}) \in \Delta \times Y \times Y$. (b-iv) By (V1) there exists $L_X > 0$ such that

$$|V(t,s,x,y) - V(t,s,\hat{x},\hat{y})| \leq L_X(1 + t^{\beta_0}(t-s)^{-\alpha})(\|x-\hat{x}\| + \|y-\hat{y}\|)$$

for $(t,s) \in \Delta$ with $t \neq s$ and $x, y \in Y$.

§ 3. Basic properties of mild solutions

The continuous dependence of mild solutions to the Cauchy problem for (SP) on their initial data is given by

Proposition 3.1. *Let* $\bar{\tau} > 0$ *and* x, $\hat{x} \in D$. *Let* u, $\hat{u} : [0,\bar{\tau}] \to X$ *be mild solutions to* (SP;x) *and* (SP;$\hat{x}$) *on* $[0,\bar{\tau}]$ *respectively. Suppose that conditions* (ii-1) *and* (ii-2) *in Theorem* 2.3 *are satisfied. Then there exist* $\overline{M} > 0$ *and* $\bar{\omega} > 0$ *such that*

$$\|u(t) - \hat{u}(t)\| \leq \overline{M}\exp(\bar{\omega}t)\|x - \hat{x}\| \quad \text{for } t \in [0,\bar{\tau}].$$

Proof. Let $\sigma \in (0,\tau]$, where $\tau > 0$ is a number satisfying condition (ii-1) in Theorem 2.3. Let l be a nonnegative integer such that $\sigma + l\tau \leq \bar{\tau}$ and $t \in (0,\sigma]$. Let $0 < \varepsilon < t$. Then, since $u \in C((0,\bar{\tau}];Y)$ we observe by (V1) and (V3) that the function

$s \mapsto V(t, s, u(s + l\tau), \hat{u}(s + l\tau))$ is continuous on $[\varepsilon, t]$. By the semigroup property of $\{T(t); t \geq 0\}$ and (2.1), we have

$$u(s + h + l\tau) = J(h)u(s + l\tau) + \int_0^h T(\xi)(Bu(s + h + l\tau - \xi) - Bu(s + l\tau))\, d\xi$$

for $s \in (0, t)$ and $h > 0$ with $s + h \leq t$. Since $Bu \in C((0, \bar{\tau}]; X)$, we deduce from condition (ii-2) that the lower right Dini derivative of the function

$$s \to V(t, s, u(s + l\tau), \hat{u}(s + l\tau)) - \int_\varepsilon^s \omega t^{\beta_0}(t - \xi)^{-\alpha_0}\xi^{-\beta_0}V(\xi, \xi, u(\xi + l\tau), \hat{u}(\xi + l\tau))\, d\xi$$

is nonpositive on $[\varepsilon, t - \varepsilon]$. It follows that

$$\begin{aligned} V(t, t-\varepsilon, &u(t + l\tau - \varepsilon), \hat{u}(t + l\tau - \varepsilon)) \\ &\leq V(t, \varepsilon, u(l\tau + \varepsilon), \hat{u}(l\tau + \varepsilon)) \\ &\quad + \int_\varepsilon^{t-\varepsilon} \omega t^{\beta_0}(t - \xi)^{-\alpha_0}\xi^{-\beta_0}V(\xi, \xi, u(\xi + l\tau), \hat{u}(\xi + l\tau))\, d\xi \end{aligned} \tag{3.1}$$

for $t \in [\varepsilon, \sigma]$. By Remark (b-ii) in Section 2 we have

$$\begin{aligned} &\xi^{-\beta_0}V(\xi, \xi, u(\xi + l\tau), \hat{u}(\xi + l\tau)) \\ &\leq L\left(\xi^{-\beta_0}(\|u(\xi + l\tau)\| + \|\hat{u}(\xi + l\tau)\|) + \|u(\xi + l\tau)\|_Y + \|\hat{u}(\xi + l\tau)\|_Y\right) \\ &\leq 2L\left(\sup\{\|u(s)\|; s \in [0, \bar{\tau}]\}\xi^{-\beta_0} + K_\alpha(\bar{\tau})(1 + \|x\| + \|\hat{x}\|)(\xi + l\tau)^{-\alpha}\right), \end{aligned}$$

where we have used Proposition 3.3 (ii) to obtain the last inequality. This implies that the function $\phi(\xi) := \xi^{-\beta_0}V(\xi, \xi, u(\xi + l\tau), \hat{u}(\xi + l\tau))$ is integrable on $(0, t)$. We use condition (V2) to obtain $\limsup_{\varepsilon\downarrow 0} V(t, \varepsilon, u(l\tau + \varepsilon), \hat{u}(l\tau + \varepsilon)) \leq M\|u(l\tau) - \hat{u}(l\tau)\|$. Passing to the limit in (3.1) as $\varepsilon \downarrow 0$, we have

$$\phi(t) \leq M\|u(l\tau) - \hat{u}(l\tau)\|t^{-\beta_0} + \int_0^t \omega(t - \xi)^{-\alpha_0}\phi(\xi)\, d\xi$$

for $t \in (0, \sigma]$. Applying Lemma 3.2 below and then using condition (V2), we have

$$\|u(t + l\tau) - \hat{u}(t + l\tau)\| \leq (M/m)K_{\beta_0,\alpha_0,\omega}(\tau)\|u(l\tau) - \hat{u}(l\tau)\| \tag{3.2}$$

for $t \in [0, \sigma]$, $l \geq 0$ with $\sigma + l\tau \leq \bar{\tau}$ and $\sigma \in (0, \tau]$.

Now, let $t \in [0, \bar{\tau}]$. Then, we have $t = [t/\tau]\tau + \sigma$ for some $\sigma \in [0, \tau)$, where $[t/\tau]$ stands for the integer part of t/τ. We apply (3.2) repeatedly to obtain

$$\|u(t) - \hat{u}(t)\| \leq ((M/m)K_{\beta_0,\alpha_0,\omega}(\tau))^{[t/\tau]+1}\|u(0) - \hat{u}(0)\|.$$

By setting $\overline{M} = (M/m)K_{\beta_0,\alpha_0,\omega}(\tau)$ and $\bar{\omega} = \tau^{-1}\log((M/m)K_{\beta_0,\alpha_0,\omega}(\tau))$, the desired result is obtained. □

Lemma 3.2. (Henry's inequality) ([9, p.190, Execise 4], [18, Lemma 2.3]) *Let $\bar{\tau} > 0$, a, $b \geq 0$ and $\sigma, \theta \in [0,1)$. Suppose that w is a nonnegative, integrable function over $(0,\bar{\tau})$ satisfying the inequality*

$$w(t) \leq at^{-\sigma} + b\int_0^t (t-s)^{-\theta} w(s)\,ds \quad \textit{for } t \in (0,\bar{\tau}).$$

Then, there exists $K_{\sigma,\theta,b} \in C(\mathbb{R}_+;\mathbb{R}_+)$ such that

$$w(t) \leq at^{-\sigma} K_{\sigma,\theta,b}(t) \quad \textit{for } t \in (0,\bar{\tau}).$$

As for the regularity of mild solutions, the following properties hold.

Proposition 3.3. ([18, Proposition 2.4]) *Let $\bar{\tau} > 0$ and $x \in D$. Let u be a mild solution to* (SP;x) *on $[0,\bar{\tau}]$. Then the following assertions hold:*

(i) *For each $\gamma \in [\alpha,1)$, the function $s \to (-A)^\gamma T(t-s)Bu(s)$ is integrable on the interval $(0,t)$, $u(t) \in D((-A)^\gamma)$ and*

$$(-A)^\gamma u(t) = (-A)^\gamma T(t)x + \int_0^t (-A)^\gamma T(t-s)Bu(s)\,ds \quad \textit{for } t \in (0,\bar{\tau}]. \tag{3.3}$$

(ii) *For each $\gamma \in [\alpha,1)$ and each $\hat{\gamma} \in [0,\alpha]$ there exists a nonnegative, nondecreasing function $K_{\gamma,\hat{\gamma}}$ on $\mathbb{R}_+$ such that if $x \in D$ then*

$$\|(-A)^\gamma u(t)\| \leq K_{\gamma,\hat{\gamma}}(\bar{\tau})(1 + \|(-A)^{\hat{\gamma}} x\|)t^{-(\gamma-\hat{\gamma})} \quad \textit{for } t \in (0,\bar{\tau}]. \tag{3.4}$$

(iii) *The mild solution u is locally Hölder continuous on $(0,\bar{\tau}]$ in Y.*

(iv) *Assume that for each $\rho > 0$ there exists $L_B(\rho) > 0$ such that*

$$\|Bu - Bv\| \leq L_B(\rho)\|u - v\|_Y$$

for $u, v \in C$ with $\|u\|_Y, \|v\|_Y \leq \rho$. Then, u is continuously differentiable over $(0,\bar{\tau}]$ in X, Au is continuous on $(0,\bar{\tau}]$ in X, and u satisfies (SP;x) *for $t \in (0,\bar{\tau}]$.*

(v) *If $x \in C$ then $u \in C([0,\bar{\tau}];Y)$ and $Bu \in C([0,\bar{\tau}];X)$.*

§ 4. Construction of approximate solutions

To discuss the construction of approximate solutions, we need the local uniformity (Proposition 4.2) of condition (ii-3) in Theorem 2.3. Without loss of generality we may assume that $\beta \leq 1-\alpha$, where β is a constant appearing in (ii-3) in Theorem 2.3.

Lemma 4.1. ([25, Lemma 3], [24, Lemma 3.1]) *There exists $K \geq 1$ depending only on α, β such that for any $\sigma \in (0,1]$ and any finite sequence $\{s_k\}_{k=0}^N$ satisfying $0 \leq s_0 < s_1 < \cdots < s_N \leq \sigma$, the following two assertions hold:*

(i) *If $M > 0$ and G is a measurable function from $[0,\sigma)$ into X satisfying $\|G(\xi)\| \leq M$ for $\xi \in [0,\sigma)$, then*

$$\int_{s_l}^{s_i} \|T(s_i - \xi)G(\xi)\|_Y \, d\xi \leq KM(s_i - s_l)^\beta \quad \text{for } 0 \leq l \leq i \leq N.$$

(ii) *Let $\varepsilon > 0$. Then, for any finite sequence $\{\zeta_i\}_{i=1}^N$ in Y satisfying $\|\zeta_i\| \leq \varepsilon(s_i - s_{i-1})$ and $\|\zeta_i\|_Y \leq \varepsilon(s_i - s_{i-1})^\beta$ for $1 \leq i \leq N$, it holds that*

$$\sum_{l=k+1}^{i} \|T(s_i - s_l)\zeta_l\|_Y \leq K\varepsilon(s_i - s_k)^\beta \quad \text{for } 0 \leq k \leq i \leq N.$$

In sections 4 through 6, K stands for a constant appearing in Lemma 4.1.

The next proposition asserts that the subtangential condition (ii-3) in Theorem 2.3 holds uniformly in a neighborhood of each element of C.

Proposition 4.2. ([18, Proposition 3.6]) *Suppose that* (ii-3) *in Theorem* 2.3 *holds. Let $v_0 \in C$. Assume that $\bar{h}$, $\varepsilon \in (0,1]$ and positive numbers ρ, M,η and ν satisfy that*

$$\begin{aligned} &\|Bx\| \leq M \quad \text{and} \quad \|Bx - Bv_0\| \leq \eta \quad \text{for } x \in U_Y[v_0,\rho] \cap C, \\ &K(M + \varepsilon + \nu)\bar{h}^\beta + \sup_{s \in [0,\bar{h}]} \|T(s)v_0 - v_0\|_Y \leq \rho. \end{aligned}$$

Let $\delta \in [0,\bar{h}]$, $w_0 \in C$ and G be a measurable function from $[0,\delta)$ into X such that

$$\begin{aligned} &\|w_0 - T(\delta)v_0\| \leq \delta(M + \nu), \quad \|G(\xi)\| \leq M \quad \text{for } \xi \in [0,\delta), \\ &\left\| w_0 - T(\delta)v_0 - \int_0^\delta T(\delta - \xi)G(\xi)\, d\xi \right\|_Y \leq K\nu\delta^\beta. \end{aligned}$$

Then, for each $\sigma > 0$ with $\sigma + \delta \leq \bar{h}$ there exist $z_0 \in C$ and $f_0 \in Y$ such that

$$z_0 = T(\sigma)w_0 + \int_0^\sigma T(\xi)Bw_0 \, d\xi + f_0, \quad \|f_0\| \leq \sigma(\varepsilon + 2\eta), \quad \|f_0\|_Y \leq K(\varepsilon + 2\eta)\sigma^\beta.$$

The following proposition establishes the existence of approximate solutions to the Cauchy problem for (SP).

