RIMS Kôkyûroku Bessatsu
B16 (2010), 75–89

On the nonrelativistic limit of Dirac operators with potentials diverging at infinity

By

Hiroshi T. ITO* and Osanobu YAMADA**

Abstract

We investigate the spectral property and the nonrelativisitic limit of the Dirac operator with a dilation analytic potential diverging at infinity by introducing two kinds of relativistic Schrödinger operators. We prove an abstract theorem on the spectrum of a self-adjoint operator defined as a boundary value of some analytic family of closed operators and then use this theorem to study the properties of the above operators. Almost all proofs are omitted in this note. For the proofs we refer the reader to [12].

§ 1. Introduction

Let us consider the Dirac operator

$$H(c) = c\alpha \cdot D + mc^2\beta + V(x) \tag{1.1}$$

in the Hilbert space $L^2(\boldsymbol{R}^3)^4$, where $c > 0$ is the velocity of light, $m > 0$ the rest mass of a relativistic particle moving in the electric field determined by an electric potential $V \in C(\boldsymbol{R}^3 \to \boldsymbol{R})$ and $\alpha \cdot D = \sum_{j=1}^{3} \alpha_j D_j$, where $D = -i\nabla_x = (D_1, D_2, D_3)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Here each α_j and β are 4×4 Hermitian matrices defined by

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

Received March 31, 2009. Revised August 19, 2009.
2000 Mathematics Subject Classification(s): Primary 35P20; Secondary 81Q15
Key Words: Dirac operator, resonances, nonrelativistic limit
The first author was supported by the Grant-in-Aid for Scientific Research (C) (No. 21540187)
The second author was supported by the Grant-in-Aid for Scientific Research (C) (No. 18540196)
*Department of Computer Science, Ehime University, Matsuyama 790-8577, Japan
e-mail: `ito@cs.ehime-u.ac.jp`
**Department of Mathematical Sciences, Ritsumeikan University, Kusatsu, Shiga 525-8577, Japan
e-mail: `yamadaos@se.ritsmei.ac.jp`

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices, and I_n is the $n \times n$ unit matrix.

It is believed that a Dirac operator converges to the corresponding Schrödinger operator acting in $L^2(\boldsymbol{R}^3)$

$$S = -\frac{1}{2m}\Delta + V(x) \tag{1.2}$$

in some sense if the velocity of light, c, goes to infinity (the nonrelativistic limit) and this expectation has been verified by many authors [3, 4, 18, 19, 21], if the potential $V(x)$ *decays* uniformly at infinity. Indeed, in this case the resolvent $(H(c) - mc^2 - z)^{-1}$, $\operatorname{Im} z \neq 0$, converges to

$$\begin{pmatrix} (S - z)^{-1} I_2 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.3}$$

as $c \to \infty$ in the operator norm (see, e.g., [18]), and the spectrum of the Dirac operator is similar to that of the Schrödinger operator in the sense that the Dirac operator has the essential spectrum $\sigma_{\mathrm{ess}}(H(c))= (-\infty, -mc^2] \cup [mc^2, \infty)$ and the discrete spectrum $\sigma_{\mathrm{d}}(H(c)) \subset (-mc^2, mc^2)$, and the Schrödinger operator has $\sigma_{\mathrm{ess}}(S) = [0, \infty)$ and $\sigma_{\mathrm{d}}(S) \subset (-\infty, 0)$.

On the other hand, if the potential *diverges at infinity* :

$$V(x) \to +\infty \quad \text{as } |x| \to \infty, \tag{1.4}$$

their spectra are quite different. Indeed, the Schödinger operator S has a purely discrete spectrum, whereas the Dirac operator $H(c)$ has a purely absolutely continuous spectrum covering the whole real line $(-\infty, +\infty)$ for a wide class of potentials including radial potentials [14]. Therefore, in this case, we cannot expect the norm resolvent convergence of $H(c) - mc^2$ to S as in the case of decaying potentials since their spectra are quite different. However, we can consider S as the nonrelativistic limit of $H(c)$ even in this case. In fact, there are two typical approaches to relate them: "spectral concentration" [11, 20] and "resonances" [1, 19]. In this work we study this problem from the standpoint of resonances. Roughly speaking, we assume that the potential $V(x)$ is dilation analytic and behaves like $const.|x|^M$ as $|x| \to \infty$ for some $M > 0$.

The following is our assumption, which is similar to the one of [1].

Assumption

(V1) $V(x)$ is a real-valued continuous function on $\boldsymbol{R}^3$and there are constants $M > 0$,

$K > 0$, a small constant $a_0 > 0$ and a $C(S^2)$-valued analytic function $V(z, \cdot)$ of z defined on S_{a_0},

$$S_{a_0} := \{ re^{i\tau} \in \boldsymbol{C} \,;\, r \in (0, \infty),\ -a_0 < \tau < a_0 \},$$

such that

$$\sup_{\omega \in S^2} |V(z, \omega)| \leq K(1 + |z|)^M \tag{1.5}$$

for all $z \in S_{a_0}$ and $V(r, \omega) = V(r\omega)$ if $r > 0$ and $\omega \in S^2$.

Define a function $V_\theta(x)$ for each $\theta \in \boldsymbol{C}$ with $|\operatorname{Im}\theta| < a_0$ by

$$V_\theta(x) := V(e^\theta |x|, \hat{x}),\ \hat{x} = x/|x|,\ x \neq 0.$$

(V2) There is a constant $R_0 > 0$ such that for each $\tau \in (-a_0, a_0)$ the function $V_{i\tau}(x)$ is C^∞ for $|x| > R_0$ and satisfies the estimate

$$|\partial_x^\alpha V_{i\tau}(x)| \leq K_\alpha |x|^{M-|\alpha|} \tag{1.6}$$

for $|\alpha| \geq 0$ and for $|x| > R_0$ uniformly in $|\tau| < a_0$.

(V3) There exist a constant $K_0 > 0$ such that

$$V(x) \geq K_0 |x|^M, \quad x \cdot \nabla V(x) \geq K_0 |x|^M \tag{1.7}$$

for $|x| \geq R_0$.

