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Schrödinger equations on scattering manifolds and microlocal singularities

By

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§ 1. Scattering theory on scattering manifold

We take a two-space approach to the scattering theory on a noncompact manifold called *scattering manifold*. Let M be a noncompact manifold with cylindrical ends with base manifold N;

$$M = M_0 \cup M_\infty : \quad M_0 \Subset M, \quad M_\infty \cong (0,\infty) \times N, \quad ((0,1) \times N \hookrightarrow M_0).$$

In what follows we denote $N = \partial M$, since N gives the topological boundary of M at infinity. We assume ∂M is a closed manifold. We put an asymptotically Euclidean metric on M. Let (r,θ) be local coordinates on $M_\infty \cong (0,\infty) \times \partial M$.

Definition 1.1. The Riemannian manifold (M, g^{sc}) with M as above is called scattering manifold if there exists a Riemannian metric g^∂ on ∂M such that

$$m = g^{\mathrm{sc}} - (dr^2 + r^2 g^\partial_{jk} d\theta^j d\theta^k)$$

satisfies for some $\mu > 0$

$$m = m^0(r,\theta)dr^2 + r m^1_j(r,\theta)(dr d\theta^j + d\theta^j dr) + r^2 m^2_{jk}(r,\theta) d\theta^j d\theta^k,$$
$$|\partial_r^j \partial_\theta^\alpha m^l(r,\theta)| \lesssim r^{-\mu-j}, \qquad (r,\theta) \in (1,\infty) \times \partial M, \quad l = 0,1,2.$$

Set

$$M_{\mathrm{fr}} = \mathbb{R} \times \partial M,$$

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and we develop the two-space scattering theory for

$$H_{\rm fr} = -\partial_r^2$$

on $\mathcal{H}_{\rm fr} = L^2(M_{\rm fr}, \sqrt{G_\partial}drd\theta)$, $G_\partial = \det(g_{jk}^\partial)$, and

$$H = -\triangle_{\rm sc} + V$$

on $\mathcal{H} = L^2(M, \sqrt{G_{\rm sc}}dx)$, $G_{\rm sc} = \det(g_{jk}^{\rm sc})$, where $\triangle_{\rm sc}$ is the Laplace-Beltrami operator on M. Note that $H_{\rm fr}$ is not derived from any Riemannian structure on $M_{\rm fr}$. As in the scattering theory on the Euclidean space, we have to put a short-range type assumption on H. For the potential V it would be natural to assume $V \in C^\infty(M;\mathbb{R})$ and

$$|\partial_r^j \partial_\theta^\alpha V(r,\theta)| \lesssim r^{-1-\mu-j} \quad \text{(subcoulomb).}$$

This is exactly what the condition $|\partial_x^\alpha V(x)| \lesssim \langle x\rangle^{-1-\mu-|\alpha|}$, $x \in \mathbb{R}^n$ implies for the polar coordinates on the Euclidean space. For the metric $g^{\rm sc}$ it automatically follows from the definition that the dual metric $g_{\rm sc}$ on T^*M is of the form

$$\begin{aligned} g_{\rm sc}^{jk} &= \partial_r \otimes \partial_r + \frac{1}{r^2} g_\partial^{jk} \partial_{\theta^j} \otimes \partial_{\theta^k} \\ &\quad + a_0 \partial_r \otimes \partial_r + \frac{1}{r} a_1^j (\partial_r \otimes \partial_{\theta^j} + \partial_{\theta^j} \otimes \partial_r) + \frac{1}{r^2} a_2^{jk} \partial_{\theta^j} \otimes \partial_{\theta^k} \end{aligned}$$

on T^*M_∞ with

$$|\partial_r^j \partial_\theta^\alpha a_l(r,\theta)| \lesssim r^{-\mu-j}, \quad l = 0,1,2. \tag{1.1}$$

We assume that $g^{\rm sc}$ is *radially short-range* in the sense that it satisfies, in addition,

$$|\partial_r^j \partial_\theta^\alpha a_0(r,\theta)| \lesssim r^{-1-\mu-j}. \tag{1.2}$$

In contrast to the potential condition, this radially short-range condition is a little weaker than the usual short-range condition on the Euclidean space that suggests $|\partial_r^j \partial_\theta^\alpha a_l(r,\theta)| \lesssim r^{-1-\mu-j}$ for $l = 0,1,2$. The reason why we can weaken the assumption is that the conservation of angular momentum is available.