Proposition 4.3. *Suppose that condition* (ii-3) *in Theorem* 2.3 *is satisfied. Let* $x_0 \in C$. *Assume that* $\bar{\tau} \in (0,1]$, $\rho_0 > 0$, $M_B > 0$ *and* $\varepsilon \in (0,1/2]$ *satisfy that*

$$\|Bx\| \leq M_B \quad \text{for } x \in U_Y[x_0,\rho_0] \cap C \text{ and } K(M_B+1)\bar{\tau}^\beta + \sup_{s\in[0,\bar{\tau}]} \|T(s)x_0 - x_0\|_Y \leq \rho_0.$$

Then there exists a sequence $\{(t_j, x_j, \zeta_j)\}_{j=1}^\infty$ *in* $[0,\bar{\tau}) \times C \times Y$ *satisfying the following:*

(i) $0 = t_0 < t_1 < \cdots < t_j < \cdots < \bar{\tau}$ *and* $t_j - t_{j-1} \leq \varepsilon$ *for* $j \geq 1$.

(ii) $x_j = T(t_j - t_{j-1})x_{j-1} + \displaystyle\int_{t_{j-1}}^{t_j} T(t_j - \xi)Bx_{j-1}\,d\xi + \zeta_j$ *for* $j \geq 1$.

(iii) $\|\zeta_j\| \leq \varepsilon(t_j - t_{j-1})$ *and* $\|\zeta_j\|_Y \leq \varepsilon(t_j - t_{j-1})^\beta$ *for* $j \geq 1$.

(iv) *If* $x \in C$ *satisfies the inequality*

$$\|x - x_{j-1}\|_Y \leq K(M_B+1)(t_j - t_{j-1})^\beta + \sup_{s\in[0,t_j - t_{j-1}]} \|T(s)x_{j-1} - x_{j-1}\|_Y,$$

then $\|Bx - Bx_{j-1}\| \leq \varepsilon/(4K)$ *for* $j \geq 1$.

(v) $K(M_B+1)(t_j - t_{j-1})^\beta + \sup_{s\in[0,t_j-t_{j-1}]} \|T(s)x_{j-1} - x_{j-1}\|_Y \leq \varepsilon$ *for* $j \geq 1$.

(vi) $\lim_{j\to\infty} t_j = \bar{\tau}$.

Outlined Proof. We shall construct inductively a sequence $\{(t_j, x_j, \zeta_j)\}_{j=1}^\infty$ in $[0,\bar{\tau}) \times C \times Y$ satisfying conditions (i) through (vi). For this purpose, let $i \geq 1$ and assume that a sequence $\{(t_j, x_j, \zeta_j)\}_{j=1}^{i-1}$ in $[0,\bar{\tau}) \times C \times Y$ can be constructed so that it satisfies (i) through (v). Then we define $\bar{h}_i$ by the supremum of numbers $h \in [0,\varepsilon]$ such that $h < \bar{\tau} - t_{i-1}$, $\|Bx - Bx_{i-1}\| \leq \varepsilon/(4K)$ for $x \in U_Y[x_{i-1},\rho] \cap C$, where $\rho := K(M_B+1)h^\beta + \sup_{s\in[0,h]} \|T(s)x_{i-1} - x_{i-1}\|_Y$ and $\rho \leq \varepsilon$. Since $\bar{h}_i > 0$ by condition (B1) and the strong continuity of $T(\cdot)$ in $B(Y)$ on $[0,\infty)$, there exists $h_i \in (0,\varepsilon]$ such that $\bar{h}_i/2 < h_i < \bar{\tau} - t_{i-1}$ and $\|Bx - Bx_{i-1}\| \leq \varepsilon/(4K)$ for $x \in U_Y[x_{i-1},\rho_i] \cap C$, where $\rho_i := K(M_B+1)h_i^\beta + \sup_{s\in[0,h_i]} \|T(s)x_{i-1} - x_{i-1}\|_Y \leq \varepsilon$. If we set $t_i = t_{i-1} + h_i$, then conditions (i), (iv) and (v) are satisfied. Next we apply Proposition 4.2 to show the existence of $x_i \in C$ and $\zeta_i \in Y$ satisfying conditions (ii) and (iii) with $j = i$. Thus, we obtain a sequence $\{(t_j, x_j, \zeta_j)\}_{j=1}^\infty$ satisfying conditions (i) through (v).

It remains to show that condition (vi) is satisfied. To this end, we assume to the contrary that $\bar{t} := \lim_{j\to\infty} t_j < \bar{\tau}$. Applying [18, Lemma 3.3 (i)], we have

$$\|x_i - x_j\|_Y \leq K(M_B+\varepsilon)((t_i - t_k)^\beta + (t_j - t_k)^\beta) + \|T(t_j - t_k)x_k - T(t_i - t_k)x_k\|_Y$$

for i, $j \geq k \geq 1$. This together with the strong continuity of $T(\cdot)$ in $B(Y)$ on $[0,\infty)$ implies that $\limsup_{i,j\to\infty} \|x_i - x_j\|_Y \leq 2K(M_B+\varepsilon)(\bar{t} - t_k)^\beta$ for all $k \geq 1$. Since

$\lim_{k\to\infty} t_k = \bar{t}$ and C is closed in Y, the inequality above shows that the sequence $\{x_j\}$ in C is convergent in Y to some $\bar{x} \in C$. Since $T(\cdot)$ is strongly continuous in $B(Y)$ on $[0,\infty)$ and B is continuous from C into X, one finds $\bar{h} \in (0,\varepsilon]$ such that $\bar{h} < \bar{\tau} - \bar{t}$ and $\|Bx - B\bar{x}\| \le \varepsilon/(8K)$ for $x \in U_Y[\bar{x}, 2\bar{\rho}] \cap C$, where $\bar{\rho} := K(M_B+1)\bar{h}^\beta + \sup_{s\in[0,\bar{h}]} \|T(s)\bar{x} - \bar{x}\|_Y \le \varepsilon/2$. Since the sequence $\{x_i\}$ converges in Y to $\bar{x}$ and since the sequence $\{\bar{\rho}_i\}$, defined by $\bar{\rho}_i = K(M_B + 1)\bar{h}^\beta + \sup_{s\in[0,\bar{h}]} \|T(s)x_{i-1} - x_{i-1}\|_Y$ for $i \ge 1$, converges to $\bar{\rho}$ as $i \to \infty$, there exists an integer $i_0 \ge 1$ such that $U_Y[x_{i-1}, \bar{\rho}_i] \subset U_Y[\bar{x}, 2\bar{\rho}]$, $\|B\bar{x} - Bx_{i-1}\| \le \varepsilon/(8K)$ and $\bar{\rho}_i \le \varepsilon$ for all $i \ge i_0$. Let $i \ge i_0$. Then we have

$$\begin{aligned} \|Bx - Bx_{i-1}\| &\le \|Bx - B\bar{x}\| + \|B\bar{x} - Bx_{i-1}\| \\ &\le \varepsilon/(8K) + \varepsilon/(8K) = \varepsilon/(4K) \qquad \text{for } x \in U_Y[x_{i-1}, \bar{\rho}_i] \cap C. \end{aligned} \tag{4.1}$$

Hence $\bar{h} \le \bar{h}_i$ for $i \ge i_0$. This contradicts the fact $\bar{h} > 0$, since $\bar{h}_i < 2h_i = 2(t_i - t_{i-1}) \to 0$ as $i \to \infty$. This proves that condition (vii) is satisfied. It is concluded that a sequence $\{(t_j, x_j, \zeta_j)\}_{j=1}^\infty$ in $[0, \bar{\tau}) \times C \times Y$ can be constructed so that conditions (i) through (vi) are satisfied. □

§ 5. Key estimate on the difference between approximate solutions

In this section we give a key estimate to showing the convergence of a sequence of approximate solutions constructed in the previous sections. The proof is similar to but more complicated than that in [17, 14, 10].

Throughout this section, condition (ii) in Theorem 2.3 is assumed to be satisfied, and let τ stand for a number appearing condition (ii-1) in Theorem 2.3. The symbols $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$ are used in the rest of this paper.

Proposition 5.1. *Let $\bar{v}_0$, $\hat{v}_0 \in C$. Assume that $\bar{h} \in (0, \tau \wedge 1]$, $\bar{\rho} > 0$, $\overline{M} > 0$, $\bar{\eta} > 0$, $\bar{\varepsilon} \in (0, 1]$, $\bar{\nu} > 0$, $\hat{h} \in (0, \tau \wedge 1]$, $\hat{\rho} > 0$, $\widehat{M} > 0$, $\hat{\eta} > 0$, $\hat{\varepsilon} \in (0, 1]$ and $\hat{\nu} > 0$ satisfy the following conditions:*

$$\begin{aligned} &\|Bx\| \le \overline{M} \quad \textit{and} \quad \|Bx - B\bar{v}_0\| \le \bar{\eta} \quad \textit{for } x \in U_Y[\bar{v}_0, \bar{\rho}] \cap C. \\ &\|Bx\| \le \widehat{M} \quad \textit{and} \quad \|Bx - B\hat{v}_0\| \le \hat{\eta} \quad \textit{for } x \in U_Y[\hat{v}_0, \hat{\rho}] \cap C. \end{aligned} \tag{5.1}$$

$$\begin{aligned} &K(\overline{M} + \bar{\varepsilon} + \bar{\nu})\bar{h}^\beta + \sup_{s\in[0,\bar{h}]} \|T(s)\bar{v}_0 - \bar{v}_0\|_Y \le \bar{\rho}. \\ &K(\widehat{M} + \hat{\varepsilon} + \hat{\nu})\hat{h}^\beta + \sup_{s\in[0,\hat{h}]} \|T(s)\hat{v}_0 - \hat{v}_0\|_Y \le \hat{\rho}. \end{aligned} \tag{5.2}$$

Let $\bar{\delta} \in [0, \bar{h}]$, $\hat{\delta} \in [0, \hat{h}]$ and $\bar{w}_0$, $\hat{w}_0 \in C$, and let $\overline{G} : [0, \bar{\delta}) \to X$ and $\widehat{G} : [0, \hat{\delta}) \to X$ be measurable functions such that they satisfy the following conditions:

$$\|\bar{w}_0 - T(\bar{\delta})\bar{v}_0\| \le \bar{\delta}(\overline{M} + \bar{\nu}), \qquad \left\| \bar{w}_0 - T(\bar{\delta})\bar{v}_0 - \int_0^{\bar{\delta}} T(\bar{\delta} - \xi)\overline{G}(\xi)\, d\xi \right\|_Y \le K\bar{\nu}\bar{\delta}^\beta. \tag{5.3}$$

(5.4) $\|\overline{G}(\xi)\| \le \overline{M}$ *for* $\xi \in [0, \bar{\delta})$.

$$\|\hat{w}_0 - T(\hat{\delta})\hat{v}_0\| \le \hat{\delta}(\widehat{M} + \hat{\nu}), \quad \left\| \hat{w}_0 - T(\hat{\delta})\hat{v}_0 - \int_0^{\hat{\delta}} T(\hat{\delta} - \xi)\widehat{G}(\xi)\, d\xi \right\|_Y \le K\hat{\nu}\hat{\delta}^{\beta}.$$

$\|\widehat{G}(\xi)\| \le \widehat{M}$ *for* $\xi \in [0, \hat{\delta})$.

Let $\tau_0 \in [0, \tau)$. *Then, for each* $\sigma > 0$ *with* $\sigma + \bar{\delta} \le \bar{h}$, $\sigma + \hat{\delta} \le \hat{h}$ *and* $\sigma + \tau_0 \le \tau$, *and for each* $t \in [\tau_0 + \sigma, \tau]$, *there exist* $\bar{z}_0$, $\hat{z}_0 \in C$ *and* $\bar{f}_0, \hat{f}_0 \in Y$ *such that*

$$\bar{z}_0 = T(\sigma)\bar{w}_0 + \int_0^{\sigma} T(\xi)B\bar{w}_0\, d\xi + \bar{f}_0, \ \|\bar{f}_0\| \le \sigma(\bar{\varepsilon} + 2\bar{\eta}), \ \|\bar{f}_0\|_Y \le K\sigma^{\beta}(\bar{\varepsilon} + 2\bar{\eta}), \tag{5.5}$$

$$\hat{z}_0 = T(\sigma)\hat{w}_0 + \int_0^{\sigma} T(\xi)B\hat{w}_0\, d\xi + \hat{f}_0, \ \|\hat{f}_0\| \le \sigma(\hat{\varepsilon} + 2\hat{\eta}), \ \|\hat{f}_0\|_Y \le K\sigma^{\beta}(\hat{\varepsilon} + 2\hat{\eta}), \tag{5.6}$$

$$\|\bar{w}_0 - \bar{v}_0\|_Y \le \bar{\rho}, \quad \|\hat{w}_0 - \hat{v}_0\|_Y \le \hat{\rho}, \quad \|\bar{z}_0 - \bar{v}_0\|_Y \le \bar{\rho}, \quad \|\hat{z}_0 - \hat{v}_0\|_Y \le \hat{\rho}, \tag{5.7}$$

$$\begin{aligned} &V(t, \sigma + \tau_0, \bar{z}_0, \hat{z}_0) - V(t, \tau_0, \bar{w}_0, \hat{w}_0) \\ &\le \omega t^{\beta_0} \left(\int_{\tau_0}^{\sigma+\tau_0} (t - \xi)^{-\alpha_0} \xi^{-\beta_0}\, d\xi \right) V(\tau_0, \tau_0, \bar{w}_0, \hat{w}_0) \\ &\quad + t^{\beta_0} \left(\int_{\tau_0}^{\sigma+\tau_0} (t - \xi)^{-\alpha_0} \xi^{-\beta_0}\, d\xi \right) \big(\omega\theta(2\sigma)(1 + \|\bar{w}_0\|_Y + \|\hat{w}_0\|_Y) \\ &\quad + 2\omega L_Y(\bar{\rho} + \hat{\rho}) + \bar{\varepsilon} + \hat{\varepsilon}\big) \\ &\quad + L_X(\bar{\varepsilon} + \hat{\varepsilon}) \left(t^{\beta_0} \int_{\sigma}^{\sigma+\tau_0} (t - \xi)^{-\alpha}\, d\xi + \sigma(1 + t^{\beta_0}(\bar{\varepsilon} + \hat{\varepsilon})) \right). \end{aligned} \tag{5.8}$$

Outlined Proof. We first construct a sequence $\{(s_j, \bar{w}_j, \hat{w}_j, \bar{\zeta}_j, \hat{\zeta}_j)\}_{j=1}^{\infty}$ in $[0, \sigma) \times C \times C \times Y \times Y$ satisfying the following conditions:

(i) $0 = s_0 < s_1 < \cdots < s_j < \cdots < \sigma$.