The *resonances* of $H(c)$ are defined as the eigenvalues of the dilated Dirac operator

$$H(c, \theta) := c\alpha \cdot e^{-\theta} D + mc^2\beta + V_\theta(x) \tag{1.8}$$

for $\theta \in \boldsymbol{C}$ with $0 < \operatorname{Im}\theta < a_0$ (see [1]). In [1] Amour, Brummelhuis and Nourrigat show that the family of $H(c, \theta)$ is an analytic family of type (A) [13, 17] with compact resolvent, and so $H(c, \theta)$ has only discrete spectrum. The standard argument of the complex scaling method shows that the resonances are independent of θ with $0 < \operatorname{Im}\theta < a_0$ [2, 6, 17]. In [1] they prove that there are resonances of $H(c) - mc^2$, eigenvalues of $H(c, \theta) - mc^2$, near each eigenvalue of S if c is large enough and the resonances converge to the eigenvalue as $c \to \infty$.

Our purpose is to clarify this mechanism by introducing two relativistic Schrödinger operators

$$L_\pm(c) := \pm\sqrt{-c^2\Delta + m^2c^4} - mc^2 + V(x) \quad \text{in } L^2(\boldsymbol{R}^3) \tag{1.9}$$

as intermediates between the Dirac operator $H(c)$ and the Schrödinger operator S.

§ 2. Relativistic Schrödinger Operators

Let $\sigma(\xi) := \sqrt{|\xi|^2 + m^2}$, $\xi \in \boldsymbol{R}^3$, and

$$A(\xi) := \left(\frac{\sigma(\xi) + m}{\sigma(\xi)} \right)^{1/2}.$$

Define a 4×4 matrix $U_c(\xi)$ by $U_c(\xi) := U(\xi/c)$, where

$$U(\xi) := \frac{1}{\sqrt{2}} \left(A(\xi) I_4 + A(\xi)^{-1} \beta\alpha \cdot \frac{\xi}{\sigma(\xi)} \right).$$

Then the operator $U_c(D)$ is unitary and diagonalizes the free Dirac operator ($V(x) = 0$):

$$U_c(D) H_0(c) U_c(D)^{-1} = \begin{pmatrix} \sqrt{-c^2\Delta + m^2c^4} I_2 & 0 \\ 0 & -\sqrt{-c^2\Delta + m^2c^4} I_2 \end{pmatrix}.$$

This transformation by $U_c(D)$ is called the FWT transformation (the Foldy-Wouthuysen-Tani transformation). It is easy to see that the FWT transformation maps $\mathcal{S}^4$ onto itself, where $\mathcal{S}$ denotes the Schwartz space $\mathcal{S}(\boldsymbol{R}^3)$. Now we apply this transformation to $H(c) - mc^2 = H_0(c) - mc^2 + V(x)$:

$$L(c) := U_c(D)(H(c) - mc^2) U_c(D)^{-1} = L_1(c) + W(c),$$

where

$$L_1(c) := \begin{pmatrix} L_+(c) I_2 & 0 \\ 0 & L_-(c) I_2 \end{pmatrix}, \tag{2.1}$$

$$L_\pm(c) := \pm\sqrt{-c^2\Delta + m^2c^4} - mc^2 + V(x), \tag{2.2}$$

and

$$W(c) := U_c(D) V(x) U_c(D)^{-1} - V(x). \tag{2.3}$$

Since the potential $V(x)$ is continuous, $H(c)$ defined on $\mathcal{S}^4$ is essentially self-adjoint (see e.g. [18]), and so $L(c)$ defined on $\mathcal{S}^4$ is also essentially self-adjoint since $U_c(D)$ maps $\mathcal{S}^4$ onto itself. Hereafter we denote by $L(c)$ and $H(c)$ the unique self-adjoint extensions of them, respectively.

In this work we shall investigate $L(c)$, instead of $H(c)$, by considering $W(c)$ as a perturbation of $L_1(c)$, since not only $L(c)$ and $H(c) - mc^2$ are unitarily equivalent but also their resonances coincide (see the remark below Proposition 4.4). However, there is a difficulty to consider $W(c)$ as a perturbation of $L_1(c)$ directly, because $L_-(c)$ has no global ellipticity though $L_+(c)$ has, and so $L_1(c)$ cannot control $W(c)$. To overcome this difficulty we introduce a complex scaling argument as in [1].

First we study the relativistic Schrödinger operators $L_\pm(c)$. Here we remark that $L_\pm(c)$ defined on $\mathcal{S}$ are essentially self-adjoint operators (see [8, 9]) under our assumption and denote again by the same notation $L_\pm(c)$ their self-adjoint extensions, respectively.

Fix a constant a with $0 < a < a_0$ in this work and let

$$\Omega := \{\, \theta \in \boldsymbol{C}\,;\, |\operatorname{Im}\theta| < a \,\},$$
$$\Omega_+ := \{\, \theta \in \boldsymbol{C}\,;\, 0 < \operatorname{Im}\theta < a \,\}.$$

Our assumptions on V make it possible to define the following operators on $\mathcal{S}$

$$L_\pm(c,\theta) := \pm\sqrt{-c^2 e^{-2\theta}\Delta + m^2c^4} - mc^2 + V_\theta(x) \tag{2.4}$$

for $\theta \in \Omega$, where $\sqrt{-c^2 e^{-2\theta}\Delta + m^2c^4}$ is considered as the pseudodifferential operator with symbol $\sqrt{c^2 e^{-2\theta}|\xi|^2 + m^2c^4}$. Here $\sqrt{z}$ is defined to have the branch on the negative real line. Note that if t is a real number, they are written as

$$L_\pm(c,t) = \mathcal{U}(t)L_\pm(c)\mathcal{U}(t)^{-1} \tag{2.5}$$

on $\mathcal{S}$, where $\mathcal{U}(t)$ is the dilation group defined by

$$\mathcal{U}(t)f(x) = e^{3t/2}f(e^t x).$$

Let us define the weighted L^2-space $L^2_M(\boldsymbol{R}^3)$ by $L^2_M(\boldsymbol{R}^3) = L^2(\boldsymbol{R}^3; \langle x\rangle^{2M}dx)$ and set $D_M := H^1(\boldsymbol{R}^3) \cap L^2_M(\boldsymbol{R}^3)$, where $H^1(\boldsymbol{R}^3)$ is the Sobolev space of order one. Here we note that according to Rellich's criterion any closed operator with domain D_M has compact resolvent if the resolvent set is not empty.

