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Asymptotics for large time of solutions for cubic nonlinear Schrödinger equations

By

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Abstract

We give a survey of our recent results concerning the large time asymptotic behavior of solutions to the Cauchy problem for the nonlinear Schrödinger equation with cubic nonlinearities in one space dimension. We consider the gauge invariant case, non-gauge invariant case, the cases including at least one derivative of unknown functions, fractional power case, with complex valued coefficients, etc.

We give a survey of our recent results concerning the large time asymptotic behavior of solutions to the Cauchy problem for the nonlinear Schrödinger equation with cubic nonlinearities in one space dimension

$$\begin{cases} iu_t + \frac{1}{2}u_{xx} = \mathcal{N}\left(u,\overline{u},u_x,\overline{u}_x\right), \ t\in\mathbf{R}, x\in\mathbf{R}, \\ \qquad u\left(0,x\right) = u_0\left(x\right), \ x\in\mathbf{R}. \end{cases} \tag{0.1}$$

Cubic nonlinear Schrödinger equations (0.1) have wide physical applications (see [23], [24], [25]).

First we consider the case of the cubic nonlinearity of the form $\mathcal{N}\left(u,\overline{u},u_x,\overline{u}_x\right) = \mathcal{N}_1 + \mathcal{N}_2$, where the first nonlinearity $\mathcal{N}_1$ consists of all gauge invariant terms

$$\mathcal{N}_1 = \lambda_1 \left|u\right|^2 u + i\lambda_2 \left|u\right|^2 u_x + i\lambda_3 u^2 \overline{u}_x + \lambda_4 \left|u_x\right|^2 u + \lambda_5 \overline{u} u_x^2 + i\lambda_6 \left|u_x\right|^2 u_x$$

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and $\mathcal{N}_2$ is a nongauge invariant nonlinearity, such that each term contains at least one derivative of the unknown function, i.e.

$$\mathcal{N}_2 = 3a_1u^2u_x + 3a_2uu_x^2 + 3a_3u_x^3 + 3b_1\overline{u}^2\overline{u}_x + 3b_2\overline{uu_x^2} + 3b_3\overline{u}_x^3$$
$$+\mu_1\overline{u}^2u_x + \mu_2 |u|^2 \overline{u}_x + \mu_3u\overline{u}_x^2 + \mu_4\overline{u} |u_x|^2 + \mu_5 |u_x|^2 \overline{u}_x.$$

We assume that the coefficients $a_j, b_j \in \mathbf{C}$, $j = 1, 2, 3$, $\mu_l \in \mathbf{C}$, $l = 1, ..., 5$, $\lambda_1, \lambda_6 \in \mathbf{R}$, and $\lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathbf{C}$ are such that $\lambda_2 - \lambda_3 \in \mathbf{R}$ and $\lambda_4 - \lambda_5 \in \mathbf{R}$.

The difficulty in the study of the global existence in time of solutions to the Cauchy problem (0.1) is that the cubic nonlinearity is in general critical for large time values, and it is already known that the usual scattering states do not exist for derivative nonlinear Schrödinger equation (0.1), when one of the coefficients λ_1, λ_6, $\lambda_2 - \lambda_3$ or $\lambda_4 - \lambda_5$ is not vanishing (see [1]). The case of the gauge invariant resonant nonlinearity $|u|^2 u$ was studied extensively (see, e.g. [20], [27], [33]). Gauge invariant cubic nonlinear Schrödinger equations $\mathcal{N}(e^{i\theta}u) = e^{i\theta}\mathcal{N}(u)$ for all $\theta \in \mathbf{R}$ were studied also in papers [6], [8], [19], [22], [28], [18]. The cubic derivative nonlinear Schrödinger equation with nongauge invariant nonlinearity was considered in paper [9], when the nonlinearity can be represented in the form of the full derivative. In [9] we used the techniques developed in [7], where it was introduced an appropriate representation of the solution and instead of the operator $\mathcal{J} = x + it\partial_x$ it was used the dilation operator $\mathcal{I}\partial_x^{-1} = x + 2t\partial_t\partial_x^{-1}$, with $\partial_x^{-1} = \int_{-\infty}^x dx$. The nonlinear Schrödinger equation with cubic nonlinearities, containing at least one derivative was studied in paper [26], where the large time asymptotics of solutions was found for small initial data $u_0 \in \mathbf{H}^{3,4}$. The special nonlinearity uu_x^2 or $\overline{uu_x}^2$ was considered in [32] and the global existence of small solutions was shown by the method of normal forms by Shatah [29] under the conditions on the data such that $u_0 \in \mathbf{H}^{1,0} \cap \mathbf{L}^{\ 1}$ for the case uu_x^2 and $(1-\Delta)^3u_0 \in \mathbf{H}^{1,0} \cap \mathbf{L}^{\ 1}$ for the case $\overline{uu_x}^2$. We improve the previous result of paper [26] obtaining estimates in a natural function space $\mathbf{H}^{3,0} \cap \mathbf{H}^{2,1}$ by using a more simple and general approach.

We give some notations. Let $\mathcal{F}\phi \equiv \hat{\phi} = \frac{1}{\sqrt{2\pi}} \int e^{-ix\xi}\phi(x)\, dx$ denote the Fourier transform of the function ϕ. The inverse Fourier transformation $\mathcal{F}^{-1}$ is defined by $\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi}\phi(\xi)d\xi$. We denote the usual Lebesgue space $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbf{R}} |\phi(x)|^p dx\right)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \sup_{x\in\mathbf{R}} |\phi(x)|$ if $p = \infty$. Weighted Sobolev space is $\mathbf{H}_p^{m,k} = \left\{\phi \in \mathbf{S}' : \|\phi\|_{\mathbf{H}_p^{m,k}} \equiv \left\|\langle x\rangle^k \langle i\partial\rangle^m \phi\right\|_{\mathbf{L}^p} < \infty\right\}$, $m, k \in \mathbf{R}$, $1 \le p \le \infty$, where $\langle x\rangle = \sqrt{1+x^2}$. We denote also for simplicity $\mathbf{H}^{m,k} = \mathbf{H}_2^{m,k}$ and the norm $\|\phi\|_{\mathbf{H}^{m,k}} = \|\phi\|_{\mathbf{H}_2^{m,k}}$. The usual Sobolev space is $\mathbf{H}^m = \mathbf{H}^{m,0}$, so the index 0 we usually omit if it does not cause a confusion.