(ii-1) $\bar{w}_j = T(s_j - s_{j-1})\bar{w}_{j-1} + \int_{s_{j-1}}^{s_j} T(s_j - \xi)B\bar{w}_{j-1}\, d\xi + \bar{\zeta}_j$ for $j \ge 1$.

(ii-2) $\hat{w}_j = T(s_j - s_{j-1})\hat{w}_{j-1} + \int_{s_{j-1}}^{s_j} T(s_j - \xi)B\hat{w}_{j-1}\, d\xi + \hat{\zeta}_j$ for $j \ge 1$.

(iii-1) $\|\bar{\zeta}_j\| \le \bar{\varepsilon}(s_j - s_{j-1})$ and $\|\bar{\zeta}_j\|_Y \le \bar{\varepsilon}(s_j - s_{j-1})^{\beta}$ for $j \ge 1$.

(iii-2) $\|\hat{\zeta}_j\| \le \hat{\varepsilon}(s_j - s_{j-1})$ and $\|\hat{\zeta}_j\|_Y \le \hat{\varepsilon}(s_j - s_{j-1})^{\beta}$ for $j \ge 1$.

(iv) $$\begin{aligned} &(V(t, s_j + \tau_0, \bar{w}_j, \hat{w}_j) - V(t, s_{j-1} + \tau_0, \bar{w}_{j-1}, \hat{w}_{j-1}))/(s_j - s_{j-1}) \\ &\le \omega t^{\beta_0}(t - s_{j-1} - \tau_0)^{-\alpha_0}(s_j + \tau_0)^{-\beta_0} V(s_{j-1} + \tau_0, s_{j-1} + \tau_0, \bar{w}_{j-1}, \hat{w}_{j-1}) \\ &\quad + t^{\beta_0}(t - s_{j-1} - \tau_0)^{-\alpha_0}(s_j + \tau_0)^{-\beta_0}(\bar{\varepsilon} + \hat{\varepsilon}) \\ &\quad + L_X(1 + t^{\beta_0}(t - (s_j + \tau_0))^{-\alpha})(\bar{\varepsilon} + \hat{\varepsilon}) \quad \text{for } j \ge 1. \end{aligned}$$

(v) $(t-(s_j+\tau_0))^{-\alpha}(s_j-s_{j-1}) \leq \int_{s_{j-1}+\tau_0}^{s_j+\tau_0} (t-\xi)^{-\alpha}\,d\xi + (s_j-s_{j-1})(\bar{\varepsilon}+\hat{\varepsilon})$ for $j \geq 1$.

(vi) $\lim_{j\to\infty} s_j = \sigma$.

The above sequence can be constructed inductively in a way similar to [18, Proposition 4.1]. Applying [18, Lemma 3.4 (ii)] to the sequence constructed above, we find $\bar{z}_0$, $\hat{z}_0 \in C$ and $\bar{f}_0, \hat{f}_0 \in Y$ such that $\lim_{j\to\infty} \|\bar{z}_0 - \bar{w}_j\|_Y = 0$, $\lim_{j\to\infty} \|\hat{z}_0 - \hat{w}_j\|_Y = 0$ and conditions (5.5) and (5.6) are satisfied. From (iv) and (v) we deduce that

$$
\begin{aligned}
(5.9)\quad & V(t, s_k+\tau_0, \bar{w}_k, \hat{w}_k) - V(t,\tau_0,\bar{w}_0,\hat{w}_0) \\
&\leq \omega t^{\beta_0} \sum_{j=1}^{k} h_j (t-s_{j-1}-\tau_0)^{-\alpha_0}(s_j+\tau_0)^{-\beta_0} V(s_{j-1}+\tau_0, s_{j-1}+\tau_0, \bar{w}_{j-1}, \hat{w}_{j-1}) \\
&\quad + t^{\beta_0} \sum_{j=1}^{k} h_j (t-s_{j-1}-\tau_0)^{-\alpha_0}(s_j+\tau_0)^{-\beta_0}(\bar{\varepsilon}+\hat{\varepsilon}) \\
&\quad + L_X(\bar{\varepsilon}+\hat{\varepsilon})\left(t^{\beta_0}\int_{\tau_0}^{s_k+\tau_0} (t-\xi)^{-\alpha}\,d\xi + \sigma(1+t^{\beta_0}(\bar{\varepsilon}+\hat{\varepsilon}))\right)
\end{aligned}
$$

for $k = 1, 2, \ldots$. To estimate the first term on the right-hand side of (5.9), let $0 \leq l \leq k-1$. By Remark 2 (b-iii) and (V3) we have

$$
\begin{aligned}
V(s_l+\tau_0, s_l+\tau_0, \bar{w}_l, \hat{w}_l) &\leq V(s_l+\tau_0, s_l+\tau_0, \bar{w}_0, \hat{w}_0) \\
&\quad + L_Y(\|\bar{w}_l - \hat{w}_0\|_Y + \|\hat{w}_l - \hat{w}_0\|_Y) \\
&\leq V(\tau_0,\tau_0,\bar{w}_0,\hat{w}_0) + \theta(2\sigma)(1+\|\bar{w}_0\|_Y + \|\hat{w}_0\|_Y) \\
&\quad + L_Y(\|\bar{w}_l - \bar{w}_0\|_Y + \|\hat{w}_l - \hat{w}_0\|_Y).
\end{aligned}
$$

Since $\bar{w}_l - \bar{v}_0 = (\bar{w}_l - T(s_l+\delta)\bar{v}_0) + (T(s_l+\delta)\bar{v}_0 - \bar{v}_0)$, we have

$$
\|\bar{w}_l - \bar{v}_0\|_Y \leq K(\bar{M}+\bar{\varepsilon}+\bar{\nu})(s_l+\delta)^{\beta} + \|T(s_l+\delta)\bar{v}_0 - \bar{v}_0\| \leq \bar{\rho}.
$$

This implies (5.7) and

$$
\begin{aligned}
V(s_l+\tau_0, s_l+\tau_0, \bar{w}_l, \hat{w}_l) &\leq V(\tau_0,\tau_0,\bar{w}_0,\hat{w}_0) + \theta(2\sigma)(1+\|\bar{w}_0\|_Y + \|\hat{w}_0\|_Y) \\
&\quad + 2L_Y(\bar{\rho}+\hat{\rho}).
\end{aligned}
$$

Substituting this inequality into (5.9) and using the inequality

$$
\sum_{j=1}^{k}\int_{s_{j-1}}^{s_j} (t-(s_{j-1}+\tau_0))^{-\alpha_0}(s_j+\tau_0)^{-\beta_0}\,d\xi \leq \int_0^{s_k} (t-\xi-\tau_0)^{-\alpha_0}(\xi+\tau_0)^{-\beta_0}\,d\xi,
$$

we have

$$
V(t, s_k+\tau_0, \bar{w}_k, \hat{w}_k) - V(t,\tau_0,\bar{w}_0,\hat{w}_0)
$$

$$
\begin{aligned}
&\le \omega t^{\beta_0}\left(\int_{\tau_0}^{s_k+\tau_0}(t-\xi)^{-\alpha_0}\xi^{-\beta_0}\,d\xi\right)V(\tau_0,\tau_0,\bar{w}_0,\hat{w}_0)\\
&\quad + t^{\beta_0}\left(\int_{\tau_0}^{s_k+\tau_0}(t-\xi)^{-\alpha_0}\xi^{-\beta_0}\,d\xi\right)(\omega\theta(2\sigma)(1+\|\bar{w}_0\|_Y+\|\hat{w}_0\|_Y)\\
&\quad + 2\omega L_Y(\bar{\rho}+\hat{\rho})+\bar{\varepsilon}+\hat{\varepsilon})\\
&\quad + L_X(\bar{\varepsilon}+\hat{\varepsilon})\left(t^{\beta_0}\int_{\tau_0}^{s_k+\tau_0}(t-\xi)^{-\alpha}\,d\xi+\sigma(1+t^{\beta_0}(\bar{\varepsilon}+\hat{\varepsilon}))\right)
\end{aligned}
$$

for $k = 1, 2, \ldots$ Passing to the limit as $k \to \infty$, we obtain the inequality (5.8). $\square$

The following is the key to showing the convergence of a sequence of approximate solutions.

Proposition 5.2. *Let $x_0 \in C$. Let $0 < \bar{\tau} \le (\tau \wedge 1)$, $\rho_0 > 0$, $M_B > 0$ and λ, $\mu \in (0, 1/2]$ and suppose that*

$$\|Bx\| \le M_B \text{ for } x \in U_Y[x_0,\rho_0]\cap C \text{ and } K(M_B+1)\bar{\tau}^{\beta} + \sup_{s\in[0,\bar{\tau}]}\|T(s)x_0 - x_0\|_Y \le \rho_0.$$

For each $\varepsilon = \lambda, \mu$, suppose that there exists a sequence $\{(t_j^\varepsilon, x_j^\varepsilon, \zeta_j^\varepsilon)\}_{j=1}^\infty$ in $[0,\bar{\tau})\times C\times Y$ satisfying the following conditions:

(i) $0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_j^\varepsilon < \cdots < \bar{\tau}$.

(ii) $x_j^\varepsilon = T(t_j^\varepsilon - t_{j-1}^\varepsilon)x_{j-1}^\varepsilon + \int_{t_{j-1}^\varepsilon}^{t_j^\varepsilon} T(t_j^\varepsilon-\xi)Bx_{j-1}^\varepsilon\,d\xi + \zeta_j^\varepsilon$ *for $j \ge 1$, where $x_0^\varepsilon = x_0$.*

(iii) $\|\zeta_j^\varepsilon\| \le \varepsilon(t_j^\varepsilon - t_{j-1}^\varepsilon)$ *and* $\|\zeta_j^\varepsilon\|_Y \le \varepsilon(t_j^\varepsilon - t_{j-1}^\varepsilon)^\beta$ *for $j \ge 1$.*

(iv) *If $x \in C$ satisfies the inequality*

$$\|x - x_{j-1}^\varepsilon\|_Y \le K(M_B+1)(t_j^\varepsilon - t_{j-1}^\varepsilon)^\beta + \sup_{s\in[0,t_j^\varepsilon - t_{j-1}^\varepsilon]}\|T(s)x_{j-1}^\varepsilon - x_{j-1}^\varepsilon\|_Y,$$

then $\|Bx - Bx_{j-1}^\varepsilon\| \le \varepsilon/(4K)$, for $j \ge 1$.

(v) $K(M_B+1)(t_j - t_{j-1})^\beta + \sup_{s\in[0,t_j-t_{j-1}]}\|T(s)x_{j-1} - x_{j-1}\|_Y \le \varepsilon$ *for $j \ge 1$.*

(vi) $\lim_{j\to\infty} t_j^\varepsilon = \bar{\tau}$.

Set $P = \{t_i^\lambda; i = 0,1,\ldots\} \cup \{t_j^\mu; j = 0,1,\ldots\}$, and define $s_0 = 0$ and $s_k = \inf(P \setminus \{s_0, s_1, \ldots, s_{k-1}\})$ for $k \ge 1$. Let N be a nonnegative integer. Then there exists a sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^N$ in $C\times C$ and a sequence $\{(\phi_k^\lambda, \phi_k^\mu)\}_{k=0}^N$ in $Y\times Y$ satisfying the following conditions:

(a) *For each $\varepsilon = \lambda, \mu$ and $0 \le k \le N$, if $s_k = t_i^\varepsilon$ for some i then $z_k^\varepsilon = x_i^\varepsilon$.*

(b) *For each $\varepsilon = \lambda, \mu$ and $0 \le k \le N$, if $s_k \ne t_i^\varepsilon$ for all i then the element f_k^ε in Y defined by*

$$f_k^\varepsilon = T(s_k - s_{k-1})z_{k-1}^\varepsilon + \int_{s_{k-1}}^{s_k} T(s_k - \xi)Bz_{k-1}^\varepsilon \, d\xi - z_k^\varepsilon \tag{5.10}$$

satisfies $\|f_k^\varepsilon\| \le \varepsilon(s_k - s_{k-1})$ and $\|f_k^\varepsilon\|_Y \le \varepsilon(s_k - s_{k-1})^\beta$.