Hereafter we suppose $a > 0$ is sufficiently small and $c \geq 1$. The following proposition is the main result in this section.

Proposition 2.1. *(a) For each $\theta \in \Omega$ and $c \geq 1$, $L_+(c,\theta)$ defined on $\mathcal{S}$ is closable, and its closure (denoted by the same notation $L_+(c,\theta)$) has domain D_M. Moreover, its resolvent set is nonempty and, in particular, $L_+(c,\theta)$ has compact resolvent.*
(b) For each $c \geq 1$ the family of closed operators $\{\, L_+(c,\theta)\,\}_{\theta\in\Omega}$ is an analytic family of type (A) (e.g. [13, 17]) with the following property:

$$L_+(c,t+\theta) = \mathcal{U}(t)L_+(c,\theta)\mathcal{U}(t)^{-1},\ t \in \boldsymbol{R},\ \theta \in \Omega. \tag{2.6}$$

(c) For each $\theta \in \Omega_+$ and $c \geq 1$, $L_-(c,\theta)$ defined on $\mathcal{S}$ is closable and its closure (denoted by the same notation $L_-(c,\theta)$) has domain D_M. Moreover, its resolvent set is nonempty and, in particular, $L_-(c,\theta)$ has compact resolvent.
(d) For each $c \geq 1$, the family of closed operators $\{\, L_-(c,\theta)\,\}_{\theta\in\Omega_+}$ is an analytic family of type (A) with the following property:

$$L_-(c,t+\theta) = \mathcal{U}(t)L_-(c,\theta)\mathcal{U}(t)^{-1},\ t \in \boldsymbol{R},\ \theta \in \Omega_+. \tag{2.7}$$

(e) There is a constant $r_0 > 0$ independent of $c \geq 1$ and $\theta \in \Omega_+$ such that $\{z \in \boldsymbol{C}; \operatorname{Im} z < -r_0\} \subset \rho(L_-(c,\theta))$, the resolvent set of $L_-(c,\theta)$.
(f) Let $c \geq 1$ and $\operatorname{Im} z < -r_0$. Then the resolvent $(L_-(c,\theta) - z)^{-1}$ converges to $(L_-(c) - z)^{-1}$ strongly as $\theta \to 0$:

$$s - \lim_{\Omega_+ \ni \theta \to 0} (L_-(c,\theta) - z)^{-1} = (L_-(c) - z)^{-1}. \tag{2.8}$$

Remarks. (i) Since $L_+(c,\theta)$ has compact resolvent, it has a purely discrete spectrum. Moreover, according to (b), with the help of the standard argument by Aguilar and Combes [2] we see that the discrete spectrum is independent of $\theta \in \Omega$ for each $c \geq 1$. In particular, it coincides with that of $L_+(c)$. On the other hand, the above argument is valid for $L_-(c,\theta)$ with only $\theta \in \Omega_+$. Actually, the structure of spectrum of $L_-(c,\theta)$ for $\operatorname{Im}\theta > 0$ and that of $L_-(c)$ are quite different (see Section 4). Thus it seems that the analysis of $L_-(c,\theta)$ for $\operatorname{Im}\theta > 0$ does not contribute to that of $L_-(c)$. But as shown in Section 3 the spectral property of $L_-(c,\theta)$ for $\operatorname{Im}\theta > 0$ helps us to determine that of $L_-(c)$ through the relation (2.8).
(ii) The result of (e) is not optimal. Indeed, combining this proposition with a result in the next section, we can prove that the whole lower half plane $\{ z \in \boldsymbol{C} ; \operatorname{Im} z < 0 \}$ is contained in the resolvent set of $L_-(c,\theta)$ for all $\theta \in \Omega_+$ and all $c \geq 1$.

§ 3. Analytic Family

In this section we study an abstract theory, which is useful to investigate the spectral properties of Dirac operators and relativistic Schrödinger operators with dilation analytic potentials (see the next section).

We will show that self-adjoint operators defined as a boundary value of some type of analytic family of closed operators can be classified into two types by following the idea of Aguilar and Combes [2] (see also [5, 17]).

Let T be a self-adjoint operator and $\{T(\theta)\}_{\theta \in \boldsymbol{\Omega}_+}$ a family of closed operators in a Hilbert space $\mathcal{H}$, where $\boldsymbol{\Omega}_+ = \{\theta \in \boldsymbol{C}; 0 < \operatorname{Im}\theta < a\}$ for some $a > 0$. We assume the following:

(**A1**) $\{T(\theta)\}_{\theta \in \boldsymbol{\Omega}_+}$ is an analytic family in the sense of Kato (see [13], [17]).

(**A2**) Each $T(\theta)$ has compact resolvent.

(**A3**) There is a strongly continuous one-parameter unitary group $\{U(t)\}_{t \in R}$ such that

$$U(t)T(\theta)U(t)^* = T(\theta + t) \tag{3.1}$$

for $t \in \boldsymbol{R}$ and $\theta \in \boldsymbol{\Omega}_+$.

By **(A1)** and **(A2)** each $T(\theta)$ has purely discrete spectrum and the eigenvalues are analytic functions or branches of one or several analytic functions. On the other hand **(A3)** implies that the eigenvalues of $T(\theta)$ are invariant when θ is changed to $\theta+t$ if t is real. Thus, each eigenvalue is a constant function of $\theta \in \boldsymbol{\Omega}_+$ (see e.g. [2, 17]). Therefore we obtain

Proposition 3.1. *Suppose* **(A1)**$\sim$**(A3)**. *Then there is a discrete set* Σ *in* $\boldsymbol{C}$ *such that* $\sigma(T(\theta)) = \sigma_{\mathrm{d}}(T(\theta)) = \Sigma$ *for all* $\theta \in \boldsymbol{\Omega}_+$.