The following result was proved in [10].

Theorem 0.1. *Let the initial data $u_0 \in \mathbf{H}^3 \cap \mathbf{H}^{2,1}$ and the norm $\|u_0\|_{\mathbf{H}^3} + \|u_0\|_{\mathbf{H}^{2,1}} = \varepsilon$ be sufficiently small. Then there exists a unique global solution u of the*

Cauchy problem (0.1) such that $u \in \mathbf{C}\left(\mathbf{R};\mathbf{H}^3 \cap \mathbf{H}^{2,1}\right)$. *Moreover there exists a unique modified final state* $W_+ \in \mathbf{L}^\infty$ *such that the following asymptotics for* $t \to \infty$

$$\begin{aligned} u(t,x) = &\frac{1}{\sqrt{it}} W_+\left(\frac{x}{t}\right) \exp\left(\frac{ix^2}{2t} + i\Lambda\left(\frac{x}{t}\right)\left|W_+\left(\frac{x}{t}\right)\right|^2 \log t\right) \\ &+ O\left(\varepsilon^3 t^{-\frac{1}{2}-\alpha}\right) \end{aligned}$$

is valid uniformly with respect to $x \in \mathbf{R}$, *where* $\Lambda(\xi) = \lambda_1 - (\lambda_2 - \lambda_3)\xi + (\lambda_4 - \lambda_5)\xi^2 - \lambda_6 \xi^3$ *and* $\alpha \in \left(0, \frac{1}{4}\right)$.

For the convenience of the reader we now briefly explain the main point of the proof. As we know the estimates of the norm $\|\mathcal{J}u\|_{\mathbf{L}^2}$ play the crucial role in obtaining the global existence and large time asymptotics of solutions to nonlinear Schrödinger type equations, (see, e.g. [7]). In order to find the estimates of the norm $\|\mathcal{J}u\|_{\mathbf{L}^2}$ we extract the full derivatives from the nonlinear term $\mathcal{N}_2$, using the identities

$$3u^2 u_x = \left(u^3\right)_x,$$

$$\begin{aligned} 3uu_x^2 &= \left(u^2 u_x\right)_x + \frac{1}{it}\left(u^3 + 3uu_x\mathcal{J}u - \left(u^2\mathcal{J}u\right)_x\right), \\ 3u_x^3 &= \left(u_x^2 u\right)_x + \frac{2}{it}\left(u_x^2\mathcal{J}u - uu_x\left(\mathcal{J}u_x\right)\right), \end{aligned}$$

representations for $\overline{u}^2\overline{u}_x$, $\overline{uu_x^2}$ and $\overline{u}_x^3$ can be obtained by the complex conjugation,

$$\begin{aligned} \overline{u}^2 u_x &= -\left(|u|^2\overline{u}\right)_x - \frac{2}{it}\left(|u|^2\overline{\mathcal{J}u} - \overline{u}^2\mathcal{J}u\right), \\ |u|^2\overline{u}_x &= \left(|u|^2\overline{u}\right)_x + \frac{1}{it}\left(|u|^2\overline{\mathcal{J}u} - \overline{u}^2\mathcal{J}u\right), \\ u\overline{u}_x^2 &= \left(|u|^2\overline{u}_x\right)_x - \frac{1}{it}\overline{u}\left(\overline{u}_x\mathcal{J}u - u\overline{\mathcal{J}u_x}\right), \\ \overline{u}|u_x|^2 &= \left(\overline{u}^2 u_x\right)_x - \frac{1}{it}\overline{u}\left(\overline{u}\mathcal{J}u_x - u_x\overline{\mathcal{J}u}\right) \end{aligned}$$

and

$$|u_x|^2\overline{u}_x = \left(|u_x|^2\overline{u}\right)_x - \frac{1}{it}\overline{u}\left(\overline{u}_x\mathcal{J}u_x - u_x\overline{\mathcal{J}u_x}\right)$$

we write $\mathcal{N}_2 = \partial_x\mathcal{N}_3 + \frac{1}{it}\mathcal{N}_4$, where

$$\begin{aligned} \mathcal{N}_3 = &a_1u^3 + a_2u^2u_x + a_3u_x^2u + b_1\overline{u}^3 + b_2\overline{u}^2\overline{u}_x + b_3\overline{u}_x^2\overline{u} \\ &+ (\mu_2 - \mu_1)|u|^2\overline{u} + \mu_3|u|^2\overline{u}_x + \mu_4\overline{u}^2u_x + \mu_5|u_x|^2\overline{u} \end{aligned}$$

and

$$\begin{aligned}\mathcal{N}_4 = & a_2\left(u^3+3uu_x\mathcal{J}u-\left(u^2\mathcal{J}u\right)_x\right)+2a_3\left(u_x^2\mathcal{J}u-uu_x\mathcal{J}u_x\right)\\ & -b_2\left(\overline{u}^3+3\overline{u}\overline{u}_x\overline{\mathcal{J}u}-\left(\overline{u}^2\overline{\mathcal{J}u}\right)_x\right)-2b_3\left(\overline{u}_x^2\overline{\mathcal{J}u}-\overline{u}\overline{u}_x\overline{\mathcal{J}u_x}\right)\\ & +\left(\mu_2-2\mu_1\right)\left(|u|^2\overline{\mathcal{J}u}-\overline{u}^2\mathcal{J}u\right)-\mu_3\overline{u}\left(\overline{u}_x\mathcal{J}u-u\overline{\mathcal{J}u_x}\right)\\ & -\mu_4\overline{u}\left(\overline{u}\mathcal{J}u_x-u_x\overline{\mathcal{J}u}\right)-\mu_5\overline{u}\left(\overline{u}_x\mathcal{J}u_x-u_x\overline{\mathcal{J}u_x}\right).\end{aligned}$$