Define the smooth cutoff $J\colon \mathcal{H}_{\rm fr} \to \mathcal{H}$ by

$$(Ju)(x) = \begin{cases} j(r(x))[G_\partial(\theta(x))/G_{\rm sc}(x)]^{1/4} u(r(x),\theta(x)), & \text{if } x \in M_\infty, \\ 0, & \text{if } x \notin M_\infty, \end{cases}$$

where $j \in C^\infty((0,\infty))$ is chosen to satisfy $j(r) = 1$ for $r \geq 2$ and $j(r) = 0$ for $r \leq 1$. Note that the factor $[G_\partial(\theta(x))/G_{\rm sc}(x)]^{1/4}$ makes J unitary on $(2,\infty) \times \partial M$.

Theorem 1.2 ([3]). *Let (M, g^{sc}) be a scattering manifold of radially short-range type, and V a smooth subcoulomb potential. Then the wave operators*

$$W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} J e^{-itH_{\mathrm{fr}}} \colon \mathcal{H}_{\mathrm{fr}} \to \mathcal{H},$$

exists, and they are partial isometries with initial spaces

$$\mathcal{H}_{\mathrm{fr},\pm} = \{u \in \mathcal{H}_{\mathrm{fr}};\ \operatorname{supp} \mathcal{F}_{\mathrm{fr}} u \subset \mathbb{R}_\pm \times \partial M\}, \quad \mathbb{R}_\pm = \{\rho \in \mathbb{R};\ \pm\rho \geq 0\},$$

respectively, where $\mathcal{F}_{\mathrm{fr}}$ is the Fourier transform in the radial direction:

$$(\mathcal{F}_{\mathrm{fr}} u)(\rho, \theta) = \int e^{-ir\rho} u(r, \theta)\, dr.$$

Moreover, $W_\pm$ are complete, i.e., $\operatorname{Ran} W_\pm = \mathcal{H}_{\mathrm{ac}}(H)$*. Hence the scattering operator $S = W_+^* W_-$ is unitary as $\mathcal{H}_{\mathrm{fr},-} \to \mathcal{H}_{\mathrm{fr},+}$.*

The proof of Theorem 1.2 is similar to the standard one on the Euclidean space except for small modifications; The existence of the wave operator follows from the Cook-Kuroda method; Apply the Mourre theory, using the conjugation operator

$$A = \frac{1}{2i}\Big(jr\frac{\partial}{\partial r} + \frac{\partial}{\partial r} jr + jr \frac{1}{2} \frac{\partial \log G_{\mathrm{sc}}}{\partial r}\Big),$$

and we obtain the limiting absorption principle; Then the abstract stationary theory ensures the completeness of the wave operators. In applying the abstract stationary theory, we encounter a difficulty that comes from the H_{fr}-unboundedness of the operator H, but it can be eluded by taking a smaller subspace than weighted L^2 space. We omit the detail here.

The restrictions

$$\mathcal{F}_{\mathrm{fr},\pm} = \mathcal{F}_{\mathrm{fr}}|_{\mathcal{H}_{\mathrm{fr},\pm}} \colon \mathcal{H}_{\mathrm{fr},\pm} \to L^2(\mathbb{R}_\pm, \mathcal{H}_\partial, \bar{d}\rho), \quad \mathcal{H}_\partial = L^2(\partial M, \sqrt{G_\partial} d\theta), \quad \bar{d}\rho = \frac{d\rho}{2\pi}$$

are unitary, and they give the *spectral representations* for $H_{\mathrm{fr}}|_{\mathcal{H}_{\mathrm{fr},\pm}}$:

$$(\mathcal{F}_{\mathrm{fr},\pm} H_{\mathrm{fr}} u)(\rho) = \rho^2 (\mathcal{F}_{\mathrm{fr},\pm} u)(\rho), \quad u \in D(H_{\mathrm{fr}}) \cap \mathcal{H}_{\mathrm{fr},\pm}.$$

Thus, from the general theory, we have the *S-matrix*:

Theorem 1.3. *For* a.e. $\rho \in \mathbb{R}_+$ *there exists a unitary operator, so-called S-matrix,*

$$\hat{S}(\rho) \colon \mathcal{H}_\partial \to \mathcal{H}_\partial$$

satisfying

$$(\mathcal{F}_{\mathrm{fr},+} S \mathcal{F}_{\mathrm{fr},-}^* f)(\rho) = \hat{S}(\rho) f(-\rho), \quad f \in L^2(\mathbb{R}_-, \mathcal{H}_\partial, \bar{d}\rho).$$