Let $\boldsymbol{C}_\pm := \{z \in \boldsymbol{C}\,; \pm \operatorname{Im} z > 0\}$. The self-adjoint operator T is related to the analytic family $\{T(\theta)\}_{\theta\in\boldsymbol{\Omega}_+}$ in the following sense.

(**A4**) There is a nonempty open set $\mathcal{O} \subset \boldsymbol{C}_- \setminus \Sigma$ such that

$$w - \lim_{t\to+0} (T(it) - z)^{-1} = (T-z)^{-1} \quad \text{(weakly)}$$

for each $z \in \mathcal{O}$.

Remark. For each $s \in \boldsymbol{R}$ define a self-adjoint operator $T(s)$ by $T(s) := U(s)TU(s)^*$. Then $T(0) = T$ and

$$\begin{aligned} w - \lim_{t\to+0} (T(s+it) - z)^{-1} &= w - \lim_{t\to+0} U(s)(T(it)-z)^{-1}U(s)^* \\ &= U(s)(T-z)^{-1}U(s)^* = (T(s)-z)^{-1} \end{aligned}$$

by **(A3)**. Thus the self-adjoint operators $T(s)$, $s \in \boldsymbol{R}$, are regarded as *boundary values* of the operator-valued function $T(\theta)$ defined on $\boldsymbol{\Omega}_+$. The following proposition shows that the eigenvalues of $T(\theta)$ (if exist) are located in the closed upper half plane $\overline{\boldsymbol{C}_+} = \{\, z \ \in \boldsymbol{C}\ : \operatorname{Im} z \geq 0\,\}$.

Proposition 3.2. *Suppose* **(A1)**$\sim$**(A4)**. *Then* $\Sigma \subset \overline{\boldsymbol{C}_+}$.

Proof. Let A be the generator of $U(t)$, i.e. $U(t) = e^{-itA}$, and let $\mathbf{P}(\cdot)$ be the spectral projection for A. Then $\mathcal{D} := \{u \in \mathcal{H};\ \mathbf{P}([-M,M])u = u \text{ for some } M\}$ is dense in $\mathcal{H}$, and $e^{-iwA}u$ is an entire function of w for each $u \in \mathcal{D}$. Moreover, $e^{-iwA}\mathcal{D} = \mathcal{D}$ for each $w \in \boldsymbol{C}$. We fix $z \in \mathcal{O}$ and f, g in $\mathcal{D}$ and write $f_\theta = U(-\theta)f$, etc., for simplicity. Then we have the identity by **(A3)**:

$$((T(\theta+t)-z)^{-1}f, g) = ((T(\theta)-z)^{-1}f_t, g_t) \tag{3.2}$$

for all $\theta \in \boldsymbol{\Omega}_+$ and all $t \in \boldsymbol{R}$, and by the use of analyticity of both sides in t we get

$$((T(\theta+\eta)-z)^{-1}f, g) = ((T(\theta)-z)^{-1}f_\eta, g_{\overline{\eta}}) \tag{3.3}$$

if $\theta \in \boldsymbol{\Omega}_+$, $\theta + \eta \in \boldsymbol{\Omega}_+$. Therefore, by **(A4)** we have

$$
\begin{aligned}
((T-z)^{-1}f, g) &= \lim_{t \to +0} ((T(it)-z)^{-1}f, g) \\
&= \lim_{t \to +0} ((T(\theta)-z)^{-1} f_{it-\theta}, g_{\overline{it-\theta}}) \\
&= ((T(\theta)-z)^{-1} f_{-\theta}, g_{-\overline{\theta}}).
\end{aligned} \tag{3.4}
$$

Since $(T(\theta)-z)^{-1}$ and $(T-z)^{-1}$ are analytic in $\boldsymbol{C}_- \setminus \Sigma$ and in $\boldsymbol{C}_-$, respectively, the above equality holds for all $z \in \boldsymbol{C}_- \setminus \Sigma$. Since both $\{f_{-\theta}; f \in \mathcal{D}\}$ and $\{g_{-\overline{\theta}}; g \in \mathcal{D}\}$ are dense in $\mathcal{H}$, we see that $(T(\theta)-z)^{-1}$ is analytic in $z \in \boldsymbol{C}_-$, and so $\boldsymbol{C}_- \cap \Sigma = \phi$. □

The equality (3.4) is important in this section.

For $E \in \boldsymbol{R}$, let γ be a positively-oriented small circle $|z-E| = \varepsilon$ enclosing E with $\{ z \in \boldsymbol{C}\,; 0 < |z-E| \le \varepsilon \} \cap \Sigma = \phi$ and let

$$P_\theta(E) = -\frac{1}{2\pi i} \int_\gamma (T(\theta)-z)^{-1} dz.$$

Then this operator is the eigenprojection associated with $E \in \sigma_{\mathrm{d}}(T(\theta)) = \Sigma$ if $E \in \Sigma$ and $P_\theta(E) = 0$ otherwise. Moreover, for each $E \in \Sigma$ the projection-valued function $P_\theta(E)$ is analytic in $\theta \in \Omega_+$. In particular, the dimension of the range of $P_\theta(E)$ is independent of θ for each E.

The following is our main result in this section. Let $\mathbf{P}_s(\cdot)$ be the spectral projection of $T(s)$ for $s \in \boldsymbol{R}$.