Therefore we have another form of the Cauchy problem (0.1)

$$\begin{cases} iu_t+\frac{1}{2}u_{xx}=\mathcal{N}_1+\partial_x\mathcal{N}_3+\frac{1}{it}\mathcal{N}_4, & t\in\mathbf{R},\ x\in\mathbf{R},\\ u\left(0,x\right)=u_0, & x\in\mathbf{R},\end{cases} \tag{0.2}$$

where $\mathcal{N}_1$ is a gauge invariant, so that the action of the operator $\mathcal{J}$ on the nonlinear term $\mathcal{N}_1$ can be easily estimated. The second term in the right-hand side of (0.2) is the full derivative and the last summand $\frac{1}{it}\mathcal{N}_4$ has an additional time decay through the operator $\mathcal{J}$. Multiplying both sides of equation (0.2) by the operator $\mathcal{J}=x+it\partial_x$, we obtain

$$\mathcal{L}\mathcal{J}u=\mathcal{J}\left(\mathcal{N}_1+\partial_x\mathcal{N}_3+\frac{1}{it}\mathcal{N}_4\right),$$

where $\mathcal{L}=i\partial_t+\frac{1}{2}\partial_x^2$. Via the relation $\mathcal{J}\partial_x=\mathcal{I}+2+2i\mathcal{L}t$, where $\mathcal{I}=x\partial_x+2t\partial_t$, the full derivative of $\mathcal{N}_3$ can be transformed as $\left(\mathcal{I}+2+2i\mathcal{L}t\right)\mathcal{N}_3$, therefore we obtain

$$\mathcal{L}\left(\mathcal{J}u-2it\mathcal{N}_3\right)=\mathcal{J}\mathcal{N}_1+\left(\mathcal{I}+2\right)\mathcal{N}_3+\frac{1}{it}\mathcal{J}\mathcal{N}_4. \tag{0.3}$$

We use equation (0.3) in order to estimate the norm $\|\mathcal{J}u\|_{\mathbf{L}^2}$ in the proof of the theorem. The proof depends also on the a-priori estimate of $\mathcal{F}\mathcal{U}(-t)u$ in $\mathbf{L}^\infty$ norm, where the Schrödinger evolution group $\mathcal{U}(t)$ is defined as $\mathcal{U}(t)=\mathcal{F}^{-1}e^{-i\xi^2t/2}\mathcal{F}$. We also use the factorization of the Schrödinger evolution group

$$\mathcal{U}\left(t\right)=M\left(t\right)\mathcal{D}\left(t\right)\mathcal{F}M\left(t\right),$$

where the multiplication factor $M\left(t\right)=\exp(ix^2/2t)$, and dilation operator $\left(\mathcal{D}\left(t\right)\varphi\right)\left(x\right)=\frac{1}{\sqrt{it}}\varphi\left(\frac{x}{t}\right)$. Also we have

$$\mathcal{U}\left(-t\right)=M\left(-t\right)i\mathcal{F}^{-1}\mathcal{D}\left(\frac{1}{t}\right)M\left(-t\right),$$

since $\left(\mathcal{D}^{-1}\left(t\right)\varphi\right)\left(x\right)=i\left(\mathcal{D}\left(\frac{1}{t}\right)\varphi\right)\left(x\right)=\sqrt{it}\varphi\left(tx\right)$. Note that the identities are true

$$\mathcal{J}=x+it\partial_x=\mathcal{U}\left(t\right)x\mathcal{U}\left(-t\right)=M\left(t\right)\left(it\partial_x\right)M\left(-t\right).$$

Denote $E = e^{it\xi^2}$, $\mathcal{V}(-t) = \mathcal{F}M(-t)\mathcal{F}^{-1}$, $\mathcal{K} = \mathcal{F}M(t)\mathcal{U}(-t)$, then via factorization of the Schrödinger evolution group we obtain

$$
\begin{aligned}
\mathcal{FU}(-t)(\varphi\phi\psi) &= -\frac{i}{t}\mathcal{V}(-t)E(\mathcal{K}\varphi)(\mathcal{K}\phi)(\mathcal{K}\psi), \\
\mathcal{FU}(-t)\left(\varphi\phi\overline{\psi}\right) &= \frac{1}{t}\mathcal{V}(-t)(\mathcal{K}\varphi)(\mathcal{K}\phi)\left(\overline{\mathcal{K}\psi}\right), \\
\mathcal{FU}(-t)\left(\varphi\overline{\phi\psi}\right) &= \frac{i}{t}\mathcal{V}(-t)\overline{E}(\mathcal{K}\varphi)\left(\overline{\mathcal{K}\phi}\right)\left(\overline{\mathcal{K}\psi}\right), \\
\mathcal{FU}(-t)\left(\overline{\varphi\phi\psi}\right) &= -\frac{1}{t}\mathcal{V}(-t)E^{-2}\left(\overline{\mathcal{K}\varphi}\right)\left(\overline{\mathcal{K}\phi}\right)\left(\overline{\mathcal{K}\psi}\right).
\end{aligned}
\tag{0.4}
$$

In order to treat the oscillation factors of the nonlinear terms represented by (0.4), we compute the commutators of $E^{\omega} = e^{i\omega t\xi^2}$ and $\mathcal{V}(-t)$. Since $\mathcal{V}(t)\varphi = \mathcal{U}\left(-\frac{1}{t}\right)$, we have by a direct calculation

$$
\mathcal{V}(-t)\left(E^{\frac{\omega-1}{2}}\varphi\right) = \sqrt{2i}\mathcal{D}_{\omega}E^{\frac{1}{2}\omega(\omega-1)}\varphi + \sqrt{2i}\mathcal{D}_{\omega}E^{\frac{1}{2}\omega(\omega-1)}\left(\mathcal{V}(-\omega t) - 1\right)\varphi,
\tag{0.5}
$$

where $\mathcal{D}_{\omega}\varphi(\xi) = \frac{1}{\sqrt{i\omega}}\varphi\left(\frac{\xi}{\omega}\right)$.