§ 2. Classical trajectories on scattering manifold

§ 2.1. Classical wave operators without potential term

Define the *(inverse of the) classical wave operators of finite time* by

$$w^*_{\mathrm{sc},t} = \exp(-tH_{K_{\mathrm{fr}}}) \circ J^*_{\mathrm{cl}} \circ \exp tH_{K_{\mathrm{sc}}},$$

where K_{fr} and K_{sc} are the classical Hamiltonians:

$$K_{\mathrm{fr}}(r,\rho,\theta,\omega) = \rho^2, \qquad\qquad K_{\mathrm{sc}}(x,\xi) = \sum_{j,k=1}^n g^{jk}_{\mathrm{sc}}(x)\xi_j\xi_k,$$

for $(r,\rho,\theta,\omega) \in T^*M_{\mathrm{fr}}$ and $(x,\xi) \in T^*M$, respectively, and

$$J_{\mathrm{cl}} \colon T^*M_{\mathrm{fr}} \supset T^*((0,\infty)\times\partial M) \overset{\cong}{\to} T^*M_\infty \subset T^*M, \quad J^*_{\mathrm{cl}} = (J_{\mathrm{cl}})^{-1}.$$

Then the limits

$$w^*_{\mathrm{sc},\pm} = \lim_{t\to\pm\infty} w^*_{\mathrm{sc},t} \colon T^*M \setminus \mathcal{T}_{\mathrm{sc},\pm} \to T^*M_{\mathrm{fr}} \setminus \mathcal{T}_{\mathrm{fr},\pm}$$

exist and are diffeomorphisms, where

$$\begin{aligned}\mathcal{T}_{\mathrm{sc},\pm} &= \{(x,\xi) \in T^*M;\ \{\exp tH_{K_{\mathrm{sc}}}(x,\xi);\ \pm t \ge 0\} \Subset T^*M\},\\ \mathcal{T}_{\mathrm{fr},\pm} &= \{(r,\rho,\theta,\omega) \in T^*M_{\mathrm{fr}};\ \pm\rho \le 0\}.\end{aligned}$$

We note that $\mathcal{T}_{\mathrm{sc},\pm}$ and $\mathcal{T}_{\mathrm{fr},\pm}$ are closed sets.

Since the Hamiltonians are homogeneous of degree 2 in the fiber variable, we have, λ^{-1} denoting the multiplication in fibers,

$$w^*_{\mathrm{sc},\lambda t}(x,\xi) = \lambda^{-1} w^*_{\mathrm{sc},t}(x,\lambda\xi), \quad \lambda > 0,$$

as long as they are well-defined. Thus we note that the classical wave operators coincide with the high energy limit of the classical wave operators of finite time:

$$w^*_{\mathrm{sc},\pm}(x,\xi) = \lim_{\lambda\to\infty} \lambda^{-1} w^*_{\mathrm{sc},t}(x,\lambda\xi), \quad \pm t > 0.$$

In particular the $w^*_{\mathrm{sc},\pm}$ is homogeneous in ξ, and the canonical relations

$$\mathcal{C}_{\mathrm{sc},\pm} = \{(x,\xi;r,\rho,\theta,\omega) \in (T^*M \setminus \mathcal{T}_{\mathrm{sc},\pm}) \times (T^*M_{\mathrm{fr}} \setminus \mathcal{T}_{\mathrm{fr},\pm});\ (r,\rho,\theta,\omega) = w^*_{\mathrm{sc},\pm}(x,\xi)\},$$

are conic.

§ 2.2. High energy limit of classical wave operators with sublinear potentials

We consider

$$w_t^* = \exp(-tH_{K_{\mathrm{fr}}}) \circ J_{\mathrm{cl}}^* \circ \exp tH_K, \qquad K(x,\xi) = \sum_{j,k=1}^{n} g_{\mathrm{sc}}^{jk}(x)\xi_j\xi_k + V,$$

here allowing the potential V to grow sublinearly:

$$|\partial_r^j \partial_\theta^\alpha V(r,\theta)| \lesssim r^{1-\mu-j}. \tag{2.1}$$

Since V may grow at infinity in every direction, $\lim_{t\to\pm\infty} w_t^*$ might not exist on any subset of T^*M except for the zero section 0. But the high energy limit exists:

Theorem 2.1. *Let (M, g^{sc}) be a scattering manifold of radially short-range type, and V a smooth sublinear potential. For any $(x_0, \xi^0) \in T^*M \setminus \mathcal{T}_{\mathrm{sc},\pm}$ and $\pm t > 0$ the following limits in the right-hand side converges in C^∞-topology, and the equalities hold:*

$$w_{\mathrm{sc},\pm}^*(x_0,\xi^0) = \lim_{\lambda\to\pm\infty} \lambda^{-1} w_t^*(x_0, \lambda\xi^0). \tag{2.2}$$