Theorem 3.3. *Suppose* **(A1)**$\sim$**(A4)**. *Then*
(a) $\sigma_{\mathrm{d}}(T(\theta)) \cap \boldsymbol{R} = \sigma_{\mathrm{p}}(T)$ *for all* $\theta \in \boldsymbol{\Omega}_+$. *Moreover, for each* $E \in \sigma_{\mathrm{p}}(T)$ *and* $s \in \boldsymbol{R}$, *we have*

$$\lim_{\boldsymbol{\Omega}_+ \ni \theta \to s} ||P_\theta(E) - \mathbf{P}_s(\{E\})|| = 0. \tag{3.5}$$

In particular, the eigenvalues of T *are discrete and each eigenvalue has finite multiplicity.*
(b) Either

(I) H *has a purely discrete spectrum, i.e.* $\sigma(T) = \sigma_{\mathrm{d}}(T)$

or

(II) $\sigma(T) = \boldsymbol{R}$, $\sigma_{\mathrm{sing}}(T) = \phi$

holds. In particular, we have $\sigma(T) \setminus \sigma_{\mathrm{p}}(T) \subset \sigma_{\mathrm{ac}}(T)$ *in the case of (II).*
(c) If $\Sigma \cap (\boldsymbol{C} \setminus \boldsymbol{R}) \neq \phi$ *or* $\Sigma = \phi$, *then the case (II) holds. Thus,* $\Sigma = \sigma_{\mathrm{p}}(T)$ *in the case of (I).*
(d) Suppose the case (I) above holds and fix $z \notin \sigma_{\mathrm{d}}(T)$. *Then the resolvent* $(T(\theta)-z)^{-1}$ *has an analytic continuation of* θ *from* $\boldsymbol{\Omega}_+$ *to* $\boldsymbol{\Omega} := \{ \theta \in \boldsymbol{C}\,; |\operatorname{Im}\theta| < a \}$.

Remarks. (i) Hereafter we call T the *boundary value* of the analytic family $\{T(\theta)\}_{\theta\in\Omega_+}$ and each element of Σ a *resonance* of T, when $\{T(\theta)\}_{\theta\in\Omega_+}$ is given.
(ii) We discuss two simple and typical examples of T from the point of view of this theorem, though the detail is omitted. Let us consider two Schrödinger operators

$$H_+ := -\Delta + |x|^2, \quad H_- := -\Delta - |x|^2 \quad \text{in } L^2(\boldsymbol{R}^3).$$

It is known that they are essentially self-adjoint on $\mathcal{S}$ (see e.g. [10], [16]). We denote by the same notation $H_\pm$ their self-adjoint extensions. The operator H_+ is known as the harmonic oscillator and has a purely discrete spectrum. Define

$$H_+(\theta) := -e^{-2\theta}\Delta + e^{2\theta}|x|^2, \quad |\operatorname{Im}\theta| < \pi/4.$$

Then we can prove that each $H_+(\theta)$ is a closed operator with domain $D(H_+(\theta)) = D(\Delta) \cap D(|x|^2)$ and has compact resolvent and that the family $\{H_+(\theta)\}$ forms an analytic family of type (A). Moreover, we can see that H_+ is a boundary value of the analytic family restricted to $0 < \operatorname{Im}\theta < \pi/4$. On the other hand, $-H_-$ can be proved to be a boundary value of the analytic family of operators

$$\begin{aligned} H_-(\theta) :&= e^{-2\theta}\Delta + e^{2\theta}|x|^2 = e^{\frac{\pi}{2}i}(-e^{-2(\theta-\frac{\pi}{4}i)}\Delta + e^{2(\theta-\frac{\pi}{4}i)}|x|^2) \\ &= iH_+(\theta - \frac{\pi}{4}i) \end{aligned}$$

for $0 < \operatorname{Im}\theta < \pi/4$. Since $H_+(\theta)$ is an analytic family of type (A) for $-\pi/4 < \operatorname{Im}\theta < \pi/4$, the theorem implies that H_+ is of type (I) in (b). In particular, $\sigma(H_+(\theta)) = \sigma_{\mathrm{d}}(H_+(\theta)) = \sigma_{\mathrm{d}}(H_+) = \{\lambda_{lmn}\,;\, l,\, m,\, n = 0, 1, 2, \cdots\}$, where

$$\lambda_{lmn} = (2l+1) + (2m+1) + (2n+1) = 2(l+m+n) + 3. \tag{3.6}$$

Furthermore, by virtue of this fact we know that

$$\sigma(H_-(\theta)) = \sigma_{\mathrm{d}}(H_-(\theta)) = \{i\lambda_{lmn}\,;\, l,\, m,\, n = 0, 1, 2, \cdots\},$$

i.e., H_- has nonreal resonances. Thus, it follows by (c) that H_- is of type (II) and has purely absolutely continuous spectrum with $\sigma(H_-) = \boldsymbol{R}$.

§ 4. Resonances

We study spectral properties of the relativistic Schrödingr operators $L_\pm(c)$ and the operator $L(c)$, which is unitarily equivalent to $H(c) - mc^2$, with the help of Theorem 3.3 in Section 3. For the relativistic Schrödinger operators $L_\pm(c)$, we apply the theorem to them as follows; $\mathcal{H} = L^2(\boldsymbol{R}^3)$, $\boldsymbol{\Omega}_+ = \Omega_+$, $T = L_\pm(c)$, $T(\theta) = L_\pm(c, \theta)$, $U(t) = \mathcal{U}(t)$.

Proposition 4.1. *(a)* $\Sigma_+(c) := \sigma_{\mathrm{d}}(L_+(c,\theta))$ *is independent of* $\theta \in \Omega$ *and coincides with* $\sigma_{\mathrm{p}}(L_+(c))$.

(b) $L_+(c)$ *has a purely discrete spectrum.*

Proposition 4.2. *(a)* $\Sigma_-(c) := \sigma_d(L_-(c,\theta))$ *is independent of* $\theta \in \Omega_+$ *and satisfies*

$$\Sigma_-(c) \subset \overline{\boldsymbol{C}_+}, \quad \Sigma_-(c) \cap \boldsymbol{R} = \sigma_{\mathrm{p}}(L_-(c)).$$

(b) The set $\sigma_{\mathrm{p}}(L_-(c))$ *of eigenvalues of* $L_-(c)$ *(if exist) is a bounded discrete set, and moreover, the multiplicity of each eigenvalue is finite.*
(c) $\sigma(L_-(c)) = \boldsymbol{R}$ *and* $\sigma_{\mathrm{sing}}(L_-(c)) = \phi$. *In particular,* $\sigma(L_-(c)) \setminus \sigma_{\mathrm{p}}(L_-(c)) = \sigma_{\mathrm{ac}}(L_-(c))$.