This method can be applied also to the case of fractional power nonlinearities. For example, in the case of the nonlinearity $\overline{u}^2 u^{\alpha} u_x$, $0 < \alpha \neq 1$, we have the identity

$$
\begin{aligned}
\overline{u}^2 u^{\alpha} u_x &= \frac{1}{\alpha - 1}\left(\overline{u}^2 u^{\alpha} u\right)_x - \frac{1}{it(\alpha-1)}\left(u^{\alpha+1}\overline{u\mathcal{J}u} - \overline{u}^2 u^{\alpha}\mathcal{J}u\right) \\
&= \partial_x \mathcal{N}_3 + \frac{1}{it}\mathcal{N}_4,
\end{aligned}
$$

where

$$
\mathcal{N}_3 = \frac{1}{\alpha-1}\left(\overline{u}^2 u^{\alpha} u\right) \quad \text{and } \mathcal{N}_4 = -\frac{1}{\alpha-1}\left(u^{\alpha+1}\overline{u\mathcal{J}u} - \overline{u}^2 u^{\alpha}\mathcal{J}u\right).
$$

Hence by the identity $\mathcal{J}\partial_x = \mathcal{I} + 2 + 2i\mathcal{L}t$ we can transform equation $\mathcal{LJ}u = \mathcal{J}\left(\overline{u}^2 u^{\alpha} u_x\right)$ into a form analogous to equation (0.3)

$$
\mathcal{L}\left(\mathcal{J}u - 2it\mathcal{N}_3\right) = (\mathcal{I} + 2)\mathcal{N}_3 + \frac{1}{it}\mathcal{J}\mathcal{N}_4.
$$

Therefore we can estimate the $\mathbf{L}^2$ norm of operator $\mathcal{J}$ via standard energy methods and evaluate the large time asymptotics of solutions.

On the other hand, this method does not work for the nonlinearities $\mathcal{N}(u)$ satisfying the following condition $\mathcal{N}\left(e^{i\theta}u\right) = \mathcal{N}(u)$, for all $\theta \in \mathbf{R}$. Indeed, we have

$$
\begin{aligned}
\mathcal{FU}(-t)\mathcal{N}(u) &= \mathcal{V}(-t)\mathcal{D}^{-1}(t)M(-t)\mathcal{N}(u) = \mathcal{V}(-t)\sqrt{it}E^{-\frac{1}{2}}\mathcal{N}\left(M(-t)u\right)(t\xi) \\
&= \mathcal{V}(-t)\sqrt{it}E^{-\frac{1}{2}}\mathcal{N}\left(\frac{1}{\sqrt{it}}\mathcal{D}^{-1}(t)M(-t)u\right) \\
&= \mathcal{V}(-t)\sqrt{it}E^{-\frac{1}{2}}\mathcal{N}\left(\frac{1}{\sqrt{it}}\mathcal{K}u\right)
\end{aligned}
$$

and formula (0.5) does not work for the case $\omega = 0$. The cubic nonlinearities with derivatives of the first order, satisfying the condition $\mathcal{N}\left(e^{i\theta}u\right) = \mathcal{N}(u)$, are $|u|^3$, $\left|u^2 u_x\right|$, $\left|u u_x^2\right|$ and $|u_x|^3$.

We next consider the Cauchy problem for the cubic nonlinear Schrödinger equation (0.1) with nonlinearity

$$\mathcal{N}(u) = \lambda_1 u^3 + \lambda_2 \overline{u}^2 u + \lambda_3 \overline{u}^3,$$

where the coefficients $\lambda_j \in \mathbf{C}$, $j = 1, 2, 3$. For the coefficients λ_j we assume that there exists $\theta_0 > 0$ such that

$$\operatorname{Re}\left(\frac{\lambda_1}{\sqrt{3}} e^{2ir} - i\lambda_2 e^{-2ir} + \frac{\lambda_3}{\sqrt{3}} e^{-4ir}\right) \geq C > 0, \tag{0.6}$$

$$\operatorname{Im}\left(\frac{\lambda_1}{\sqrt{3}} e^{2ir} - i\lambda_2 e^{-2ir} + \frac{\lambda_3}{\sqrt{3}} e^{-4ir}\right) r \geq C r^2, \tag{0.7}$$

for all $|r| < \theta_0$, and also we suppose that the initial data $u_0(x)$ are such that

$$\left|\arg \widehat{u_0}(\xi)\right| < \theta_0, \quad \inf_{|\xi| \leq 1} \left|\widehat{u_0}(\xi)\right| \geq C\varepsilon, \tag{0.8}$$

where $\varepsilon > 0$ is a small constant depending on the size of the initial function in a suitable norm defined later. Note that conditions (0.6) - (0.7) can be fulfilled, for example, if we choose $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 0$ and $\theta_0 = \frac{\pi}{4}$. We will show below that due to special oscillating properties of the nonlinearity $\mathcal{N}(u)$, the solution has an additional logarithmic time decay in the short range region $|x| < \sqrt{t}$ and in the far region $|x| >> \sqrt{t}$ it has a quasi linear behavior as in the super critical cases.

All types of cubic nonlinearities, including derivatives and the critical term $|u|^2 u$ without derivatives of the unknown function were considered previously (see [19], [10], [28], [32]). The final state problem for (0.1) was studied extensively (see paper [16] and references cited therein). In [11] we proved global existence and obtained the large time behavior of solutions for the case of the nonlinear term $\overline{u}^3$ if the initial data $u_0 \in \mathbf{H}^{1,0} \cap \mathbf{H}^{0,1}$ are small, and their Fourier transforms are such that $\cos\left(4 \arg \widehat{u_0}(\xi)\right) \geq C > 0, \inf_{|\xi| \leq 1} \left|\widehat{u_0}(\xi)\right| \geq C\varepsilon$. We applied in [11] a method of the normal forms of Shatah [29], making a transformation of the cubic nonlinear Schrödinger equation $\mathcal{L}u = \overline{u}^3$ to the form

$$\mathcal{L}\left(u + t\mathcal{Q}_3\left(\overline{u}, \overline{u}, \overline{u}\right)\right) = \mathcal{Q}_3\left(\overline{u}, \overline{u}, \overline{u}\right) + 3t\mathcal{Q}_3\left(\overline{u}, \overline{u}, \overline{\mathcal{L}u}\right),$$

where the trilinear operators $\mathcal{Q}_3$ have symbols $\left(1 + it\left(\xi^2 + b_3\eta^2 + c_3\zeta^2\right)\right)^{-1}$, with $b_3 > 0$ and $c_3 > 0$. The fact that b_3 and c_3 are positive implies that the nonlinearities $\mathcal{Q}_3\left(\overline{u}, \overline{u}, \overline{u}\right)$, $t\mathcal{Q}_3\left(\overline{u}, \overline{u}, \overline{\mathcal{L}u}\right)$ have better time decay properties, so we can obtain the global existence result. If we try to apply the same approach to the case of the nonlinearities u^3