Moreover, if $\pm t > 0$ is fixed, then

$$w_{\mathrm{sc},\pm}^* = (r_{\mathrm{sc},\pm}, \rho^{\mathrm{sc},\pm}, \theta_{\mathrm{sc},\pm}, \omega^{\mathrm{sc},\pm}), \qquad w_t^* = (r_t, \rho^t, \theta_t, \omega^t)$$

satisfy locally in $(x_0, \xi^0/|\xi^0|) \in T^*M \setminus \mathcal{T}_{\mathrm{sc},\pm}$ and for large $|\xi^0|$

$$\begin{aligned}
|\partial_{x_0}^\alpha \partial_{\xi^0}^\beta (r_{\mathrm{sc},\pm} - r_t)| &\le C\langle \xi^0\rangle^{-\mu-|\alpha|},\\
|\partial_{x_0}^\alpha \partial_{\xi^0}^\beta (\rho^{\mathrm{sc},\pm} - \rho^t)| &\le C\langle \xi^0\rangle^{1-\mu-|\alpha|},\\
|\partial_{x_0}^\alpha \partial_{\xi^0}^\beta (\theta_{\mathrm{sc},\pm} - \theta_t)| &\le C\langle \xi^0\rangle^{-\mu-|\alpha|},\\
|\partial_{x_0}^\alpha \partial_{\xi^0}^\beta (\omega^{\mathrm{sc},\pm} - \omega^t)| &\le C\langle \xi^0\rangle^{1-\mu-|\alpha|}.
\end{aligned} \tag{2.3}$$

From the estimates 2.3 it follows that the canonical relation

$$\mathcal{C}_t = \{(x,\xi; r,\rho,\theta,\omega) \in (T^*M \setminus \mathcal{T}_{\mathrm{sc},\pm}) \times (T^*M_{\mathrm{fr}} \setminus \mathcal{T}_{\mathrm{fr},\pm});\ (r,\rho,\theta,\omega) = w_t^*(x,\xi)\}$$

is not necessarily conic but *asymptotically conic* with asymptotes $\mathcal{C}_{\mathrm{sc},\pm}$ for $\pm t > 0$, respectively.

§ 2.3. Classical wave operators at infinity

We study the asymptotics of the classical wave operator $w_{\mathrm{sc},\pm}$ (without potential term) at spatial infinity, that is,

$$\lim_{\lambda\to\infty} \lambda^{-1} w_{\mathrm{sc},\pm}^*(\lambda r, \rho, \theta, \lambda\omega), \tag{2.4}$$

where the multiplication λ^{-1} here acts on the (r,ω)-variables. Since the scattering metric approaches the underlying conic metric $g^{\mathrm{cn}} = dr^2 + r^2 g^{\partial}_{jk} d\theta^j d\theta^k$, the limits (2.4) would approach

$$w^*_{\mathrm{cn},\pm} = \lim_{t\to\pm\infty} \exp(-tH_{K_{\mathrm{fr}}}) \circ J^*_{\mathrm{cl}} \circ \exp tH_{K_{\mathrm{cn}}}, \quad K_{\mathrm{cn}}(x,\xi) = \sum g^{jk}_{\mathrm{cn}}(x)\xi_j\xi_k,$$

respectively. Precisely,

Theorem 2.2. *Let (M, g^{sc}) be a scattering manifold of radially short-range type. For any*

$$(r_0, \rho^0, \theta_0, \omega^0) \in \mathcal{U}_0 = \{(r,\rho,\theta,\omega) \in T^*M_\infty;\ r > 0,\ \omega \neq 0\},$$

the following limits in the right-hand side converge in C^∞-topology, and the equalities hold:

$$w^*_{\mathrm{cn},\pm}(r_0, \rho^0, \theta_0, \omega^0) = \lim_{\lambda\to\infty} \lambda^{-1} w^*_{\mathrm{sc},\pm}(\lambda r_0, \rho^0, \theta_0, \lambda\omega^0).$$

Moreover, if one denotes

$$w^*_{\mathrm{cn},\pm} = (r_{\mathrm{cn},\pm}, \rho^{\mathrm{cn},\pm}, \theta_{\mathrm{cn},\pm}, \omega^{\mathrm{cn},\pm}), \qquad w^*_{\mathrm{sc},\pm} = (r_{\mathrm{sc},\pm}, \rho^{\mathrm{sc},\pm}, \theta_{\mathrm{sc},\pm}, \omega^{\mathrm{sc},\pm}),$$

then, locally in $(r_0/|(r_0,\omega^0)|, \rho^0, \theta_0, \omega^0/|(r_0,\omega^0)|) \in \mathcal{U}_0$,