(c) *For each $\varepsilon = \lambda, \mu$ and $0 \le k \le N$, if $s_k = t_i^\varepsilon$ for some i, then*

$$\|\phi_k^\varepsilon\| \le 3(t_i^\varepsilon - t_{i-1}^\varepsilon)\varepsilon, \qquad \|\phi_k^\varepsilon\|_Y \le 3K(t_i^\varepsilon - t_{i-1}^\varepsilon)^\beta \varepsilon.$$

For each $\varepsilon = \lambda, \mu$ and $1 \le k \le N$, if $s_k \ne t_i^\varepsilon$ for all i, then $\phi_k^\varepsilon = 0$.

(d) *For $0 \le k \le N$, $V(s_k, s_k, z_k^\lambda, z_k^\mu) \le L_Y(2\rho_0 + (1 + 3K\bar{\tau}^\beta)(\lambda + \mu))$.*

(e) *For $1 \le k \le N$,*

$$\begin{aligned} &V(s_N, s_k, z_k^\lambda, z_k^\mu) - V(s_N, s_{k-1}, z_{k-1}^\lambda, z_{k-1}^\mu) \\ &\le L(\|\phi_k^\lambda\| + \|\phi_k^\mu\| + \bar{\tau}^{\beta_0}\|T(s_N - s_k)\phi_k^\lambda\|_Y + \bar{\tau}^{\beta_0}\|T(s_N - s_k)\phi_k^\mu\|_Y) \\ &\quad + \omega s_N^{\beta_0}\left(\int_{s_{k-1}}^{s_k} (s_N - \xi)^{-\alpha_0}\xi^{-\beta_0}\, d\xi\right) V(s_{k-1}, s_{k-1}, z_{k-1}^\lambda, z_{k-1}^\mu) \\ &\quad + s_N^{\beta_0}\left(\int_{s_{k-1}}^{s_k} (s_N - \xi)^{-\alpha_0}\xi^{-\beta_0}\, d\xi\right) \delta_{\lambda,\mu}(x_0, \rho_0) \\ &\quad + L_X(\lambda + \mu)\left(s_N^{\beta_0}\int_{s_{k-1}}^{s_k} (s_N - \xi)^{-\alpha}\, d\xi + (s_k - s_{k-1})(1 + s_N^{\beta_0}(\lambda + \mu))\right), \end{aligned} \tag{5.11}$$

where $\delta_{\lambda,\mu}(x_0, \rho_0) = \omega\theta(2\lambda + 2\mu)(1 + 2\|x_0\|_Y + 2\rho_0 + \lambda + \mu) + (2\omega L_Y + 1)(\lambda + \mu)$.

Outlined Proof. We shall construct inductively a sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^N$ in $C \times C$ satisfying conditions (a) through (d). For this purpose, set $(z_0^\lambda, z_0^\mu) = (x_0^\lambda, x_0^\mu)$. Let $1 \le l \le N$ and assume that a sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^{l-1}$ in $C \times C$ can be chosen so that conditions (a) through (d) are satisfied. Then we want to find a pair $(z_l^\lambda, z_l^\mu) \in C \times C$ as required, by applying Proposition 5.1. Since all the assumptions of Proposition 5.1 can be checked in a way similar to the proof of [18, Proposition 4.2], we apply Proposition 5.1 with $t = s_N$ to find $y_l^\lambda, y_l^\mu \in C$ and $g_l^\lambda, g_l^\mu \in Y$ satisfying the following conditions:

$$\begin{aligned} &y_l^\lambda = T(s_l - s_{l-1})z_{l-1}^\lambda + \int_0^{s_l - s_{l-1}} T(\xi)Bz_{l-1}^\lambda \, d\xi + g_l^\lambda, \\ &\|g_l^\lambda\| \le \lambda(s_l - s_{l-1}), \qquad \|g_l^\lambda\|_Y \le \lambda(s_l - s_{l-1})^\beta, \\ &y_l^\mu = T(s_l - s_{l-1})z_{l-1}^\mu + \int_0^{s_l - s_{l-1}} T(\xi)Bz_{l-1}^\mu \, d\xi + g_l^\mu, \end{aligned} \tag{5.12}$$

$$\|g_l^\mu\| \le \mu(s_l - s_{l-1}), \qquad \|g_l^\mu\|_Y \le \mu(s_l - s_{l-1})^\beta,$$

$$\|z_{l-1}^\lambda - x_{i-1}^\lambda\|_Y \le \lambda, \qquad \|z_{l-1}^\mu - x_{j-1}^\mu\|_Y \le \mu, \tag{5.13}$$
$$\|y_l^\lambda - x_{i-1}^\lambda\|_Y \le \lambda, \qquad \|y_l^\mu - x_{j-1}^\mu\|_Y \le \mu,$$

$$\begin{aligned} &V(s_N, s_l, y_l^\lambda, y_l^\mu) - V(s_N, s_{l-1}, z_{l-1}^\lambda, z_{l-1}^\mu) \\ &\le \omega s_N^{\beta_0} \left(\int_{s_{l-1}}^{s_l} (s_N - \xi)^{-\alpha_0} \xi^{-\beta_0} \, d\xi \right) V(s_{l-1}, s_{l-1}, z_{l-1}^\lambda, z_{l-1}^\mu) \\ &+ s_N^{\beta_0} \left(\int_{s_{l-1}}^{s_l} (s_N - \xi)^{-\alpha_0} \xi^{-\beta_0} \, d\xi \right) (\omega\theta(2\lambda + 2\mu)(1 + \|z_{l-1}^\lambda\|_Y + \|z_{l-1}^\mu\|_Y) \\ &+ (2\omega L_Y + 1)(\lambda + \mu)) \\ &+ L_X(\lambda + \mu) \left(s_N^{\beta_0} \int_{s_{l-1}}^{s_l} (s_N - \xi)^{-\alpha} \, d\xi + (s_l - s_{l-1})(1 + s_N^{\beta_0}(\lambda + \mu)) \right). \end{aligned} \tag{5.14}$$

Now, we consider the pair (z_l^λ, z_l^μ) in $C \times C$ defined by

$$z_l^\lambda = \begin{cases} y_l^\lambda & \text{if } s_l < t_i^\lambda \\ x_i^\lambda & \text{if } s_l = t_i^\lambda \end{cases} \quad \text{and} \quad z_l^\mu = \begin{cases} y_l^\mu & \text{if } s_l < t_j^\mu \\ x_j^\mu & \text{if } s_l = t_j^\mu, \end{cases}$$

and the pair $(\phi_l^\lambda, \phi_l^\mu)$ in $Y \times Y$ defined by $\phi_l^\lambda = z_l^\lambda - y_l^\lambda$ and $\phi_l^\mu = z_l^\mu - y_l^\mu$. Then, conditions (a) and (b) are clearly satisfied. Note by (5.13) that $\|z_{l-1}^\lambda\|_Y \le \|x_0\|_Y + \rho_0 + \lambda$ and $\|z_{l-1}^\mu\|_Y \le \|x_0\|_Y + \rho_0 + \mu$. Substituting these inequalities into (5.14), and using condition (V1), we verify condition (e). Once condition (c) with $k = l$ is proved, condition (d) can be shown by using (5.13),

$$\begin{aligned} V(s_l, s_l, z_l^\lambda, z_l^\mu) &\le V(s_l, s_l, y_l^\lambda, y_l^\mu) + L_Y(\|\phi_l^\lambda\|_Y + \|\phi_l^\mu\|_Y) \\ &\le L_Y(\|y_l^\lambda - y_l^\mu\|_Y + \|\phi_l^\lambda\|_Y + \|\phi_l^\mu\|_Y) \end{aligned}$$

and $\|x_{i-1}^\lambda - x_{j-1}^\mu\|_Y \le 2\rho_0$. The fact that the pair (z_l^λ, z_l^μ) satisfies conditions (c) for $k = l$ is proved similarly to [18, Proposition 4.2]. We here omit the detail. Thus, the desired sequences $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^N$ in $C \times C$ and $\{(\phi_k^\lambda, \phi_k^\mu)\}_{k=0}^N$ in $Y \times Y$ can be constructed inductively. □

§ 6. Proof of Theorem 2.3

We begin by showing the implication "(i) ⇒ (ii)". To do this, assume that there exists a semigroup $\{S(t); t \ge 0\}$ of Lipschitz operators on D such that for each $x \in D$,

$S(\cdot)x$ is a global mild solution to (SP;x). It is known [11, Theorem 4.1] that there exist $\omega_0 \geq 0$, $M \geq m > 0$, $L > 0$ and a nonnegative functional V_0 on $X \times X$ such that

$$
\begin{aligned}
&|V_0(x,y) - V_0(\hat{x},\hat{y})| \leq L(\|x-y\| + \|\hat{x}-\hat{y}\|) \quad \text{for } (x,y),(\hat{x},\hat{y}) \in X \times X,\\
&m\|x-y\| \leq V_0(x,y) \leq M\|x-y\| \quad \text{for } x,y \in D,\\
&V_0(S(t)x, S(t)y) \leq e^{\omega_0 t} V_0(x,y) \quad \text{for } x,y \in D \text{ and } t \geq 0.
\end{aligned}
$$

Let $\tau > 0$ and define a nonnegative functional V on $\Delta \times Y \times Y$ by $V(t,s,x,y) = V_0(x,y)$ for $(t,s,x,y) \in \Delta \times Y \times Y$. Let $\alpha_0, \beta_0 \in [0,1)$. Then, it is obvious that the functional V satisfies conditions (V1) through (V3). It remains to check conditions (ii-2) and (ii-3). To do this, let $(t,s) \in \Delta$ with $t \neq s$ and $(x,y) \in C \times C$. Then we have, for $h > 0$ such that $s + h \leq t$,

$$
\begin{aligned}
&((s+h)/t)^{\beta_0}(t-s)^{\alpha_0}(V(t,s+h,J(h)x,J(h)y) - V(t,s,x,y))/h\\
&\quad \leq (t-s)^{\alpha_0}(V_0(J(h)x,J(h)y) - V_0(x,y))/h\\
&\quad \leq \tau^{\alpha_0}\{h^{-1}(e^{\omega_0 h}-1)V_0(x,y) + L(\|J(h)x - S(h)x\| + \|J(h)y - S(h)y\|)/h\},
\end{aligned}
$$

and the last term on the right-hand side vanishes as $h \downarrow 0$ because $S(\cdot)x$ is a mild solution to (SP;x) and $\lim_{h\downarrow 0} h^{-1}(S(h)x - J(h)x) = \lim_{h\downarrow 0} h^{-1}\int_0^h T(h-s)(BS(s)x - Bx)\,ds = 0$. Here we have used the continuity of $BS(\cdot)x$ in X at $t = 0$ (by Proposition 3.3 (v)). Condition (ii-2) is thus shown to be satisfied with $\omega = \tau^{\alpha_0}\omega_0$. To check condition (ii-3), let $x \in C$. Then, we have $S(\cdot)x \in C([0,\infty);Y)$ and $BS(\cdot)x \in C([0,\infty);X)$ by Proposition 3.3 (v). Since $J(h)x - S(h)x \in D((-A)^\alpha)$ and

$$
\begin{aligned}
\|(-A)^\alpha(J(h)x - S(h)x)\| &\leq \int_0^h \|(-A)^\alpha T(h-s)(Bx - BS(s)x)\|\,ds\\
&\leq (1-\alpha)^{-1}M_\alpha \sup_{0\leq s\leq h}\|Bx - BS(s)x\| \cdot h^{1-\alpha}
\end{aligned}
$$

for all $h > 0$, and since $S(h)x \in C$ for all $h > 0$, we observe that condition (ii-3) is satisfied with $\beta = 1-\alpha$ and $x_\delta = S(\delta)x$.

In order to prove that (ii) implies (i), by [18, Propositions 2.5 and 2.6] we have only to show the existence of a local mild solution with initial value in C. To this end, let $x_0 \in C$. Then, condition (B) ensures the existence of $\rho_0 > 0$ and $M_B > 0$ satisfying $\|Bx\| \leq M_B$ for $x \in U_Y[x_0,\rho_0] \cap C$. By continuity, there exists $a > 0$ such that $K(M_B+1)a^\beta + \sup_{\xi\in[0,a]}\|T(\xi)x_0 - x_0\|_Y \leq \rho_0$. Let b be a positive number satisfying $\omega b^{1-\alpha}B(1-\alpha_0, 1-\beta_0) < 1$, where $B(\cdot,\cdot)$ is the beta function. Set $\bar{\tau} = a \wedge b \wedge \tau \wedge 1$. Then we have $K(M_B+1)\bar{\tau}^\beta + \sup_{\xi\in[0,\bar{\tau}]}\|T(\xi)x_0 - x_0\|_Y \leq \rho_0$. Proposition 4.3 asserts that for each $\varepsilon \in (0,\varepsilon_0]$ there exists a sequence $\{(t_j^\varepsilon, x_j^\varepsilon, \zeta_j^\varepsilon)\}_{j=1}^\infty$ in $[0,\bar{\tau}) \times C \times Y$ satisfying (i) through (vii) in Proposition 4.3. For each $\varepsilon \in (0,\varepsilon_0]$, we define a family $\{u^\varepsilon\}$ of step functions by $u^\varepsilon(t) = x_i^\varepsilon$ for $t \in [t_i^\varepsilon, t_{i+1}^\varepsilon)$ and $i = 0,1,2,\ldots$ Once it is demonstrated that the family $\{u^\varepsilon\}$ converges in the space $C([0,\bar{\tau});X)$ as $\varepsilon \downarrow 0$, the proof of the implication "(ii) $\Rightarrow$ (i)" is completed just as in the proof of [18, Theorem 5.2].