and $\overline{u}^2 u$, then the coefficients a_m and b_m of the symbols $\left(1+it\left(\xi^2+b_m\eta^2+c_m\zeta^2\right)\right)^{-1}$ of the trilinear operators $\mathcal{Q}_m$, $m=1,2$, are not positive and it is difficult to obtain better time decay estimates for the nonlinearities $\mathcal{Q}_m$, $m=1,2$. So we use another method based on the following identity

$$\partial_\xi = i\Upsilon\left(1-i\Upsilon\xi\right)^{-1}\left(-\widehat{\mathcal{I}}+2t\partial_t\right)+\left(1-i\Upsilon\xi\right)^{-1}\partial_\xi$$

with $\widehat{\mathcal{I}}=-\xi\partial_\xi+2t\partial_t, \Upsilon>0$. Repeating this formula we obtain

$$\partial_\xi = i\Upsilon\left(1-i\Upsilon\xi\right)^{-1}\left(1+\left(1-i\Upsilon\xi\right)^{-1}\right)\left(-\widehat{\mathcal{I}}+2t\partial_t\right)+\left(1-i\Upsilon\xi\right)^{-2}\partial_\xi.$$

We choose Υ with a time growth of order $\sqrt{t}$, so that the terms $\left(1-i\Upsilon\xi\right)^{-1}$ appearing in this identity help us to obtain better time decay properties of the solution. Unfortunately this identity involves differential operator $\widehat{\mathcal{I}}$. Therefore the application of this formula implies some derivative loss (with respect to $\widehat{\mathcal{I}}$). That is why we need some analytic function spaces to get large time estimates of solutions.

We let $\mathcal{P}=\mathcal{FU}\left(-t\right)$ and the analytic function space $\mathbf{A}_1$ be defined

$$\mathbf{A}_1=\left\{\phi\in\mathbf{L}^2:\left\|\phi\right\|_{\mathbf{A}_1}\equiv\sum_{n=0}^{\infty}\frac{\varepsilon^{\varkappa n}}{n!}\left(\left\|\mathcal{I}^n\phi\right\|+\left\|\mathcal{P}\mathcal{I}^n\phi\right\|_\infty\right)<\infty\right\},$$

where $0<\varkappa<\frac{1}{10}$.

In paper [12] it was proved the following result.

Theorem 0.2. *Suppose that conditions (0.6) - (0.8) are valid with sufficiently small $\varepsilon>0$. Assume that the initial data $u_0\in\mathbf{A}_1$. Then there exists $\varepsilon_0>0$ such that if $\left\|u_0\right\|_{\mathbf{A}_1}\leq\varepsilon$ for $0<\varepsilon<\varepsilon_0$, then there exists a unique solution $u\in\mathbf{C}\left(\left[0,\infty\right),\mathbf{L}^\infty\right)$ of the Cauchy problem (0.1) with $\mathcal{N}\left(u\right)=\lambda_1u^3+\lambda_2\overline{u}^2u+\lambda_3\overline{u}^3$. Moreover there exist unique final states $W_+,r_+\in\mathbf{L}^\infty$ such that the following asymptotics for $t\to\infty$*

$$u\left(t,x\right)=\frac{\left(it\right)^{-\frac{1}{2}}W_+\left(\frac{x}{t}\right)e^{\frac{ix^2}{2t}}}{\sqrt{1+\chi\left(\frac{x}{t}\right)\left|W_+\left(\frac{x}{t}\right)\right|^2\log\frac{t^2}{t+x^2}}}+O\left(t^{-\frac{1}{2}}\left(1+\log\frac{t^2}{t+x^2}\right)^{-\frac{1}{2}-\gamma}\right) \tag{0.9}$$

is valid uniformly with respect to $x\in\mathbf{R}$, where $0<\gamma<\frac{1}{20}$, and $\chi\left(\xi\right)$ is given by

$$\chi\left(\xi\right)=Re\left(\frac{\lambda_1}{\sqrt{3}}\exp\left(2ir_+\left(\xi\right)\right)-i\lambda_2\exp\left(-2ir_+\left(\xi\right)\right)+\frac{\lambda_3}{\sqrt{3}}\exp\left(-4ir_+\left(\xi\right)\right)\right).$$

Note that the solution u given by the asymptotic formula (0.9) has an additional logarithmic time-decay in the short range region $|x|\leq\sqrt{t}$. In the far region $|x|\gg\sqrt{t}$ the asymptotics has a quasi - linear character.

In the case of two space dimensions, quadratic nonlinearities are considered as critical and the global existence and asymptotic behavior of solutions were studied in papers [2], [4] under suitable assumptions on the nonlinearities and regularity and smallness of the initial data. However for the case of quadratic nonlinearities such as u^2 or $\overline{u}^2$ or $u\overline{u}$, the problem of asymptotic behavior of solutions is still open even if the data are analytic.

For the convenience of the reader we now give a strategy of the proof briefly. We shift the time $t' = t+1$ (we will omit the prime below), so the initial data are given at the time $t' = 1$. We use the factorization technique for the Schrödinger evolution group $\mathcal{U}(t)\phi = M\mathcal{D}\mathcal{F}M\phi$, where $M = e^{\frac{ix^2}{2t}}$, $E = e^{\frac{i}{2}t\xi^2}$, $\mathcal{D}(t)$ is the dilation operator $\mathcal{D}(t)\phi = \frac{1}{\sqrt{it}}\phi\left(\frac{\cdot}{t}\right)$, $\mathcal{D}(t)^{-1} = i\mathcal{D}\left(\frac{1}{t}\right)$. Also we have $\mathcal{U}(-t) = \overline{M}\mathcal{F}^{-1}\mathcal{D}^{-1}\overline{M}$. Also we use the following identity

$$\mathcal{F}\overline{M}\mathcal{F}^{-1}\mathcal{D}^{-1}\overline{M}\phi = \sqrt{i}\mathcal{D}(\rho)E^{\rho(\rho-1)}\mathcal{F}\overline{M}^{\frac{1}{\rho}}\mathcal{F}^{-1}\mathcal{D}^{-1}\overline{M}^{\rho}\phi \tag{0.10}$$

for any $\rho \in \mathbf{R}\setminus\{0\}$. So that applying the operator $\mathcal{F}\mathcal{U}(-t)$ to both sides of the equation $\mathcal{L}u = \mathcal{N}(u)$, with $\mathcal{N}(u) = \lambda_1 u^3 + \lambda_2\overline{u}^2 u + \lambda_3\overline{u}^3$, putting $v = \mathcal{F}\mathcal{U}(-t)u$ we get

$$\begin{aligned} i\partial_t v &= i\left(\mathcal{F}\mathcal{U}(-t)u\right)_t = \mathcal{F}\mathcal{U}(-t)\left(\lambda_1 u^3 + \lambda_2\overline{u}^2 u + \lambda_3\overline{u}^3\right) \\ &= \mathcal{F}\overline{M}\mathcal{F}^{-1}\mathcal{D}^{-1}\overline{M}\left(\lambda_1 u^3 + \lambda_2\overline{u}^2 u + \lambda_3\overline{u}^3\right). \end{aligned}$$