$$\begin{aligned}
|\partial^\alpha_{r_0}\partial^\beta_{\rho^0}\partial^\gamma_{\theta_0}\partial^\delta_{\omega^0}(r_{\mathrm{cn},\pm} - r_{\mathrm{sc},\pm})| &\leq C|(r_0,\omega^0)|^{1-\mu-|\alpha|-|\delta|},\\
|\partial^\alpha_{r_0}\partial^\beta_{\rho^0}\partial^\gamma_{\theta_0}\partial^\delta_{\omega^0}(\rho^{\mathrm{cn},\pm} - \rho^{\mathrm{sc},\pm})| &\leq C|(r_0,\omega^0)|^{-\mu-|\alpha|-|\delta|},\\
|\partial^\alpha_{r_0}\partial^\beta_{\rho^0}\partial^\gamma_{\theta_0}\partial^\delta_{\omega^0}(\theta_{\mathrm{cn},\pm} - \theta_{\mathrm{sc},\pm})| &\leq C|(r_0,\omega^0)|^{-\mu-|\alpha|-|\delta|},\\
|\partial^\alpha_{r_0}\partial^\beta_{\rho^0}\partial^\gamma_{\theta_0}\partial^\delta_{\omega^0}(\omega^{\mathrm{cn},\pm} - \omega^{\mathrm{sc},\pm})| &\leq C|(r_0,\omega^0)|^{1-\mu-|\alpha|-|\delta|}
\end{aligned}$$

hold for large $|(r_0,\omega^0)|$.

Note that $w^*_{\mathrm{cn},\pm}$ is explicitly computed and is a diffeomorphism as

$$\begin{aligned}
&\mathcal{U}_0 = \{(r,\rho,\theta,\omega) \in T^*M_\infty;\ r > 0,\ \omega \neq 0\}\\
&\quad \to \mathcal{U}_{\mathrm{fr},\pm} = \{(r,\rho,\theta,\omega) \in T^*M_{\mathrm{fr}};\ \pm\rho > 0,\ \omega \neq 0\}.
\end{aligned}$$

We define the *classical scattering operators* analogously to the scattering operator $S = W^*_+ W_-$ by

$$\begin{aligned}
s_{\mathrm{sc}} &= w^*_{\mathrm{sc},+} \circ w_{\mathrm{sc},-} : T^*M_{\mathrm{fr}} \setminus \mathcal{T}_{\mathrm{fr},-} \to T^*M_{\mathrm{fr}} \setminus \mathcal{T}_{\mathrm{fr},+}, & w_{\mathrm{sc},-} &= (w^*_{\mathrm{sc},-})^{-1},\\
s_{\mathrm{cn}} &= w^*_{\mathrm{cn},+} \circ w_{\mathrm{cn},-} : \mathcal{U}_{\mathrm{fr},-} \to \mathcal{U}_{\mathrm{fr},+}, & w_{\mathrm{cn},-} &= (w^*_{\mathrm{cn},-})^{-1}.
\end{aligned}$$

Since we have the explicit formula:

$$s_{\mathrm{cn}}(r,\rho,\theta,\omega)=(-r,-\rho,\exp\pi H_{\sqrt{K_\partial}}(\theta,\omega)),\qquad K_\partial(\theta,\omega)=\sum_{j,k=1}^{n-1}g_\partial^{jk}(\theta)\omega_j\omega_k,$$

the canonical relation

$$\begin{aligned}\hat{\mathcal{D}}_{\mathrm{cn}}=\{&(\rho,r,\theta,\omega;\rho',r',\theta',\omega')\in(T^*\hat{M}_{\mathrm{fr},+}\setminus 0)\times(T^*\hat{M}_{\mathrm{fr},-}\setminus 0);\\&(r,\rho,\theta,\omega)=s_{\mathrm{cn}}(r',\rho',\theta',\omega'),\ \omega\neq 0,\ \omega'\neq 0\},\end{aligned}$$

where $\hat{M}_{\mathrm{fr},\pm}=\{\pm\rho>0\}\times\partial M$, is (r,ω)-conic. Theorem 2.2 implies that the canonical relation

$$\begin{aligned}\hat{\mathcal{D}}_{\mathrm{sc}}=\{&(\rho,r,\theta,\omega;\rho',r',\theta',\omega')\in(T^*\hat{M}_{\mathrm{fr},+}\setminus 0)\times(T^*\hat{M}_{\mathrm{fr},-}\setminus 0);\\&(r,\rho,\theta,\omega)=s_{\mathrm{sc}}(r',\rho',\theta',\omega'),\ \omega\neq 0,\ \omega'\neq 0\},\end{aligned}$$

is asymptotically (r,ω)-conic with asymptote $\hat{\mathcal{D}}_{\mathrm{cn}}$.