Now, let $\lambda, \mu \in (0, \varepsilon_0]$, and let $\{s_k\}_{k=0}^\infty$ be a sequence constructed as in Proposition 5.2. Then, in order to show that the family $\{u^\varepsilon\}$ converges in the space $C([0,\bar{\tau});X)$ as $\varepsilon \downarrow 0$, we use the step function $\Psi : [0,\bar{\tau}) \to \mathbb{R}_+$ defined by

$$\Psi(s) = V(s_{k-1}, s_{k-1}, u^\lambda(s_{k-1}), u^\mu(s_{k-1})) \quad \text{for } s \in [s_{k-1}, s_k) \text{ and } k = 1,2,3,\ldots$$

Let $t \in [0,\bar{\tau})$. Then there exists a nonnegative integer N such that $t \in [s_N, s_{N+1})$. Applying Proposition 5.2, we find a sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^N$ in $C \times C$ and a sequence $\{(\phi_k^\lambda, \phi_k^\mu)\}_{k=0}^N$ in $Y \times Y$ satisfying (a) through (e) in Proposition 5.2. In order to estimate $\Psi(t)$ we need the following inequality

$$|V(s_{k-1}, s_{k-1}, z_{k-1}^\lambda, z_{k-1}^\mu) - V(s_{k-1}, s_{k-1}, u^\lambda(s_{k-1}), u^\mu(s_{k-1}))| \le L_Y(\lambda + \mu) \tag{6.1}$$

for $1 \le k \le N+1$, which is derived in a way similar to [18, Theorem 5.2]. By Proposition 5.2 (c) we have $\sum_{k=1}^N \|\phi_k^\lambda\| = 3\lambda \sum_{t_i^\lambda \in \{s_1,\ldots,s_K\}} (t_i^\lambda - t_{i-1}^\lambda)$. We shall apply Lemma 4.1 (ii) to estimate $\sum_{k=1}^N \|T(s_N - s_k)\phi_k^\lambda\|_Y$. Let $\{k_j\}_{j=1}^l$ be the increasing sequence consisting of all $1 \le k \le N$ such that $s_k = t_i^\lambda$ for some i. Then there exists an increasing sequence $\{i_j\}_{j=1}^l$ such that $s_{k_j} = t_{i_j}^\lambda$ for $1 \le j \le l$. By Proposition 5.2 (c) we have $\phi_k^\lambda = 0$ for $k \notin \{k_1, \ldots, k_l\}$; hence $\sum_{k=1}^N \|T(s_N - s_k)\phi_k^\lambda\|_Y = \sum_{j=1}^l \|T(s_N - t_{i_j})\phi_{k_j}^\lambda\|_Y$. Set $t_{i_0}^\lambda = 0$. Since $t_{i_{j-1}}^\lambda \le t_{i_j - 1}^\lambda < t_{i_j}^\lambda$ for $1 \le j \le l$, Proposition 5.2 (c) implies that $\|\phi_{k_j}^\lambda\| \le 3\lambda(t_{i_j}^\lambda - t_{i_{j-1}}^\lambda)$ and $\|\phi_{k_j}^\lambda\|_Y \le 3K\lambda(t_{i_j}^\lambda - t_{i_{j-1}}^\lambda)^\beta$ for $1 \le j \le l$. We therefore apply Lemma 4.1 (ii) to obtain $\sum_{k=1}^N \|T(s_N - s_k)\phi_k^\lambda\|_Y \le 3K^2\lambda s_N^\beta$.

Using (6.1) to estimate the second term on the right-hand side of (5.11) and adding the resulting inequality from $k = 1$ and $k = N$, we find

$$\begin{aligned}\Psi(t) \le & L_Y(\lambda+\mu) + 3KL(\bar{\tau} + K\bar{\tau}^{\beta+\beta_0})(\lambda+\mu) + \omega s_N^{\beta_0} \int_0^{s_N} (s_N - \xi)^{-\alpha_0} \xi^{-\beta_0} \Psi(\xi)\, d\xi \\ & + s_N^{\beta_0} \left(\int_0^{s_N} (s_N - \xi)^{-\alpha_0} \xi^{-\beta_0}\, d\xi \right) (\omega L_Y(\lambda+\mu) + \delta_{\lambda,\mu}(x_0, \rho_0)) \\ & + L_X(\lambda+\mu) \left(\bar{\tau}^{\beta_0} \int_0^{s_N} (s_N - \xi)^{-\alpha}\, d\xi + \bar{\tau}(1 + \bar{\tau}^{\beta_0}(\lambda+\mu)) \right).\end{aligned}$$

The third term on the right-hand side is estimated in such a way that

$$s_N^{\beta_0} \int_0^{s_N} (s_N - \xi)^{-\alpha_0} \xi^{-\beta_0} \Psi(\xi)\, d\xi \le \bar{\tau}^{1-\alpha_0} B(1-\alpha_0, 1-\beta_0) \sup_{s \in [0,\bar{\tau})} \Psi(s).$$

Therefore, there exists a family $\{\varepsilon_{\lambda,\mu}\}$ of positive numbers such that $\limsup_{\lambda,\mu \downarrow 0} \varepsilon_{\lambda,\mu} = 0$ and

$$\Psi(t) \le \varepsilon_{\lambda,\mu} + \omega \bar{\tau}^{1-\alpha_0} B(1-\alpha_0, 1-\beta_0) \sup_{s \in [0,\bar{\tau})} \Psi(s) \quad \text{for } t \in [0,\bar{\tau}).$$

Since $\omega\bar{\tau}^{1-\alpha_0}B(1-\alpha_0,1-\beta_0)<1$, this implies that

$$\sup\{\|u^\lambda(t)-u^\mu(t)\|;t\in[0,\bar{\tau})\}\to 0 \quad \text{as } \lambda,\,\mu\downarrow 0$$

and hence that there exists a measurable function $u:[0,\bar{\tau})\to X$ such that $\lim_{\varepsilon\downarrow 0}u^\varepsilon(t)=u(t)$ in X, uniformly for $t\in[0,\bar{\tau})$. Thus the proof is complete. □

§ 7. An application to drift-diffusion systems

This section is devoted to an application of Theorem 2.3 to the Cauchy problem for the drift-diffusion system

$$\text{(DD)} \qquad \begin{cases} \partial_t u-\Delta u-\nabla\cdot(u\nabla\psi)=0 & \text{in } \mathbb{R}^N\times(0,\infty), \\ -\Delta\psi=u & \text{on } \mathbb{R}^N\times(0,\infty), \\ u(x,0)=u_0(x)\geq 0, \end{cases}$$

where $N\geq 2$. Let $L^p_+(\mathbb{R}^N)$ be the set of all nonnegative functions in $L^p(\mathbb{R}^N)$. The following theorem will be obtained through our abstract results (Theorem 2.3 and Proposition 3.3).

Theorem 7.1. *Let $p\in[N/2,N)$. For each initial data $u_0\in L^p_+(\mathbb{R}^N)$, the Cauchy problem to* (DD) *has a unique solution u in the class*

$$C([0,\infty);L^p_+(\mathbb{R}^N))\cap C^1((0,\infty);L^p(\mathbb{R}^N))\cap C((0,\infty);D(\Delta_p)),$$

where $D(\Delta_p)$ is the domain of the Laplace operator Δ in $L^p(\mathbb{R}^N)$ defined by

$$D(\Delta_p)=W^{2,p}(\mathbb{R}^N) \quad \textit{for } N/2<p<N,$$

$$D(\Delta_{N/2})=\begin{cases} W^{2,N/2} & \textit{for } N\geq 3, \\ \{u\in L^1;u\in W^{1,p} \ \ \textit{for } 1\leq p<2 \textit{ and } \Delta u\in L^1\} & \textit{for } N=2. \end{cases}$$

Let $p\in[N/2,N)$. Then the operator A in $X=L^p(\mathbb{R}^N)$ defined by

$$Au=\Delta u-u \qquad \text{for } u\in D(A):=D(\Delta_p)$$

is the infinitesimal generator of an analytic (C_0) semigroup on X of negative type. By using the operator B from $D\cap D((-A)^\alpha)$ into X defined by

$$Bu=\nabla\cdot(u\nabla\psi)+u \quad \text{for } u\in D\cap D((-A)^\alpha), \text{ where } -\Delta\psi=u,$$

the Cauchy problem for (DD) can be converted into the semilinear problem (SP) if the set $D\subset X$ and $\alpha\in(0,1)$ appearing in the abstract setting are determined appropriately. The arguments are divided into the following two cases:

$$\text{(7.1)} \qquad \text{(I)} \quad N/2<p<N, \qquad \text{(II)} \quad p=N/2.$$

For simplicity in notation we write L^p, L^p_+ and $W^{k,p}$ instead of $L^p(\mathbb{R}^N)$, $L^p_+(\mathbb{R}^N)$ and $W^{k,p}(\mathbb{R}^N)$, respectively. The usual L^p norm is denoted by $\|\cdot\|_p$. The symbol K stands for various constants throughout this section.

We begin by considering the case (I). Let $X = L^p$ and $\|\cdot\| = \|\cdot\|_p$. Then, the analytic (C_0) semigroup $\{T(t); t \geq 0\}$ on X generated by the operator A satisfies $\|T(t)\| \leq e^{-t}$ for $t \geq 0$ and the L^p-$L^{p'}$ estimate

$$\|T(t)(\nabla\cdot v)\| \leq Kt^{-N/(2p)}\|v\|_{p'} \quad \text{for } v \in (L^{p'})^N \text{ with } \nabla\cdot v \in X, \tag{7.2}$$

where $1/p' = 1/p + 1/q$ and $1/q = 1/p - 1/N$. By the Gagliardo-Nirenberg inequality we have $D(A) \subset L^\infty \cap W^{1,N}$ and

$$\|u\|_\infty \leq K\|u\|^{1-N/(2p)}\|u\|_{W^{2,p}}^{N/(2p)} \leq K\|u\|^{1-N/(2p)}\|Au\|^{N/(2p)} \quad \text{for } u \in D(A),$$
$$\|u\|_{W^{1,N}} \leq K\|u\|^{1-N/(2p)}\|u\|_{W^{2,p}}^{N/(2p)} \leq K\|u\|^{1-N/(2p)}\|Au\|^{N/(2p)} \quad \text{for } u \in D(A).$$

By (7.1) (I) we choose $\alpha \in (N/(2p), 1)$ and set $Y = D((-A)^\alpha)$. Then, by [18, Lemma A.1] we have $Y \subset L^\infty \cap W^{1,N}$ and

$$\|u\|_\infty \leq K\|u\|^{1-N/(2p\alpha)}\|(-A)^\alpha u\|^{N/(2p\alpha)} \quad \text{for } u \in Y, \tag{7.3}$$
$$\|u\|_{W^{1,N}} \leq K\|u\|^{1-N/(2p\alpha)}\|(-A)^\alpha u\|^{N/(2p\alpha)} \quad \text{for } u \in Y. \tag{7.4}$$

Let $r_0 > 0$ and set

$$D = \{u \in L^p;\ u \geq 0,\ \|u\|_p \leq r_0\} \tag{7.5}$$

and $C = D \cap Y$. Then, we define an operator B from C into X by

$$Bu = \nabla\cdot(u\nabla\psi) + u\ (= \nabla u\cdot\nabla\psi - u^2 + u) \quad \text{for } u \in C, \tag{7.6}$$

where $\psi = (-\Delta)^{-1}u$. The definition makes sense because $\nabla u \in (L^N)^N$ for $u \in C$, $\nabla\psi \in (L^q)^N$ and $1/q = 1/p - 1/N$, by the following lemma.

Lemma 7.2. [13, Corollary 2.3] *Let $r \in (1, N)$ and $1/s = 1/r - 1/N$. Then there exists a positive constant $K_{N,r}$ depending only on N and r such that*

$$\|\nabla(-\Delta)^{-1}f\|_s \leq K_{N,r}\|f\|_r \quad \textit{for } f \in L^r.$$

This lemma immediately follows from the Hardy-Littlewood-Sobolev inequality. The following lemma asserts that the operator B satisfies conditions (B1) and (B2).

Lemma 7.3.