Outline of the proof of Proposition 4.2. We apply Theorem 3.3 to $L_-(c)$. By investigating the numerical range of $L_-(c,\theta)$, we can see that there exists a constant $K > 0$ such that $\Sigma_-(c) \cap ((-\infty,-K] \cup [K,\infty)) = \phi$. Hence the set of eigenvalues (if exist) is bounded and each multiplicty is finite. Since the dimension of $L^2(\boldsymbol{R}^3)$ is infinite, this implies that $L_-(c)$ should not be type (I). Consequently, we have proved the proposition.

Remark. Each element of $\Sigma_\pm(c)$ is called a *resonance* of $L_\pm(c)$, respectively. Hence $L_+(c)$ has no nonreal resonance. On the other hand, $L_-(c)$ may have a nonreal resonance (see the remark below Proposition 5.5).

We define

$$L(c,\theta) = L_1(c,\theta) + W(c,\theta), \tag{4.1}$$

where

$$L_1(c,\theta) := \begin{pmatrix} L_+(c,\theta)I_2 & 0 \\ 0 & L_-(c,\theta)I_2 \end{pmatrix}$$

and

$$W(c,\theta) := U_c(e^{-\theta}D)V_\theta(x)U_c(e^{-\theta}D)^{-1} - V_\theta(x).$$

Proposition 4.3.

(a) $L(c,\theta)$ *defined on* $\mathcal{S}^4$ *is closable and its closure (denoted by the same notation* $L(c,\theta)$*) has domain* $D(L(c,\theta)) = (D_M)^4$ *for* $\theta \in \Omega_+$.

(b) The resolvent set of $L(c,\theta)$, $\theta \in \Omega_+$, *is not empty and its resolvent* $(L(c,\theta)-z)^{-1}$ *is compact. Moreover,* $L(c,\theta)$ *is an analytic family of type (A) in* $\theta \in \Omega_+$.

(c) There is a large constant $T > 0$ such that

$$s - \lim_{\Omega_+ \ni \theta \to 0} (L(c,\theta) - z)^{-1} = (L(c) - z)^{-1} \tag{4.2}$$

for all $z \in \boldsymbol{C}$ with $\operatorname{Re} z < -T$ and $\operatorname{Im} z < -T$.

In the proof of (c) we introduce

$$\tilde{H}(c,\theta) := ce^{-\theta}\alpha \cdot D + \beta mc^2 - mc^2 + V_\theta(x)$$

for $\theta \in \Omega_+$. Then

$$L(c,\theta) = U_c(e^\theta D)\tilde{H}(c,\theta)U_c(e^\theta D)^{-1}. \tag{4.3}$$

This proposition shows that $L(c)$ is the boundary value of the analytic family $\{ L(c,\theta) \}_{\theta\in\Omega_+}$. Thus by Theorem 3.3 we have

Proposition 4.4. *(a) The set $\Sigma(c) := \sigma_{\mathrm{d}}(L(c,\theta))$ is independent of $\theta \in \Omega_+$ and satisfies*

$$\Sigma(c) \subset \overline{\boldsymbol{C}_+}, \quad \Sigma(c) \cap \boldsymbol{R} = \sigma_{\mathrm{p}}(L(c)).$$

Moreover, the multiplicity of each eigenvalue (if exists) of $L(c)$ is finite.
(b) $\sigma(L(c)) = \boldsymbol{R}$ and $\sigma_{\mathrm{sing}}(L(c)) = \phi$. In particular, $\sigma(L(c)) \setminus \sigma_{\mathrm{p}}(L(c)) = \sigma_{\mathrm{ac}}(L(c))$.

Proof. It is known that $\sigma(H(c) - mc^2) = \sigma(L(c)) = \boldsymbol{R}$ (see e.g.[18]). Thus, by applying Theorem 3.3 to $L(c)$, we see that $L(c)$ is of type (II) in (b) of Theorem 3.3, and thus obtain the desired result. □

Remark. We call an element of $\Sigma(c)$ a *resonance* of $L(c)$. On the other hand, in [1] eigenvalues of $\tilde{H}(c,\theta)$, which independent $\theta \in \Omega_+$, are called resonances of $H(c) - mc^2$. Then (4.3) shows that the the resonances of $H(c) - mc^2$ coincide with those of $L(c)$.

Theorem 4.5. *(a) The resonances of the Dirac operator $H(c)$ are contained in the upper half plane $\overline{\boldsymbol{C}}_+$, and the real resonances coincide with the eigenvalues of $H(c)$. In particular, the set of eigenvalues (if exist) is a discrete set. Moreover, the multiplicity of each eigenvalue is finite.*
(b) $\sigma(H(c)) = \boldsymbol{R}$ and $\sigma_{\mathrm{sing}}(H(c)) = \phi$. In particular, $\sigma(H(c)) \setminus \sigma_{\mathrm{p}}(H(c)) \subset \sigma_{\mathrm{ac}}(H(c))$.

Remarks. (i) It is a natural question whether there is a resonance of $H(c)$ or not. In the next section we will show that there are resonances near the eigenvalue of the Schrödinger operator $S = -(2m)^{-1}\Delta + V(x)$ if c is large enough.

(ii) We can see that the resonances of $L(c)$ are contained in a half-plane having no intersection with $(-\infty, -K)$ for large $K > 0$. Hence, the set of the eigenvalues of $H(c)$ (if exist) is bounded from below.

(iii) If the potential $V(x)$ satisfies some mild condition, then the Dirac operator $H(c)$ has a purely absolutely continuous spectrum and $\sigma(H(c)) = \boldsymbol{R}$ [14, 23].

§ 5. Nonrelativistic Limit

In this section we shall show that there exist resonances of $L(c)$ (and so $H(c) - mc^2$) near each eigenvalue of S if c is sufficiently large. Furthermore, we study the nonrelativistic limit of the spectral projection of $H(c) - mc^2$ at the end of this section.