§ 3. Microlocal structure of the wave operators

In this and the following section we state the theorems concerning the microlocal structure of the wave operators and the S-matrix. The wave front set of $u\in\mathcal{S}'(\mathbb{R}^n)$ is characterized as follows: Let $(x_0,\xi^0)\in T^*\mathbb{R}^n\setminus 0\cong\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$. Then $(x_0,\xi^0)\notin$ WF (u) is equivalent to that for some $a\in C_0^\infty(T^*\mathbb{R}^n)$ we have

$$a(x_0,\xi^0)\neq 0,\qquad \|a^{\mathrm{w}}(x,hD_x)u\|_{L^2}=O(h^\infty)(=O(h^N)\ \text{ for any } N>0)\text{ as } h\downarrow 0,$$

where

$$a^{\mathrm{w}}(x,hD_x)u(x)=\frac{1}{(2\pi)^n}\int e^{i(x-y)\xi}a\Big(\frac{x+y}{2},\xi\Big)u(y)\,dyd\xi.$$

Hence the wave front set is the set of directions in the phase space $T^*\mathbb{R}^n$ in which the function u is decaying rapidly. Note that u is a function of *only* x, but the fact that the wave front set is well-defined means that we may consider u as a function of x *and* ξ modulo small errors for large $|\xi|$. The wave front set for a function on a manifold is characterized similarly by using the local coordinates.

We let $I_\rho^m(M,N;\mathcal{C})$ be the set of Fourier integral operators from functions on N to those on M that have amplitudes in $S_\rho^m=S_{\rho,1-\rho}^m$ and a canonical relation $\mathcal{C}$. Note that, in general, Fourier integral operators move the wave front set around according to the associated canonical relations: If $\mathcal{A}$ is a Fourier integral operator and $\mathcal{C}$ is the associated canonical relation, then we have

$$\mathrm{WF}\,(\mathcal{A}u)\subset\mathcal{C}\circ\mathrm{WF}\,(u)=\{(x,\xi);\ \exists(y,\eta)\ \text{ s.t. }(x,\xi,y,\eta)\in\mathcal{C}\}$$

Theorem 3.1 ([5], cf. [4]). *Suppose (M, g^{sc}) is a scattering manifold of radially short-range type, and V is a smooth sublinear potential, and let μ be as in (1.1), (1.2) and (2.1). Then for $u \in \mathcal{H}_{\mathrm{fr}}$ and $\pm t > 0$*

$$\mathrm{WF}\,(W_t u) \setminus \mathcal{T}_{\mathrm{sc},\pm} = (w^*_{\mathrm{sc},\pm})^{-1}[\mathrm{WF}\,(u) \setminus \mathcal{T}_{\mathrm{fr},\pm}], \tag{3.1}$$

respectively. In addition, if g^{sc} is nontrapping, that is, $\mathcal{T}_{\mathrm{sc},+} = \mathcal{T}_{\mathrm{sc},-} = 0$, then the wave operator W_t of finite time $\pm t > 0$ belongs to $I^0_1(M, M_{\mathrm{fr}}; \mathcal{C}_t) \cap I^0_\mu(M, M_{\mathrm{fr}}; \mathcal{C}_{\mathrm{sc},\pm})$ with

$$\mathcal{C}_t = \{(x, \xi; r, \rho, \theta, \omega) \in (T^*M \setminus 0) \times (T^*M_{\mathrm{fr}} \setminus 0);\ (r, \rho, \theta, \omega) = w^*_t(x, \xi)\},$$
$$\mathcal{C}_{\mathrm{sc},\pm} = \{(x, \xi; r, \rho, \theta, \omega) \in (T^*M \setminus 0) \times (T^*M_{\mathrm{fr}} \setminus 0);\ (r, \rho, \theta, \omega) = w^*_{\mathrm{sc},\pm}(x, \xi)\},$$

Remarks. 1. Theorem 3.1 is an analogue of the result by Hassell-Wunsch [1], and would actually be a refinement.

2. Since W_t is classically described by $w_t = (w^*_t)^{-1}$, it should be natural to use the canonical relation $\mathcal{C}_t$ to conclude $W_t \in I^0_1(M, M_{\mathrm{fr}}; \mathcal{C}_t)$. However, $\mathcal{C}_t$ is just *asymptotically* conic. If we are forced to use the *exactly* conic canonical relations $\mathcal{C}_{\mathrm{sc},\pm}$, which are the asymptotes of $\mathcal{C}_t$, then the amplitudes get worse and we have $W_t \in I^0_\mu(M, M_{\mathrm{fr}}; \mathcal{C}_{\mathrm{sc},\pm})$ for $\pm t > 0$, respectively.