(i) *For each $\rho > 0$ there exists $L_B(\rho) > 0$ such that*

$$\|Bu - Bv\| \leq L_B(\rho)\|u - v\|_Y \quad \textit{for } u,\ v \in C \textit{ with } \|u\|_Y \leq \rho \textit{ and } \|v\|_Y \leq \rho.$$

(ii) *There exists $M_B > 0$ such that $\|Bu\| \le M_B\|u\|_Y$ for $u \in C$.*

Proof. Let u, $\hat{u} \in C$ and put $\hat{\psi} = (-\Delta)^{-1}\hat{u}$. Since $1/q = 1/p - 1/N$, by the Hölder inequality we have

$$\begin{aligned}\|Bu - B\hat{u}\| &\le \|\nabla u \cdot \nabla\psi - \nabla\hat{u}\cdot\nabla\hat{\psi}\|_p + \|u^2 - \hat{u}^2\|_p + \|u - \hat{u}\|_p \\ &\le \|u - \hat{u}\|_{W^{1,N}}\|\nabla\psi\|_q + \|\hat{u}\|_{W^{1,N}}\|\nabla(-\Delta)^{-1}(u - \hat{u})\|_q \\ &\quad + 2(\|u\|_p + \|\hat{u}\|_p)\|u - \hat{u}\|_\infty + \|u - \hat{u}\|_p.\end{aligned}$$

Using (7.3), (7.4), Lemma 7.2 with $(r, s) = (p, q)$ and the inequality $\|u\| \le K\|u\|_Y$, we see that assertion (i) holds. To show assertion (ii), let $u \in C$. Then, we have

$$\begin{aligned}\|Bu\| &\le \|\nabla u\|_N\|\nabla\psi\|_q + \|u^2\|_p + \|u\|_p \\ &\le K\|u\|_p\|u\|_Y + \|u\|_\infty\|u\|_p + \|u\|_p \le K(1 + r_0)\|u\|_Y\end{aligned}$$

by (7.3), (7.4) and Lemma 7.2. □

To check condition (ii-3), let $v_0 \in C$ and consider the linear operator B_0 on Y into X defined by $B_0u = \nabla u \cdot \nabla\psi_0 - uv_0 + u$ for $u \in Y$, where $\psi_0 = (-\Delta)^{-1}v_0$. Similarly to the verification of (ii) of Lemma 7.3, the operator B_0 is bounded on Y into X. This implies the existence of $u \in C([0,\infty); Y) \cap C^1((0,\infty); X)$ such that $u(0) = v_0$ and $u'(t) = Au(t) + B_0u(t)$ for $t > 0$.

Lemma 7.4. *For each $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon]$ such that $u(\delta) \in C$ and*

$$\|u(\delta) - J(\delta)v_0\| \le \varepsilon\delta, \qquad \|u(\delta) - J(\delta)v_0\|_Y \le \varepsilon\delta^{1-\alpha}, \tag{7.7}$$

where $J(t)v_0 = T(t)v_0 + \int_0^t T(\xi)Bv_0\,d\xi$ for $t \ge 0$.

Proof. To prove $u(t) \in C$ for $t > 0$, let ρ_ε be a mollifier and define h_ε and Φ_ε by $h_\varepsilon(\sigma) = \int_{-\varepsilon}^\sigma \rho_\varepsilon(\xi)\,d\xi$ and $\Phi_\varepsilon(\sigma) = \int_{-\varepsilon}^\sigma h_\varepsilon(\xi)\,d\xi$ for $\sigma \in \mathbb{R}$. Then

$$\begin{aligned}&(d/dt)\int_{\mathbb{R}^N} \Phi_\varepsilon(-u - \varepsilon)^p\,dx \\ &= -p\int_{\mathbb{R}^N} \Phi_\varepsilon(-u - \varepsilon)^{p-1}h_\varepsilon(-u - \varepsilon)(\Delta u + \nabla u \cdot \nabla\psi_0 - uv_0)\,dx\end{aligned}$$

for $t > 0$. If $f \in C^2(\mathbb{R})$ satisfies that $f(\xi) \ge 0$ for $\xi \in [a, b]$ and that $|f(\xi)| \le K|\xi|$, $|f'(\xi)| \le K$ and $f''(\xi) \ge 0$ for $\xi \in \mathbb{R}$, then

$$\int_{\mathbb{R}^N} f(v)^{p-1}f'(v)\Delta v\,dx \le 0 \quad \text{for } v \in W^{2,p} \text{ with } a \le v \le b.$$

An application of this fact yields that

$$
\begin{aligned}
&(d/dt)\int_{\mathbb{R}^N} \Phi_\varepsilon(-u-\varepsilon)^p\,dx \\
&\le \int_{\mathbb{R}^N} \nabla(\Phi_\varepsilon(-u-\varepsilon)^p)\cdot\nabla\psi_0\,dx - p\int_{\mathbb{R}^N} \Phi_\varepsilon(-u-\varepsilon)^{p-1}h_\varepsilon(-u-\varepsilon)(-u)v_0\,dx \\
&\le (1-p)\int_{\mathbb{R}^N} \Phi_\varepsilon(-u-\varepsilon)^p v_0\,dx
\end{aligned}
$$

for $t>0$. To obtain the last inequality we have used the fact that $-\Delta\psi_0 = v_0$ and $\Phi_\varepsilon(\xi-\varepsilon)\le h_\varepsilon(\xi-\varepsilon)\xi$ for $\xi\in\mathbb{R}$. Notice that $v_0\ge 0$. Since the right-hand side is less than or equal to zero, we have $\Phi_\varepsilon(-u-\varepsilon)=0$ for $t>0$; hence $u(t)\ge 0$ for $t>0$. Similarly, we find, by noting that $u(t)\ge 0$ for $t\ge 0$,

$$
(d/dt)\|u(t)\|_p^p \le (1-p)\int_{\mathbb{R}^N} u(x,t)^p v_0(x)\,dx \le 0 \quad \text{for } t>0,
$$

which implies that $\|u(t)\|_p\le\|v_0\|_p\le r_0$ for $t>0$. Hence $u(t)\in C$ for $t>0$. Since $u(t)=T(t)v_0+\int_0^t T(t-\xi)B_0u(\xi)\,d\xi$ for $t\ge 0$, we have

$$
u(t)-J(t)v_0 = \int_0^t T(t-\xi)(B_0u(\xi)-Bv_0)\,d\xi
$$

for $t\ge 0$. By (7.3), (7.4) and Lemma 7.2 with $(r,s)=(p,q)$ we have

$$
\begin{aligned}
\|B_0u(\xi)-Bv_0\| &\le \|\nabla(u(\xi)-v_0)\|_N\|\nabla\psi_0\|_q + \|u(\xi)-v_0\|_\infty\|v_0\|_p + \|u(\xi)-v_0\|_p \\
&\le K(1+\|v_0\|_p)\|u(\xi)-v_0\|_Y
\end{aligned}
$$

for $\xi\ge 0$. Since $\|T(t)v\|_Y\le M_\alpha t^{-\alpha}\|v\|$ for $v\in Y$ and $t>0$, we find that

$$
\begin{aligned}
\|u(t)-J(t)v_0\|_p &\le K(1+\|v_0\|_p)\int_0^t \|u(\xi)-v_0\|_Y\,d\xi, \\
\|u(t)-J(t)v_0\|_Y &\le K(1+\|v_0\|_p)t^{1-\alpha}\sup_{\xi\in[0,t]}\|u(\xi)-v_0\|_Y.
\end{aligned}
$$

Since $u\in C([0,\infty);Y)$, the assertion of the lemma is true. □

Let $\tau>0$ and $\Delta=\{(t,s);0\le s\le t\le\tau\}$. To check conditions (ii-1) and (ii-2), we employ the nonnegative functional V on $\Delta\times Y\times Y$ defined by

$$
V(t,s,v,w)=\|T(t-s)(v-w)\| \quad \text{for } (t,s,v,w)\in\Delta\times Y\times Y.
$$

Then it is easily checked that condition (V1) and (V2) in (ii-1) is satisfied with $\beta_0=0$. Condition (V3) follows from the contractivity of $T(t)$ in $B(X)$ and the inequality

$$
\|(-A)^\gamma(T(h)v-v)\|\le Kh^{\alpha-\gamma}\|v\|_Y \tag{7.8}
$$

for $h \geq 0$, $\gamma \in [0, \alpha]$ and $v \in Y$. To verify condition (ii-2), let $s \in [0, \tau)$ and $v, \hat{v} \in C$, and set $\hat{\psi} = (-\Delta)^{-1}\hat{v}$. Then

$$\begin{aligned} T(t-(s+h))(J(h)v - J(h)\hat{v}) =& T(t-s)(v-\hat{v}) + hT(t-s)(Bv - B\hat{v}) \\ &+ T(t-(s+h)) \int_0^h (T(\xi) - T(h))(Bv - B\hat{v})\, d\xi \end{aligned}$$

and an application of (7.2) yields that

$$\begin{aligned} \|T(t-s)(Bv - B\hat{v})\| &\leq K(t-s)^{-N/(2p)} \|(v-\hat{v})\nabla\psi + \hat{v}(\nabla\psi - \nabla\hat{\psi})\|_{p'} + \|v - \hat{v}\|_p \\ &\leq K(t-s)^{-N/(2p)} (\|v\|_p + \|\hat{v}\|_p + 1)\|v - \hat{v}\|_p \end{aligned}$$

for $s + h \leq t \leq \tau$. Here we have used the Hölder inequality and Lemma 7.2 with $(r, s) = (p, q)$ to obtain the last inequality, since p, q, p' satisfy $1/p' = 1/p + 1/q$ and $1/q = 1/p - 1/N$. Since $V(s, s, v, \hat{v}) = \|v - \hat{v}\|_p$, we have

$$\begin{aligned} &(V(t, s+h, J(h)v, J(h)\hat{v}) - V(t, s, v, \hat{v}))/h \\ &\quad \leq K(t-s)^{-N/(2p)} (\|v\|_p + \|\hat{v}\|_p + 1) V(s, s, v, \hat{v}) \\ &\quad\quad + h^{-1} \int_0^h \|(T(\xi) - T(h))(Bv - B\hat{v})\|\, d\xi \end{aligned}$$

for $s + h \leq t \leq \tau$. Since $\|v\|_p \leq r_0$ and $\|\hat{v}\|_p \leq r_0$, the inequality above shows that condition (ii-2) is satisfied with $\alpha_0 = N/(2p)$. Since (ii-3) follows from Lemma 7.4, we apply Theorem 2.3 to obtain a semigroup $\{S_0(t); t \geq 0\}$ of Lipschitz operators on D such that for each $u_0 \in D$, $S_0(t)u_0$ is a global mild solution to (SP; u_0), where D is the set defined by (7.5).

In the case of (I), Theorem 7.1 is a direct consequence of the following theorem.

Theorem 7.5. *There exists a semigroup $\{S(t); t \geq 0\}$ on L^p_+ satisfying the following conditions:*

(i) *For each $\tau, r > 0$ there exists $M(\tau, r) > 0$ such that $\|S(t)u_0 - S(t)v_0\|_p \leq M(\tau, r)\|u_0 - v_0\|_p$ for $t \in [0, \tau]$ and u_0, $v_0 \in L^p_+$ with $\|u_0\|_p \leq r$, $\|v_0\|_p \leq r$.*

(ii) *$\|S(t)u_0\|_p \leq \|u_0\|_p$ for $t \geq 0$ and $u_0 \in L^p_+$.*

(iii) *For each $u_0 \in L^p_+$, the* (DD) *has a unique global C^1-solution u given by $u(t) = S(t)u_0$ for $t \geq 0$, where by a C^1-solution is meant a solution in the class $C([0, \infty); L^p) \cap C^1((0, \infty); L^p) \cap C((0, \infty); D(\Delta_p))$.*

Proof. To prove the existence of a semigroup $\{S(t); t \geq 0\}$ on L^p_+ as required, let $r > 0$. Then, by the fact shown above, there exists a semigroup $\{S_r(t); t \geq 0\}$ of

Lipschitz operators on $D_r = \{v \in L^p_+; \|v\| \le r\}$ such that for each $u_0 \in D_r$, $S_r(t)u_0$ is a global mild solution to (SP; u_0). By the uniqueness of mild solutions (Theorem 3.1), the family $\{S(t); t \ge 0\}$ defined by $S(t)v := S_r(t)v$ for $v \in D_r$ and $t \ge 0$ forms a semigroup on L^p_+. From Propositions 3.1 and 3.3 (iv) we deduce that the semigroup $\{S(t); t \ge 0\}$ satisfies condition (i) and the condition that for each $u_0 \in L^p_+$, $S(t)u_0$ is a global C^1-solution of (DD). To prove (ii), let $u_0 \in L^p_+$ and set $r = \|u_0\|_p$. Then we have $S(t)u_0 = S_r(t)u_0 \in D_r$ for $t \ge 0$. This implies condition (ii). To show the uniqueness of C^1-solutions, let $u_0 \in L^p_+$ and u be any C^1-solution with $u(0) = u_0$. Let $\bar{\tau} > 0$. Then there exists $r_0 > 0$ such that $\|u(t)\|_p \le r_0$ for $t \in [0, \bar{\tau}]$; namely $u(t) \in D$ for $t \in [0, \bar{\tau}]$. Let $\varepsilon > 0$. Since the function $u_\varepsilon(t) := u(t+\varepsilon)$ is a mild solution of (SP; $u(\varepsilon)$) on $[0, \bar{\tau} - \varepsilon]$, Proposition 3.1 asserts that $\|u_\varepsilon(t) - S(t+\varepsilon)u_0\| \le M_{\bar{\tau}}\|u(\varepsilon) - S(\varepsilon)u_0\|$ for $t \in [0, \bar{\tau} - \varepsilon]$. This implies that $u(t) = S(t)u_0$ for $t \ge 0$. □

Next we shall consider the case (II). In this case the arguments will be divided into two parts. In fact, a semigroup on $L^{N/2}_+ \cap L^{2N/3}$ will be constructed in Step 1 and extended to a semigroup on $L^{N/2}_+$ in Step 2.