Then using identity (0.10) we obtain

$$\begin{aligned} i\partial_t v &= \lambda_1\frac{1}{it}E^{\frac{2}{3}}\frac{1}{\sqrt{3}}\left(\mathcal{F}\overline{M}^{\frac{1}{3}}\mathcal{F}^{-1}\left(\mathcal{D}^{-1}\overline{M}u\right)^3\right)\left(\frac{\xi}{3}\right) \\ &\quad -\lambda_2\frac{1}{t}E^2\mathcal{F}M\mathcal{F}^{-1}\left(\overline{\mathcal{D}^{-1}\overline{M}u}\right)^2\left(\mathcal{D}^{-1}\overline{M}u\right)(-\xi) \\ &\quad +\lambda_3\frac{1}{it}E^{\frac{4}{3}}\frac{1}{\sqrt{3}}\mathcal{F}M^{\frac{1}{3}}\mathcal{F}^{-1}\left(\overline{\mathcal{D}^{-1}\overline{M}u}\right)^3\left(-\frac{\xi}{3}\right). \end{aligned}$$

Hence asymptotically for large values of time we obtain

$$\begin{aligned} \partial_t v(t,\xi) &\simeq -\frac{\lambda_1}{\sqrt{3}}\frac{1}{t}E^{\frac{2}{3}}v\left(t,\frac{\xi}{3}\right)^3 + i\lambda_2\frac{1}{t}E^2\overline{v(t,-\xi)}^2 v(t,-\xi) \\ &\quad -\frac{\lambda_3}{\sqrt{3}}\frac{1}{t}E^{\frac{4}{3}}\overline{v\left(t,-\frac{\xi}{3}\right)}^3, \end{aligned}$$

and using the oscillating factors $E^{\frac{2}{3}}$, E^2 and $E^{\frac{4}{3}}$ and integration by parts with respect to time, we get

$$\begin{aligned} \partial_t v(t,\xi) &\simeq -\frac{\frac{1}{\sqrt{3}}\lambda_1 e^{i\frac{1}{3}t\xi^2}}{t\left(1+i\frac{1}{3}t\xi^2\right)}v(t,\xi)^3 + \frac{i\lambda_2 e^{it\xi^2}}{t\left(1+it\xi^2\right)}\overline{v(t,\xi)}^2 v(t,\xi) \\ &\quad -\frac{\frac{1}{\sqrt{3}}\lambda_3 e^{i\frac{2}{3}t\xi^2}}{t\left(1+i\frac{2}{3}t\xi^2\right)}\overline{v(t,\xi)}^3, \end{aligned}$$

where $A \simeq B$ means $A = B+$ remainder term for our functional space. For the case $\xi^2 \geq 1$ the right-hand side of the above formula has a sufficient time decay, so we only consider the case $\xi^2 \leq 1$. We change the dependent variable $v = fe^{-\phi+ir}$, where ϕ and r are the real valued functions. Then

$$\begin{aligned} & f_t e^{-\phi+ir} + (-\phi_t + ir_t) f e^{-\phi+ir} \\ \simeq & -\frac{1}{t} e^{-2\phi} \left(\frac{\frac{1}{\sqrt{3}} \lambda_1 e^{i\frac{1}{3}t\xi^2}}{\left(1 + i\frac{1}{3}t\xi^2\right)} f^2 e^{2ir} - \frac{i\lambda_2 e^{it\xi^2}}{(1+it\xi^2)} \overline{f}^2 e^{-2ir} \right. \\ & \left. + \frac{\frac{1}{\sqrt{3}} \lambda_3 e^{i\frac{2}{3}t\xi^2}}{\left(1 + i\frac{2}{3}t\xi^2\right)} \frac{\overline{f}^3}{f} e^{-4ir} \right). \end{aligned}$$

We let

$$\phi_t = \frac{1}{t} e^{-2\phi} \mathrm{Re} \left(\frac{\frac{1}{\sqrt{3}} \lambda_1 e^{i\frac{1}{3}t\xi^2}}{\left(1 + i\frac{1}{3}t\xi^2\right)} f^2 e^{2ir} - \frac{i\lambda_2 e^{it\xi^2}}{(1+it\xi^2)} \overline{f}^2 e^{-2ir} + \frac{\frac{1}{\sqrt{3}} \lambda_3 e^{i\frac{2}{3}t\xi^2}}{\left(1 + i\frac{2}{3}t\xi^2\right)} \frac{\overline{f}^3}{f} e^{-4ir} \right)$$

and

$$r_t = -\frac{1}{t} e^{-2\phi} \mathrm{Im} \left(\frac{\frac{1}{\sqrt{3}} \lambda_1 e^{i\frac{1}{3}t\xi^2}}{\left(1 + i\frac{1}{3}t\xi^2\right)} f^2 e^{2ir} - \frac{i\lambda_2 e^{it\xi^2}}{(1+it\xi^2)} \overline{f}^2 e^{-2ir} + \frac{\frac{1}{\sqrt{3}} \lambda_3 e^{i\frac{2}{3}t\xi^2}}{\left(1 + i\frac{2}{3}t\xi^2\right)} \frac{\overline{f}^3}{f} e^{-4ir} \right)$$

with the initial conditions

$$\phi(1,\xi) = 0, r(1,\xi) = \arg v(1,\xi).$$

Then we have $f_t \simeq 0$, which implies that $f(t,\xi) \simeq f(1,\xi)$. Hence we have

$$\phi_t \simeq \frac{1}{t} e^{-2\phi} \frac{\left|\widehat{u_0}(\xi)\right|^2}{|1+it\xi^2|} \mathrm{Re} \left(\frac{\lambda_1}{\sqrt{3}} e^{2ir} - i\lambda_2 e^{-2ir} + \frac{\lambda_3}{\sqrt{3}} e^{-4ir} \right)$$

and

$$r_t \simeq -\frac{1}{t} e^{-2\phi} \left|\widehat{u_0}(\xi)\right|^2 \mathrm{Im} \left(\frac{\lambda_1}{\sqrt{3}} e^{2ir} - i\lambda_2 e^{-2ir} + \frac{\lambda_3}{\sqrt{3}} e^{-4ir} \right). \tag{0.11}$$