3. If the potential V is subconstant, i.e.,

$$|\partial^j_r \partial^\alpha_\theta V(r, \theta)| \le C_{j\alpha} r^{-\mu - j},$$

then we have $W_t \in I^0_1(M, M_{\mathrm{fr}}; \mathcal{C}_t) \cap I^0_\mu(M, M_{\mathrm{fr}}; \mathcal{C}_{\mathrm{sc},\pm}) \cap I^0_1(M, M_{\mathrm{fr}}; \mathcal{C}_{\mathrm{sc},t})$, where

$$\mathcal{C}_{\mathrm{sc},t} = \{(x, \xi; r, \rho, \theta, \omega) \in (T^*M \setminus 0) \times (T^*M_{\mathrm{fr}} \setminus 0);\ (r, \rho, \theta, \omega) = w^*_{\mathrm{sc},t}(x, \xi)\}.$$

4. Even if g^{sc} is not nontrapping, W_t composed with a microlocal cut off of the trapping region would belong to $I^0_1(M, M_{\mathrm{fr}}; \mathcal{C}_t) \cap I^0_\mu(M, M_{\mathrm{fr}}; \mathcal{C}_{\mathrm{sc},\pm})$.

Combining Theorem 3.1 with the microlocal smoothing property of the Schrödinger propagator, we can restate the former part of Theorem 3.1 as follows:

Corollary 3.2 ([2]). *If g^{sc} is radially short-range and V is sublinear, then for any $(x_0, \xi^0) \in T^*M \setminus \mathcal{T}_{\mathrm{sc},\mp}$, $\pm t > 0$ and $u \in \mathcal{H}$*

$$(x_0, \xi^0) \in \mathrm{WF}\,(e^{-itH} u) \iff w^*_{\mathrm{sc},\mp}(x_0, \xi^0) \in \mathrm{WF}\,(e^{-itH_{\mathrm{fr}}} J^* u),$$

respectively.

$\mathrm{WF}\,(e^{-itH_{\mathrm{fr}}} J^* u)$ can be computed explicitly from u by using

$$\|a^{\mathrm{w}}(r, hD_r, \theta, hD_\theta) e^{-itH_{\mathrm{fr}}} J^* u\|_{\mathcal{H}_{\mathrm{fr}}} = \|a^{\mathrm{w}}(r + 2tD_r, hD_r, \theta, hD_\theta) J^* u\|_{\mathcal{H}_{\mathrm{fr}}},$$

and hence, Corollary 3.2 gives a characterization of $\mathrm{WF}\,(e^{-itH} u)$ in terms of the initial data u.

Theorem 3.1 holds also for $t = \pm\infty$.

Theorem 3.3 ([5], cf. [4]). *Suppose (M, g^{sc}) is a scattering manifold of radially short-range type and V is a smooth subcoulomb potential, and let $u \in \mathcal{H}_{\mathrm{fr}}$. Then*

$$\mathrm{WF}\,(W_{\pm}u) \setminus \mathcal{T}_{\mathrm{sc},\pm} = (w^*_{\mathrm{sc},\pm})^{-1}[\mathrm{WF}\,(u) \setminus \mathcal{T}_{\mathrm{fr},\pm}], \tag{3.2}$$

respectively. In addition, if g^{sc} is nontrapping, then the wave operators $W_{\pm}$ belong to $I^0_1(M, M_{\mathrm{fr}}; \mathcal{C}_{\mathrm{sc},\pm})$, respectively.

§ 4. Microlocal structure of the scattering matrix

The S-matrix $\hat{S}(\rho)$ is obtained by restricting the scattering operator to the fixed energy ρ^2. (Such a restriction is possible since we have the conservation of energy.) Fixing ρ, we lose freedom in the radial direction, and we can say that the S-matrix is the scattering operator S at infinity. Thus, instead of studying the wave front set of the S-matrix directly, we study the *scattering wave front set* of the scattering operator:

Definition 4.1. Let $u \in \mathcal{S}'(M_{\mathrm{fr}})$ be a tempered distribution. The *scattering wave front set* $\mathrm{WF}_{\mathrm{sc,fr}}(u) \subset T^*M_{\mathrm{fr}}$ is the complement of the set of $(r_0, \rho^0, \theta_0, \omega^0) \in T^*M_{\mathrm{fr}}$ satisfying for some $a \in C^\infty_0(T^*M_{\mathrm{fr}})$

$$a(r_0, \rho^0, \theta_0, \omega^0) \neq 0, \qquad \|a^{\mathrm{w}}(hr, D_r, \theta, hD_\theta)u\|_{\mathcal{H}_{\mathrm{fr}}} = O(h^\infty) \quad \text{as } h \downarrow 0. \tag{4.1}$$

Let $\chi \in C^\infty_0(\mathbb{R})$ be equal 1 near the origin and set

$$\eta(r, \theta, \omega) = \bigl[1 - \chi\bigl(r^{-2} g^{jk}_{\partial}(\theta)\omega_j\omega_k\bigr)\bigr]\bigl[1 - \chi\bigl(r^2 + g^{jk}_{\partial}(\theta)\omega_j\omega_k\bigr)\bigr]. \tag{4.2}$$

The first factor in the right-hand side equals 0 or 1 for $g^{jk}_{\partial}(\theta)\omega_j\omega_k < cr^2$ or $g^{jk}_{\partial}(\theta)\omega_j\omega_k > Cr^2$, respectively, while the second kills the singularity of the first near $(r, \omega) = 0$. Hence (4.2) is a cutoff function of an (r, ω)-conic neighborhood of $T^*M_{\mathrm{fr}} \setminus \mathcal{U}_{\mathrm{fr},\pm}$.