Step 1. The purpose is to prove the existence of a semigroup $\{S(t); t \ge 0\}$ on $L^{N/2}_+ \cap L^{2N/3}$ satisfying the following conditions:

(i) For each $\tau, r > 0$ there exists $M(\tau, r) > 0$ such that

$$\|S(t)u_0 - S(t)v_0\|_{N/2} \le M(\tau, r)\|u_0 - v_0\|_{N/2}$$

for $t \in [0, \tau]$ and u_0, $v_0 \in L^{N/2}_+ \cap L^{2N/3}$ satisfying $\|u_0\|_{2N/3} \le r$ and $\|v_0\|_{2N/3} \le r$.

(ii) $\|S(t)u_0\|_{2N/3} \le \|u_0\|_{2N/3}$ for $t \ge 0$ and $u_0 \in L^{N/2}_+ \cap L^{2N/3}$.

(iii) For each $u_0 \in L^{N/2}_+ \cap L^{2N/3}$, $S(t)u_0$ is a unique C^1-solution satisfying $S(t)u_0 \in C([0, \infty); L^{2N/3})$.

For this purpose, let $X = L^{N/2}$, $\|\cdot\| = \|\cdot\|_{N/2}$ and define a linear operator A_0 by $A_0u = \Delta u$ for $u \in D(A_0) = D(\Delta_{N/2})$. Then the operator A in X defined by $Au = A_0u - u$ for $u \in D(A) = D(A_0)$ generates an analytic (C_0) semigroup $\{T(t); t \ge 0\}$ on X satisfying $\|T(t)\| \le e^{-t}$ for $t \ge 0$. Let $v \in D(A)$ and $t > 0$. By the identity $v = T(t)v - \int_0^t T(\xi)Av\, d\xi$ and the $L^{N/2}$-$L^{2N/3}$ estimate

$$\|T(t)v\|_{2N/3} \le Kt^{-1/4}\|v\| \qquad \text{for } v \in L^{N/2} \text{ and } t > 0$$

we have $\|v\|_{2N/3} \le K(t^{-1/4}\|v\| + t^{3/4}\|Av\|)$. Letting $t = \|v\|_{N/2}/\|Av\|_{N/2}$ gives the estimate $\|v\|_{2N/3} \le K\|v\|^{3/4}\|Av\|^{1/4}$ for $v \in D(A)$. Similarly, we obtain $\|\nabla v\|_{2N/3} \le$

$K\|v\|^{1/4}\|Av\|^{3/4}$ for $v \in D(A)$. By the two inequalities above and the Gagliardo-Nirenberg inequality we have $D(A) \subset W^{1,2N/3} \subset L^{2N}$ and

$$\|u\|_{2N} \le K\|u\|_{W^{1,2N/3}} \le K\|u\|^{1/4}\|Au\|^{3/4} \quad \text{for } u \in D(A).$$

Let $\alpha \in (3/4, 1)$ and $Y = D((-A)^\alpha)$. Then, by [18, Lemma A.1] we have $Y \subset W^{1,2N/3}$ and

$$\|u\|_{2N/3} \le K\|u\|^{1/2}\|(-A)^{1/2}u\|^{1/2} \quad \text{for } u \in Y, \tag{7.9}$$

$$\|u\|_{2N} \le K\|u\|_{W^{1,2N/3}} \le K\|u\|^{1-3/(4\alpha)}\|(-A)^\alpha u\|^{3/(4\alpha)} \quad \text{for } u \in Y. \tag{7.10}$$

Let $r_0 > 0$ and set $D = \{u \in L^{N/2} \cap L^{2N/3};\ u \ge 0,\ \|u\|_{2N/3} \le r_0\}$ and $C = D \cap Y$. Define a nonlinear operator B from C into X as in the case (I). Then, the definition of the operator B makes sense because $\nabla u \in (L^{2N/3})^N$ and $\nabla\psi \in (L^{2N})^N$ for $u \in C$. The proof of Lemma 7.3 is also valid with L^p, L^q, $W^{1,N}$ and L^∞ replaced by $L^{N/2}$, L^{2N}, $W^{1,2N/3}$ and L^{2N}, respectively. Thus, the operator B satisfies conditions (B1) and (B2).

To check condition (ii-3), let $v_0 \in C$ and consider the operators $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathfrak{X} := L^{N/2} \cap L^{2N/3}$ defined by $\mathfrak{A}u = \Delta u - u$ for $D(\mathfrak{A}) := D(A_{N/2}) \cap D(A_{2N/3})$ and $\mathfrak{B}u = \nabla u \cdot \nabla\psi_0 - uv_0 - u$ for $D(\mathfrak{B}) := D((-\mathfrak{A})^\alpha)$, where $\psi_0 = (-\Delta)^{-1}v_0$. Notice that $D((-\mathfrak{A})^\alpha) \subset D((-A_{N/2})^\alpha) \cap D((-A_{2N/3})^\alpha$ and $(-\mathfrak{A})^\alpha u = (-A_{N/2})^\alpha u = (-A_{2N/3})^\alpha u$ for $u \in D((-\mathfrak{A})^\alpha)$. Similarly to the verification of condition (B2) we find

$$\|\mathfrak{B}u\|_{N/2} \le K(1 + \|v_0\|_{2N/3})\|(-A_{N/2})^\alpha u\|_{N/2} \quad \text{for } u \in D((-\mathfrak{A})^\alpha),$$
$$\|\mathfrak{B}u\|_{2N/3} \le K(1 + \|v_0\|_{2N/3})\|(-A_{2N/3})^\alpha u\|_{2N/3} \quad \text{for } u \in D((-\mathfrak{A})^\alpha);$$

hence $\|\mathfrak{B}u\|_{\mathfrak{X}} \le K(1 + \|v_0\|_{2N/3})\|(-\mathfrak{A})^\alpha u\|_{\mathfrak{X}}$ for $u \in D((-\mathfrak{A})^\alpha)$. It follows that the abstract Cauchy problem for $\mathfrak{A} + \mathfrak{B}$ has a solution $u \in C([0,\infty); Y) \cap C^1((0,\infty); X) \cap C((0,\infty); W^{2,2N/3}) \cap C^1((0,\infty); L^{2N/3}))$. By using this fact, Lemma 7.4 is also proved in the present setting.

To check conditions (ii-1) and (ii-2), we employ the nonnegative functional V on $\Delta \times Y \times Y$ defined by

$$V(t,s,v,w) = \|T(t-s)(v-w)\| + t^{1/4}\|T(t-s)(v-w)\|_{2N/3}$$

for $(t,s,v,w) \in \Delta \times Y \times Y$. Then, conditions (V1) and (V2) in (ii-1) are satisfied with $\beta_0 = 1/4$. To check condition (V3), let (t,s,v,w), $(\hat{t},\hat{s},v,w) \in \Delta \times Y \times Y$. Then we infer from (7.8) and (7.9) that

$$\begin{aligned}
&|V(t,s,v,w) - V(\hat{t},\hat{s},v,w)| \\
&\quad \le |\|T(t-s)(v-w) - T(\hat{t}-\hat{s})(v-w)\| + |t^{1/4} - \hat{t}^{1/4}|\|T(t-s)(v-w)\|_{2N/3}
\end{aligned}$$

$$
\begin{aligned}
&+ \hat{t}^{1/4}\|(-A)^{1/2}(T(t-s)(v-w) - T(\hat{t}-\hat{s})(v-w))\| \\
&\le K|t-s-(\hat{t}-\hat{s})|^{\alpha}\|v-w\|_Y + K|t-\hat{t}|^{1/4}\|v-w\|_Y \\
&\quad + K\tau^{1/4}|t-s-(\hat{t}-\hat{s})|^{\alpha-1/2}\|(v-w))\|_Y \\
&\le \theta(|t-\hat{t}|+|s-\hat{s}|)(\|v\|_Y+\|w\|_Y),
\end{aligned}
$$

where $\theta(\xi) = K(\xi^{\alpha}+\xi^{1/4}+\xi^{\alpha-1/2})$. This shows that condition (H3) is satisfied.

To verify condition (ii-2), let $(t,s)\in\Delta$ with $t\neq s$ and v, $\hat{v}\in C$, and set $\hat{\psi} = (-\Delta)^{-1}\hat{v}$. For any Banach space $(X_0, \|\cdot\|_{X_0})$ we have

$$
\begin{aligned}
&\|T(t-s-h)(J(h)v - J(h)\hat{v})\|_{X_0} \\
&\quad \le \|T(t-s)(v-\hat{v})\|_{X_0} + h\|T(t-s)(Bv-B\hat{v})\|_{X_0} \\
&\qquad + \|T(t-s-h)\|_{X\to X_0}\int_0^h \|(T(\xi)-T(h))(Bv-B\hat{v})\|_X\,d\xi
\end{aligned}
\tag{7.11}
$$

for $h>0$ such that $s+h\le t$. Let $h>0$ and $s+h\le t$. Considering the cases $X_0 = X$ and $X_0 = L^{2N/3}$ in the above estimate, we obtain

$$
\begin{aligned}
&(V(t,s+h,J(h)v,J(h)\hat{v}) - V(t,s,v,\hat{v}))/h \\
&\quad \le \|T(t-s)(Bv-B\hat{v})\| + t^{1/4}\|T(t-s)(Bv-B\hat{v})\|_{2N/3} + g(h),
\end{aligned}
$$

where

$$
\begin{aligned}
g(h) &= h^{-1}\int_0^h \|(T(\xi)-T(h))(Bv-B\hat{v})\|\,d\xi \\
&\quad + t^{1/4}K(t-s-h)^{-1/4}h^{-1}\int_0^h \|(T(\xi)-T(h))(Bv-B\hat{v})\|\,d\xi.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&(t-s)^{3/4}(s+h)^{1/4}(V(t,s+h,J(h)v,J(h)\hat{v}) - V(t,s,v,\hat{v}))/h \\
&\quad \le K(t-s)^{1/4}(s+h)^{1/4}\|v\nabla\psi - \hat{v}\nabla\hat{\psi}\|_{N/2} + K(t-s)^{3/4}(s+h)^{1/4}\|v-\hat{v}\|_{N/2} \\
&\qquad + Kt^{1/4}(s+h)^{1/4}\|v\nabla\psi - \hat{v}\nabla\hat{\psi}\|_{N/2} + Kt^{1/4}(t-s)^{3/4}(s+h)^{1/4}\|v-\hat{v}\|_{2N/3} \\
&\qquad + (t-s)^{3/4}(s+h)^{1/4}g(h).
\end{aligned}
\tag{7.12}
$$

By the Hölder inequality and Lemma 7.2 we find that

$$
\begin{aligned}
\|v\nabla\psi - \hat{v}\nabla\hat{\psi}\|_{N/2} &\le \|v-\hat{v}\|_{2N/3}\|\nabla\psi\|_{2N} + \|\hat{v}\|_{2N/3}\|\nabla\psi - \nabla\hat{\psi}\|_{L^{2N}} \\
&\le K(\|v\|_{2N/3} + \|\hat{v}\|_{2N/3})\|v-\hat{v}\|_{2N/3}.
\end{aligned}
\tag{7.13}
$$

It follows from (7.12) that

$$
\liminf_{h\downarrow 0}\,(t-s)^{3/4}(s+h)^{1/4}(V(t,s+h,J(h)v,J(h)\hat{v}) - V(t,s,v,\hat{v}))/h
$$

$$\le Kt^{1/4}V(s,s,v,\hat{v}),$$

and so condition (ii-2) is checked to be satisfied. Condition (ii-3) follows from Lemma 7.4. Similarly to the proof in Case (I), Theorem 2.3 guarantees the existence of a desired semigroup $\{S(t); t \ge 0\}$ on $L_+^{N/2} \cap L^{2N/3}$, if it is proved that $S(t)u_0 \in C([0,\infty); L^{2N/3})$ for each $u_0 \in L_+^{N/2} \cap L^{2N/3}$. This fact will be shown in the following way: We note that

$$S(t)v_0 = U(t)v_0 + \int_0^t U(t-s)FS(s)v_0\,ds \tag{7.14}$$

for $t \ge 0$ and $v_0 \in L_+^{N/2} \cap L^{2N/3}$, where $\{U(t); t \ge 0\}$ is the analytic (C_0) semigroup on $X = L_+^{N/2}$ generated by A_0 and F is the operator from $L_+^{N/2} \cap Y$ into X defined by $Fv = \nabla \cdot (v\nabla\psi)$ for $v \in L_+^{N/2} \cap Y$, where $\psi = (-\Delta)^{-1}v$. The following L^p-L^q estimates will be often used:

$$\|U(t)v\|_{2N/3} \le K_0 t^{-1/4}\|v\| \quad \text{for } v \in L^{N/2}, \tag{7.15}$$

$$\|U(t)(\nabla \cdot v)\| \le K_1 t^{-1/2}\|v\| \quad \text{for } v \in (L^{N/2})^N \text{ with } \nabla \cdot v \in L^{N/2}, \tag{7.16}$$

$$\|U(t)(\nabla \cdot v)\|_{2N/3} \le K_2 t^{-3/4}\|v\| \quad \text{for } v \in (L^{N/2})^N \text{ with } \nabla \cdot v \in L^{N/2}. \tag{7.17}$$

Hereafter K_i stand for constants depending on N. Now, let $u_0 \in L_+^{N/2} \cap L^{2N/3}$. Then we have $U(t)u_0 \in L^{2N/3}$ for $t \ge 0$ and $\lim_{t\downarrow 0} U(t)u_0 = u_0$ in $L^{2N/3}$. Since $Y \subset L^{2N/3}$, the last term on the right-hand side of (7.14) converges in $L^{2N/3}$ and

$$\left\| \int_0^t U(t-s)FS(s)u_0\,ds \right\|_{2N/3} \le \int_0^t K(t-s)^{-3/4}\|S(s)u_0\|_{2N/3}^2\,ds$$

for $t \ge 0$. Here we have used (7.17) and $\|\nabla(-\Delta)^{-1}v\|_{2N} \le K_3\|v\|_{2N/3}$ for $v \in L^{2N/3}$. By (ii) of Step 1 we have $\lim_{t\downarrow 0}\int_0^t U(t-s)FS(s)u_0\,ds = 0$ in $L^{2N/3}$. It follows that $S(t)u_0 \in C([0,\infty); L^{2N/3})$.