From the first equation if we assume that

$$|r(t,\xi)| \leq \left|\arg \widehat{u_0}(\xi)\right| + \varepsilon^{\frac{1}{4}},$$

then in view of the condition (0.6) we get

$$\left(e^{2\phi}\right)_t \geq \frac{1}{t} \left|\widehat{u_0}(\xi)\right|^2 \frac{1}{|1+it\xi^2|} C \geq \frac{1}{t} C\varepsilon^2 \frac{1}{|1+it\xi^2|}.$$

Integrating this estimate with respect to time, we obtain

$$e^{2\phi(t,\xi)} \geq 1 + C\varepsilon^2 \log\left(\min\left(t, \frac{1}{\xi^2}\right)\right)$$

which implies

$$e^{-\phi(t,\xi)} \leq \frac{1}{\sqrt{1 + C\varepsilon^2 \log\left(\min\left(t, \frac{1}{\xi^2}\right)\right)}}$$

since

$$\int_1^t \frac{1}{\tau} \frac{1}{1+\tau\xi^2} d\tau \geq \log \frac{1}{\frac{1}{t} + \xi^2} \geq C \log\left(\min\left(t, \frac{1}{\xi^2}\right)\right).$$

Thus we have an additional time decay for $\xi^2 \leq 1$ such that

$$e^{-\phi(t,\xi)} \leq \frac{C}{\sqrt{1+\varepsilon^2 \log t}}.$$

Hence by virtue of the equation (0.11) and by assumption (0.7) we get

$$\begin{aligned}\partial_t\left(r^2\right) &\simeq -\frac{1}{t} e^{-2\phi} \left|\widehat{u_0}\left(\xi\right)\right|^2 \operatorname{Im}\left(\frac{\lambda_1}{\sqrt{3}} e^{2ir} - i\lambda_2 e^{-2ir} + \frac{\lambda_3}{\sqrt{3}} e^{-4ir}\right) r \\ &\leq -\frac{C}{t} e^{-2\phi} \left|\widehat{u_0}\left(\xi\right)\right|^2 r^2\end{aligned}$$

which gives us

$$|r(t,\xi)| \leq |r(1,\xi)| + C\varepsilon^2 \leq |\theta_0| + \varepsilon^{\frac{1}{4}}.$$

Collecting these estimates we see that there exist unique functions W_+ and r_+ such that

$$\begin{aligned}u(t) &= \frac{e^{\frac{ix^2}{2t}}}{\sqrt{it}} f\left(t, \frac{x}{t}\right) e^{-\phi\left(t,\frac{x}{t}\right) + ir\left(t,\frac{x}{t}\right)} \\ &\simeq \frac{e^{\frac{ix^2}{2t}}}{\sqrt{it}} \frac{W_+\left(\frac{x}{t}\right)}{\sqrt{1 + \chi\left(\frac{x}{t}\right)\left|W_+\left(\frac{x}{t}\right)\right|^2 \log \frac{t^2}{t+x^2}}},\end{aligned}$$

where

$$\chi\left(\xi\right) = \operatorname{Re}\left(\frac{\lambda_1}{\sqrt{3}} \exp\left(2ir_+\left(\xi\right)\right) - i\lambda_2 \exp\left(-2ir_+\left(\xi\right)\right) + \frac{\lambda_3}{\sqrt{3}} \exp\left(-4ir_+\left(\xi\right)\right)\right)$$

and

$$\left|r_+\left(\frac{x}{t}\right)\right| \leq |\theta_0| + \varepsilon^{\frac{1}{4}}, \left|W_+\left(\frac{x}{t}\right)\right| \leq 2\varepsilon.$$

Next we consider the Cauchy problem for the cubic nonlinear Schrödinger equation (0.1) with a nongauge invariant nonlinearity $\mathcal{N}(u,\overline{u}) = \lambda \overline{u}^\alpha u^{3-\alpha}$, where $\alpha \in [0,1)$ and $\lambda = -i^{1-\alpha}\sqrt{3-2\alpha}$. We chose this particular value of λ for the convenience of the forthcoming formulas (note that λ can be excluded from the equation by the scaling). Applying the transformation of [21] we now consider problem (0.1) in the usual Sobolev spaces.

Next result was proved in [13].

Theorem 0.3. *Let the initial data $u_0 \in \mathbf{H}^2 \cap \mathbf{H}^{0,2}$ with a norm $\|u_0\|_{\mathbf{H}^2} + \|u_0\|_{\mathbf{H}^{0,2}} \leq \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Also suppose that*

$$\sup_{|\xi|\leq 1} |\arg \widehat{u_0}(\xi)| < \pi \min\left\{1, \frac{1}{8\omega}\right\}, \quad \inf_{|\xi|\leq 1} |\widehat{u_0}(\xi)| \geq \delta,$$

where $\omega = 1-\alpha$ and $\delta = \varepsilon^{\frac{6}{5}}$. Then there exists a unique solution $u \in \mathbf{C}\left([0,\infty);\mathbf{H}^2 \cap \mathbf{H}^{0,2}\right)$ of the Cauchy problem (0.1) with nonlinearity $\mathcal{N}(u,\overline{u}) = \lambda \overline{u}^{\alpha} u^{3-\alpha}$, where $\alpha \in [0,1)$. Moreover the asymptotics is valid

$$u(t,x) = \frac{(it)^{-\frac{1}{2}} \left|\widehat{u_0}\left(\frac{x}{t}\right)\right| \exp\left(\frac{ix^2}{2t}\right)}{\sqrt{1 + \left|\widehat{u_0}\left(\frac{x}{t}\right)\right|^2 \log \frac{t^2}{t+x^2}}} + O\left(t^{-\frac{1}{2}} \left(\log \frac{t^2}{t+x^2}\right)^{-1}\right)$$

for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$.