We fix a constant $L > 0$ and an interval $I \subset \boldsymbol{R}$ with $I \cap \sigma(S) \neq \phi$ and define

$$\begin{aligned} \mathcal{O} &:= \{\, z \in \boldsymbol{C}\,; \operatorname{Re} z \in I,\ |\operatorname{Im} z| < L \,\}, \\ \mathcal{O}^+ &:= \{\, z \in \boldsymbol{C}\,; \operatorname{Re} z \in I,\ 0 < \operatorname{Im} z < L \,\} \end{aligned}$$

For small $\varepsilon > 0$ we also define

$$\begin{aligned} \mathcal{O}_\varepsilon &:= \mathcal{O} \setminus \cup_{j=1}^N B_\varepsilon(\lambda_j), \\ \mathcal{O}_\varepsilon^+ &:= \mathcal{O}^+ \setminus \cup_{j=1}^N B_\varepsilon(\lambda_j), \end{aligned}$$

where $I \cap \sigma_d(S) = \{\, \lambda_j \,\}_{j=1}^N$ and $B_\varepsilon(\lambda) = \{z \in \boldsymbol{C}; |z - \lambda| \le \varepsilon\}$. Let m_j be the multiplicity of the eigenvalue λ_j of S.

Proposition 5.1. *Fix $\theta \in \Omega_+$. There is a constant $c_0 > 0$ and $K > 0$ such that $\mathcal{O} \subset \rho(L_-(c,\theta))$ and*

$$||(L_-(c,\theta) - z)^{-1}|| \le Kc^{-2} \tag{5.1}$$

for all $c > c_0$ and all $z \in \mathcal{O}$.

Let $S(\theta) := -(2m)^{-1} e^{-2\theta} \Delta + V_\theta(x)$, $\theta \in \Omega$.

Proposition 5.2. *(a) $S(\theta)$ defined on $\mathcal{S}$ is closable and its closure (denoted by the same notation $S(\theta)$) has the domain $D(S(\theta)) = D(-\Delta) \cap L^2_M(\boldsymbol{R}^3)$.*
(b) The resolvent set of $S(\theta)$ is not empty and its resolvent is compact.
(c) $\{\, S(\theta) \,\}_{\theta \in \Omega}$ is an analytic family of type (A).
(d) The spectrum of $S(\theta)$ is independent of θ and consists of only a discrete spectrum which coincides with that of S.

Proposition 5.3. *Let G be a compact set in $\rho(S)$ and fix $\theta \in \Omega$. Then there are constants $c_0 > 0$ and $K > 0$ such that $G \subset \rho(L_+(c,\theta))$ for $c \ge c_0$ and*

$$\sup_{z \in G} ||(L_+(c,\theta) - z)^{-1} - (S(\theta) - z)^{-1}|| \le Kc^{-2}$$

for $c \ge c_0$.

This result implies that for each eigenvalue λ (with multiplicity n) of S there exists n eigenvalues (counting multiplicity) $\lambda_j(c)$, $j = 1, \ldots, n$, of $L_+(c)$ near λ for large c and $\lambda_j(c) \to \lambda$ as $c \to \infty$.

A similar argument is useful to prove the existence of nonreal resonances of $L_-(c)$. Let $\widetilde{S}(\theta) := (2m)^{-1}e^{-2\theta}\Delta + V_\theta(x)$, $\theta \in \Omega_+$.

Proposition 5.4. *(a) $\widetilde{S}(\theta)$ defined on $\mathcal{S}$ is closable and its closure (denoted by the same notation $\widetilde{S}(\theta)$) has the domain $D(\widetilde{S}(\theta)) = D(-\Delta) \cap L^2_M(\boldsymbol{R}^3)$.*
(b) The resolvent set of $\widetilde{S}(\theta)$ is not empty and its resolvent is compact.
(c) $\{\widetilde{S}(\theta)\}_{\theta \in \Omega_+}$ is an analytic family of type (A).
(d) $\widetilde{S}(\theta)$ has a purely discrete spectrum $\widetilde{\Sigma}$, which is independent of θ.

Moreover, we have

Proposition 5.5. *Let G be a compact set in $\boldsymbol{C} \setminus \widetilde{\Sigma}$ and fix $\theta \in \Omega_+$. Then there are constants $c_0 > 0$ and $K > 0$ such that $G \subset \rho(L_-(c,\theta) + 2mc^2)$ for $c \geq c_0$ and*

$$\sup_{z \in G} ||(L_-(c,\theta) + 2mc^2 - z)^{-1} - (\widetilde{S}(\theta) - z)^{-1}|| \leq Kc^{-2}$$

for $c \geq c_0$.

Remark. This proposition implies that if $\widetilde{\Sigma}$ has a nonreal element λ there exists a nonreal resonance $\lambda(c)$ of $L_-(c)$ for large c such that $\lambda(c) + 2mc^2 \to \lambda$ as $c \to \infty$. For example, if $V(x) = |x|^2$ and $m = 1/2$, then $\widetilde{S}(\theta)$ with $m = 1/2$ coincides with $H_-(\theta)$ in Section 3 and has nonreal eigenvalues $i\lambda_{lmn}$ (see (3.6)). Thus, in this case $L_-(c)$ has nonreal resonances for large c.

Proposition 5.6. *Fix $\theta \in \Omega_+$. Then for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that $\mathcal{O}_\varepsilon \subset \rho(L(c,\theta)) \cap \rho(S)$ for $c > c_\varepsilon$ and*

$$\lim_{c \to \infty} \sup_{z \in \mathcal{O}_\varepsilon} \| (L(c,\theta) - z)^{-1} - \begin{pmatrix} (S(\theta) - z)^{-1} I_2 & 0 \\ 0 & 0 \end{pmatrix} \| = 0. \tag{5.2}$$

The following is our main result on the nonrelativisitic limit. A similar result has already been obtained in [1].