Step 2. The purpose is to extend the semigroup $\{S(t); t \ge 0\}$ on $L_+^{N/2} \cap L^{2N/3}$ obtained in Step 1 to a semigroup $\{\widetilde{S}(t); t \ge 0\}$ on $L_+^{N/2}$. Our argument is similar to [3, 4, 7, 8]. To construct a family $\{D_r; r > 0\}$ of subsets of X such that $L_+^{N/2} = \cup_{r>0} D_r$, choose $\delta_0 > 0$ so that $16K_2K_3B(1/4,1/2)\delta_0 < 1$ and put

$$R_0 = (1 - \sqrt{1 - 16K_2K_3B(1/4,1/2)\delta_0})/(4K_2K_3B(1/4,1/2)) > 0.$$

Since R_0 satisfies $2\delta_0 + 2K_2K_3B(1/4,1/2)R_0^2 = R_0$, we observe that

$$2K_2K_3B(1/4,1/2)R_0 < 1, \quad 2\delta_0 + K_2K_3B(1/4,1/2)R_0^2 < R_0. \tag{7.18}$$

Since $t^{1/4}\|U(t)v\|_{2N/3} \to 0$ as $t \downarrow 0$ for any $v \in C_0^\infty$ dense in $L^{2N/3}$ and since $t^{1/4}\|U(t)v\|_{2N/3}$ is bounded as $t \downarrow 0$ by (7.15), we have

$$\lim_{t\downarrow 0} t^{1/4}\|U(t)v\|_{L^{2N/3}} = 0 \tag{7.19}$$

for $v \in L^{N/2}$. For each $r > 0$ set $D_r = \{v \in L_+^{N/2}; t^{1/4}\|U(t)v\|_{2N/3} \le \delta_0$ for $t \in (0,r]\}$ and $C_r = D_r \cap Y$. Then we observe that $L_+^{N/2} = \cup_{r>0} D_r$ by (7.19) and that $C_r \subset L^{2N/3}$ and C_r is dense in D_r for each $r > 0$.

Let $\{S(t); t \ge 0\}$ be the semigroup on $L_+^{N/2} \cap L^{2N/3}$ obtained in Step 1. Then we want to show that for each $\tau > 0$ and $r > 0$ there exist $K_{\tau,r} > 0$ and $\widetilde{K}_{\tau,r} > 0$ such that

$$\|S(t)v_0 - S(t)\hat{v}_0\|_{2N/3} \le K_{\tau,r} t^{-1/4} \|v_0 - \hat{v}_0\|, \tag{7.20}$$

$$\|S(t)v_0 - S(t)\hat{v}_0\| \le \widetilde{K}_{\tau,r} \|v_0 - \hat{v}_0\| \tag{7.21}$$

for $t \in (0,\tau]$ and v_0, $\hat{v}_0 \in C_r$. For this purpose we shall demonstrate that

$$\|S(t)v\|_{2N/3} \le (t \wedge r)^{-1/4} R_0 \quad \text{for } t \ge 0 \text{ and } v \in C_r. \tag{7.22}$$

To do this, let $v \in C_r$ and define $w(t) := t^{1/4}\|S(t)v\|_{2N/3}$ for $t \ge 0$. Clearly, $w \in C([0,\infty); \mathbb{R}_+)$ and $w(0) = 0$. We use (7.15), (7.17) and (7.13) with $\hat{v} = 0$ to estimate (7.14). This yields that

$$w(t) \le 2\delta_0 + K_2 K_3 t^{1/4} \int_0^t (t-s)^{-3/4} s^{-1/2} w(s)^2 \, ds \tag{7.23}$$

for $t > 0$. Let $\bar{t} = \sup\{t \in [0,r]; w(s) \le R_0 \text{ for } s \in [0,t]\}$. To show that $\bar{t} = r$, assume to the contrary that $\bar{t} < r$. Then by the definition of $\bar{t}$ and the continuity of w we see that $w(t) \le R_0$ for $t \in [0,\bar{t}]$ and $w(\bar{t}) = R_0$. Setting $t = \bar{t}$ in (7.23), we have by (7.18)

$$\begin{aligned} w(\bar{t}) &\le 2\delta_0 + K_2 K_3 \bar{t}^{1/4} \int_0^{\bar{t}} (\bar{t}-s)^{-3/4} s^{-1/2} w(s)^2 \, ds \\ &\le 2\delta_0 + K_2 K_3 \bar{t}^{1/4} \int_0^{\bar{t}} (\bar{t}-s)^{-3/4} s^{-1/2} R_0^2 \, ds < R_0. \end{aligned}$$

This is a contradiction to the maximality of $\bar{t}$. Hence $t^{1/4}\|S(t)v\|_{2N/3} \le R_0$ for $t \in [0,r]$. By condition (ii) shown in Step 1 we have $\|S(t)v\|_{2N/3} \le \|S(r)v\|_{2N/3} \le r^{-1/4} R_0$ for $t \ge r$. Combining these inequalities we obtain the desired inequality (7.22).

To prove (7.20), let $\tau > 0$, $r > 0$ and v_0, $\hat{v}_0 \in C_r$. Then we use (7.14) to represent the difference $S(t)v_0 - S(t)\hat{v}_0$, and then estimate it by (7.15), (7.17), (7.13) and (7.22). This yields that

$$\begin{aligned} & t^{1/4}\|S(t)v_0 - S(t)\hat{v}_0\|_{2N/3} \\ & \le K_0\|v_0 - \hat{v}_0\| + 2K_2 K_3 \int_0^t t^{1/4}(t-s)^{-3/4} s^{-1/4} R_0 \|S(s)v_0 - S(s)\hat{v}_0\|_{2N/3} \, ds \\ & \le K_0\|v_0 - \hat{v}_0\| + 2K_2 K_3 R_0 B(1/4, 1/2) \sup_{t \in [0,r]} t^{1/4}\|S(t)v_0 - S(t)\hat{v}_0\|_{2N/3} \end{aligned} \tag{7.24}$$

for $t \in (0, r]$. By (7.18) one finds $K_4 > 0$ such that

$$\|S(t)v_0 - S(t)\hat{v}_0\|_{2N/3} \le K_4 t^{-1/4}\|v_0 - \hat{v}_0\| \quad \text{for } t \in (0, r]. \tag{7.25}$$

Similarly, we deduce from (7.25) that

$$\begin{aligned}
&\|S(t)v_0 - S(t)\hat{v}_0\|_{2N/3} \\
&\quad \le K_0 t^{-1/4}\|v_0 - \hat{v}_0\| + 2K_2K_3\int_0^r (t-s)^{-3/4}s^{-1/4}R_0\|S(s)v_0 - S(s)\hat{v}_0\|_{2N/3}\,ds \\
&\qquad + 2K_2K_3\int_r^t (t-s)^{-3/4}r^{-1/4}R_0\|S(s)v_0 - S(s)\hat{v}_0\|_{2N/3}\,ds \\
&\quad \le K_0 t^{-1/4}\|v_0 - \hat{v}_0\| + 2K_2K_3K_4R_0B(1/4, 1/2)t^{-1/4}\|v_0 - \hat{v_0}\| \\
&\qquad + 2K_2K_3 r^{-1/4}R_0\int_r^t (t-s)^{-3/4}\|S(s)v_0 - S(s)\hat{v}_0\|_{2N/3}\,ds
\end{aligned}$$

for $t > r$. Combining these inequalities we have

$$\begin{aligned}
\|S(t)v_0 - S(t)\hat{v}_0\|_{2N/3} &\le K_5 t^{-1/4}\|v_0 - \hat{v}_0\| \\
&\qquad + 2K_2K_3 r^{-1/4}R_0\int_0^t (t-s)^{-3/4}\|S(s)v_0 - S(s)\hat{v}_0\|_{2N/3}\,ds
\end{aligned}$$

for $t \in (0, \tau]$, where $K_5 = K_0 + K_4 + 2K_2K_3K_4R_0B(1/4, 1/2)$. Applying Henry' inequality, we obtain the inequality (7.20). Similarly to the derivation of (7.24) we find by (7.20) that

$$\begin{aligned}
&\|S(t)v_0 - S(t)\hat{v}_0\| \\
&\quad \le \|v_0 - \hat{v}_0\| + 2K_1K_3R_0K_{\tau,r}\left(\int_0^t (t-s)^{-1/2}(s\wedge r)^{-1/4}s^{-1/4}\,ds\right)\|v_0 - \hat{v}_0\|
\end{aligned}$$

for $t \in [0, \tau]$. This implies that the inequality (7.21) holds.

Now, we shall extend the semigroup $\{S(t); t \ge 0\}$ to a semigroup on $L_+^{N/2}$. Let $v_0 \in L_+^{N/2}$. Then there exists $r > 0$ such that $v_0 \in D_r$. Since C_r is dense in D_r, we find a sequence $\{v_{0,n}\} \subset C_r$ such that $\|v_{0,n} - v_0\| \to 0$ as $n \to \infty$. Let $\tau > 0$. Then by (7.20) and (7.21) we observe that the sequence $\{S(\cdot)v_{0,n}\}$ converges in $C([0,\tau]; X) \cap C((0,\tau]; L^{(2N)/3})$ as $n \to \infty$. Since the limit function does not depend on the choice of sequences $\{v_{0,n}\}$ by (7.20) and (7.21), we can define a one-parameter family $\{\widetilde{S}(t); t \ge 0\}$ from $L_+^{N/2}$ into itself by $\widetilde{S}(t)v_0 := \lim_{n\to\infty} S(t)v_{0,n}$ for $t \ge 0$. Clearly, the family $\{\widetilde{S}(t); t \ge 0\}$ is a semigroup on $L_+^{N/2}$ such that $\widetilde{S}(\cdot)v_0 \in C((0,\infty); L^{2N/3})$ for $v_0 \in L_+^{N/2}$ and

$$\|\widetilde{S}(t)v_0 - \widetilde{S}(t)\hat{v}_0\| \le \widetilde{K}_{\tau,r}\|v_0 - \hat{v}_0\| \quad \text{for } t \in [0, \tau] \text{ and } v_0,\, \hat{v}_0 \in D_r.$$

Finally, we shall prove that for each $u_0 \in L_+^{N/2}$, $\widetilde{S}(t)u_0$ gives a unique global C^1-solution to (DD). Let $u_0 \in L_+^{N/2}$ and set $u(t) = \widetilde{S}(t)u_0$ for $t \geq 0$. Since $\widetilde{S}(\varepsilon)u_0 \in L_+^{N/2} \cap L^{2N/3}$ and $u(t) = S(t-\varepsilon)\widetilde{S}(\varepsilon)u_0$ for any $\varepsilon > 0$ and $t \geq \varepsilon$, u is a global C^1-solution to (DD) by the fact shown in Step 1. To prove the uniqueness of C^1-solutions, let v be any global C^1-solutions to (DD). Since $D(\Delta_{N/2}) \subset L^{2N/3}$, $v(t)$ is continuous in $L^{2N/3}$ for $t > 0$. The uniqueness result in Step 1 assures that $v(t+\varepsilon) = S(t)v(\varepsilon) = \widetilde{S}(t)v(\varepsilon)$ for $\varepsilon > 0$ and $t \geq 0$. Thus, Theorem 7.1 for $p = N/2$ follows from the fact shown in Step 1. The proof of the case (II) is now complete.

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