The case when the nonlinearity $\mathcal{N}(u,\overline{u})$ is a sum of terms of the form $\lambda \overline{u}^{\alpha} u^{3-\alpha}$ with $\alpha \in [0,3]$, $\alpha \neq 1, \frac{3}{2}$ also can be considered by this method. When $\alpha = 1$, then the nonlinearity is $\mathcal{N}(u,\overline{u}) = -|u|^2 u$, and represents a resonance term.

We next consider the Cauchy problem for the cubic nonlinear Schrödinger equation (0.1) with the nonlinearity of the form $\mathcal{N} = \sum_{k=-2}^{1} \lambda_k u^{2+k} \overline{u}^{1-k}$, where $\lambda_0 \in \mathbf{R}$ and $\lambda_{-2}, \lambda_{-1}, \lambda_1 \in \mathbf{C}$ in the case of an odd initial data $u_0(x)$. So we consider all the types of cubic nonlinearities in Schrödinger equation, combining gauge invariant term $\lambda_0 |u|^2 u$ with nongauge invariant terms. If we restrict our attention to odd initial data u_0 then the solution is an odd function for all $t > 0$. This implies that the Fourier transform $\widehat{u_0}$ vanishes at the origin, and as a consequence the nonlinearity obtains better time decay properties similar to the case of the nonlinear Schrödinger equation with cubic nonlinearities, containing at least one derivative. Note that the condition $\inf_{|\xi|\leq 1} |\widehat{u_0}(\xi)| \geq \delta$ which was assumed previously cannot be fulfilled for the odd solutions under the consideration since $\widehat{u_0}(0) = 0$.

The following result was proved in paper [15].

Theorem 0.4. *Let the initial data $u_0 \in \mathbf{H}^2 \cap \mathbf{H}^{0,2}$ be odd functions and the norm $\|u_0\|_{\mathbf{H}^2} + \|u_0\|_{\mathbf{H}^{0,2}}$ be sufficiently small. Then there exists a unique global solution $u \in \mathbf{C}\left([0,\infty);\mathbf{H}^2 \cap \mathbf{H}^{0,2}\right)$ of the Cauchy problem (0.1) with the nonlinearity $\mathcal{N} = \sum_{k=-2}^{1} \lambda_k u^{2+k} \overline{u}^{1-k}$. Moreover there exists a unique modified final state $W_+ \in \mathbf{L}^2 \cap \mathbf{L}^\infty$*

such that the following asymptotics for $t \to \infty$ is valid

$$\begin{aligned}&\left\| u\left(t\right) - \frac{1}{\sqrt{it}} W_{+}\left(\frac{\cdot}{t}\right) \exp\left(\frac{ix^2}{2t} + i\lambda_0 \left|W_{+}\left(\frac{\cdot}{t}\right)\right|^2 \log t\right)\right\|_{\mathbf{L}^\infty} \\ &+t^{\frac{1}{4}} \left\| u\left(t\right) - \frac{1}{\sqrt{it}} W_{+}\left(\frac{\cdot}{t}\right) \exp\left(\frac{ix^2}{2t} + i\lambda_0 \left|W_{+}\left(\frac{\cdot}{t}\right)\right|^2 \log t\right)\right\|_{\mathbf{L}^2} \\ &\leq C\varepsilon t^{-\frac{1}{2}-\nu},\end{aligned}$$

where $\nu \in \left(0, \frac{1}{4}\right)$.

In [16], for the final data belonging to $\mathbf{H}^2 \cap \mathbf{H}^{0,2} \cap \overset{\circ}{\mathbf{H}}{}^{-\gamma,0}$, the modified wave operator was constructed, where $\overset{\circ}{\mathbf{H}}{}^{-\gamma,0}$ is the homogeneous Sobolev space of negative order and $\gamma > \frac{1}{2}$. It is easy to see that if $\phi \in \mathbf{H}^{0,2}$ is an odd function, then $\phi \in \mathbf{H}^{0,2} \cap \overset{\circ}{\mathbf{H}}{}^{-\gamma,0}$ for $\gamma < 1$. Therefore our restrictions on the initial data are stronger than that for the final data in paper [16]. For the convenience of the reader we now briefly explain the main point of the proof. In the same way as in paper [21] we change the variables $u\left(t,x\right) = t^{-\frac{1}{2}} Ev\left(t,\xi\right), E = e^{\frac{it}{2}\xi^2}$ and $\xi = \frac{x}{t}$ in equation (0.1). As we know the estimates of the norm $\|v_\xi\|_{\mathbf{L}^2} = \|x\mathcal{U}(-t)u\|_{\mathbf{L}^2} = \|\mathcal{J}u\|_{\mathbf{L}^2}$ play the crucial role in obtaining the global existence and large time asymptotics of solutions to nonlinear Schrödinger type equations (see, for example, [6], [13], [17], [18], [22]). In order to find the estimates of the norm $\|v_\xi\|_{\mathbf{L}^2}$ we use the transformation similar to the normal forms of Shatah [29] introducing the operator

$$\mathcal{I}\left(v\right) = v_\xi - 2t\xi \sum_{k=-2,k\neq 0}^{1} A_k E^{2k} \mathcal{N}^{(k)}\left(v\right),$$

where

$$A_k = \left(1 + \left(1+2k\right) it\xi^2\right)^{-1},$$

and $\mathcal{N}^{(k)}\left(v\right) = \lambda_k v^{2+k}\overline{v}^{1-k}$ for $k = -2,-1,0,1$. Our proof also depends on the a-priori estimate of $\mathcal{F}\mathcal{U}(-t)u$ in $\mathbf{L}^\infty$ norm. In order to get the desired estimate we need to take into account the oscillating properties of the nonlinearity.

The Klein-Gordon equation is considered as the relativistic version of the Schrödinger equation and solutions of the Klein-Gordon equation have the same time decay property as those of the Schrödinger equation. Therefore one can expect the similar results as ours are obtained in nonlinear Klein-Gordon equations with cubic nonlinearities. Indeed nonlinear Klein-Gordon equations with cubic nonlinearities were studied by many authors, and in particular, sharp time decay of solutions was obtained by [3], [5], [14], [30], [31] and phase modifications were introduced in [3], [14], [31] to show global asymptotics of solutions like ours.

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