Theorem 5.7. *For any small $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that $\mathcal{O}_\varepsilon \subset \rho(L(c,\theta))$, $\theta \in \Omega_+$, and there exist $2m_j$ eigenvalues of $L(c,\theta)$ (counting their algebraic multiplicities) in $B_\varepsilon(\lambda_j) \cap \widetilde{\mathcal{O}}^+$ for each $j = 1, \ldots, N$ if $c > c_\varepsilon$, where $\widetilde{\mathcal{O}}^+ = \{ z \in \boldsymbol{C}, ; \operatorname{Re} z \in I,\ 0 \leq \operatorname{Im} z < M \}$. That is to say, there is no resonance of $H(c) - mc^2$ in $\mathcal{O}_\varepsilon$ and there are $2m_j$ resonances of $H(c) - mc^2$ in $B_\varepsilon(\lambda_j) \cap \widetilde{\mathcal{O}}^+$ for each $j = 1, \ldots, N$, if c is sufficiently large.*

The following theorem implies that if the Dirac operator has no eigenvalue then there exist nonreal resonances.

Theorem 5.8. *Suppose $\sigma_{\mathrm{p}}(H(c)) = \phi$ for all large c. Then for any small $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that there is no resonance of $H(c) - mc^2$ in $\mathcal{O}_\varepsilon$ and there exist $2m_j$ nonreal resonances of $H(c) - mc^2$ in $B_\varepsilon(\lambda_j) \cap \mathcal{O}^+$ for each $j = 1, \ldots, N$ if $c > c_\varepsilon$.*

Finally, we state a result on the nonrelativistic limit of the spectral projection of the Dirac operator.

Proposition 5.9. *Let $I = [\,\alpha,\, \beta\,]$ be an interval such that $I \cap \sigma(S) = \{\, \lambda_0 \,\}$ with $\alpha < \lambda_0 < \beta$ or $I \cap \sigma(S) = \phi$. Then we have*

$$s - \lim_{c \to \infty} \mathbf{P}_{H(c)-mc^2}(I) f = \mathbf{P} f, \tag{5.3}$$

for each $f \in L^2(\boldsymbol{R}^3)^4$, where

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_S(\{\lambda_0\}) I_2 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.4}$$

if $I \cap \sigma(S) = \{\, \lambda_0 \,\}$ and $\mathbf{P} = 0$ if $I \cap \sigma(S) = \phi$, where $\mathbf{P}_A(\cdot)$ denotes the spectral projection of a self-adjoint operator A.

Remark. A similar result for a wide class of electromagnetic potentials has already been proved in [11], in which f is replaced by

$$\begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} f.$$

References

[1] Amour, L., Brummelhuis, R. and Nourrigat, J., Resonances of the Dirac Hamiltonian in the non relativistic limit, Ann. Henri Poincaré, **2**(2001), 583–603.
[2] Aguilar, J., and Combes, J. M., A class of analytic perturbations for one-body Schrödinger hamiltonians, Comm. Math. Phys., **22**(1971), 269–276.
[3] Ciricione R.J. and Chernoff, P.R., Dirac and Klein–Gordon equations : Convergence of solutions in the nonrelativistic limit, Comm. Math. Phys., **79**(1981), 33–46.
[4] Grigore, D.R., Nenciu, G and Purice, R., On the nonrelativistic limit of the Dirac hamiltonian, Ann. Inst. H. Poincaré, **51**(1989), 231–263.
[5] Herbst, I. W., Dilation analyticity in constant electric field, Comm. Math. Phys., **64**(1979), 279–298.

[6] Hislop, P.D. and Sigal, I. M., *Introduction to Spectral of Theory*, Springer, Berlin, 1996.
[7] Hunziker W., On the nonrelativistic limit of the Dirac theory, Comm. Math. Phys., **40**(1975), 215–222.
[8] Ichinose, W., On essential selfadjointness of the relativistic Hamiltonian of a spinless particle in a negative scalar potential, Ann. Inst. H. Poincaré, **60**(1994), 241–252.
[9] Ichinose, T. and Ichinose, W., On the essential self-adjointness of the relativistic Hamiltonian with a negative scalar potential, Rev. Math. Phys., **7**(1995), 709–721.
[10] Ikebe, T. and Kato, T., Uniqueness of the self-adjoint extension of singular elliptic differential operators, Arch. Rational Mech. Anal., **9**(1962), 77–92.
[11] Ito, H. T. and Yamada, O., A note on the nonrelativistic limit of Dirac operators and spectral concentration, Proceeding of the Japan Academy, **81**(2005), 157–161.
[12] Ito, H. T. and Yamada, O., Relativistic Hamiltonians with potentials diverging at infinity, preprint.
[13] Kato, T,, *Perturbation theory for linear operators*, Springer-Verlag, 1996.
[14] Kalf, H., Ōkaji, T. and Yamada, O., Absence of eigenvalues of Dirac operators with potentials diverging at infinity, Math. Nachr., **259**(2003), 19–41.
[15] Reed, M. and B. Simon, *Methods of Mathematical Physics, I : Functional Analysis*, Academic Press, New York, 1972.
[16] Reed, M. and B. Simon, *Methods of Mathematical Physics, II : Fourier Analysis, Self-Adjointness*, 1975. Academic Press, New York, 1972.
[17] Reed, M. and B. Simon, *Methods of Mathematical Physics, IV : Analysis of Operators*, Academic Press, London (1978).
[18] Thaller, B., *The Dirac equation*, Springer, Berlin, 1992.
[19] Titchmarsh, E.C., A problem in relativistic quantum mechanics, Proc. London Math. Soc., **11**(1961), 169–192.
[20] Veselić, K., The nonrelativistic limit of the Dirac equation and the spectral concentration, Glasnik Mat., **4**(1969), 231–241.
[21] Yajima, K., Nonrelativistic limit of the Dirac theory, scattering theory, J. Fac. Sci. Univ. of Tokyo, Sect. A1, **23**(1976), 517–523.
[22] Yamada, O., The nonrelativistic limit of modified wave operators for Dirac operators, Proc. Japan Acad., **59**(1983), 71–74.
[23] Yamada, O., On the spectrum of Dirac operators with the unbounded potential at infinity, Hokkaido Math. J., **26**(1997), 439–449.