Theorem 4.2 ([6], cf. [4]). *Suppose (M, g^{sc}) is a scattering manifold of radially short-range type and V is a smooth subcoulomb potential. Let $S = W^*_+W_-$ be the scattering operator, and $s_{\mathrm{cn}} = w^*_{\mathrm{cn},+} \circ w_{\mathrm{cn},-} \colon \mathcal{U}_{\mathrm{fr},-} \to \mathcal{U}_{\mathrm{fr},+}$ be the classical scattering operator with respect to the underlying conic structure. Then, for any $u \in \mathcal{H}_{\mathrm{fr}}$,*

$$\mathrm{WF}_{\mathrm{sc,fr}}(Su) \cap \mathcal{U}_{\mathrm{fr},+} = s_{\mathrm{cn}}(\mathrm{WF}_{\mathrm{sc,fr}}(u) \cap \mathcal{U}_{\mathrm{fr},-}). \tag{4.3}$$

Moreover, set

$$\hat{S} = \mathcal{F}_{\mathrm{fr}} S \mathcal{F}^*_{\mathrm{fr}} \colon \hat{\mathcal{H}}_{\mathrm{fr},-} \to \hat{\mathcal{H}}_{\mathrm{fr},+}, \qquad s_{\mathrm{sc}} = w^*_{\mathrm{sc},+} \circ w_{\mathrm{sc},-}, \qquad \hat{M}_{\mathrm{fr},\pm} = \{\pm\rho > 0\} \times \partial M.$$

Then, for any microlocal cutoff function $\eta \in C^\infty(T^*\hat{M}_{\mathrm{fr}})$ given by (4.2), the operator (composed with restrictions)

$$\hat{S} \circ \eta^{\mathrm{w}}(D_\rho, \theta, D_\theta) \colon C^\infty_0(\hat{M}_{\mathrm{fr},-}) \to C^\infty(\hat{M}_{\mathrm{fr},+})$$

belongs to $I_1^0(\hat{M}_{\mathrm{fr},+}, \hat{M}_{\mathrm{fr},-}; \hat{\mathcal{D}}_{\mathrm{sc}}) \cap I_\mu^0(\hat{M}_{\mathrm{fr},+}, \hat{M}_{\mathrm{fr},-}; \hat{\mathcal{D}}_{\mathrm{cn}})$ *with*

$$\begin{aligned}\hat{\mathcal{D}}_{\mathrm{sc}} = \{&(\rho, r, \theta, \omega; \rho', r', \theta', \omega') \in (T^*\hat{M}_{\mathrm{fr},+} \setminus 0) \times (T^*\hat{M}_{\mathrm{fr},-} \setminus 0);\\ &(r, \rho, \theta, \omega) = s_{\mathrm{sc}}(r', \rho', \theta', \omega'),\ \omega \neq 0,\ \omega' \neq 0\},\\ \hat{\mathcal{D}}_{\mathrm{cn}} = \{&(\rho, r, \theta, \omega; \rho', r', \theta', \omega') \in (T^*\hat{M}_{\mathrm{fr},+} \setminus 0) \times (T^*\hat{M}_{\mathrm{fr},-} \setminus 0);\\ &(r, \rho, \theta, \omega) = s_{\mathrm{cn}}(r', \rho', \theta', \omega'),\ \omega \neq 0,\ \omega' \neq 0\}.\end{aligned}$$

Remark. As in Theorem 3.1, the canonical relation $\hat{\mathcal{D}}_{\mathrm{sc}}$ is just asymptotically conic, and, if we replace it with the exactly conic asymptote $\hat{\mathcal{D}}_{\mathrm{cn}}$, then the class of the amplitude gets worse. If g^{sc} is *short-range* in the sense that

$$|\partial_r^j \partial_\theta^\alpha a_l(r, \theta)| \leq C_{j\alpha} r^{-1-\mu-j} \quad l = 0, 1, 2, \tag{4.4}$$

then we obtain $\hat{S} \circ \eta^{\mathrm{w}}(D_\rho, \theta, D_\theta) \in I_1^0(\hat{M}_{\mathrm{fr},+}, \hat{M}_{\mathrm{fr},-}; \hat{\mathcal{D}}_{\mathrm{cn}})$.

Corollary 4.3. *The S-matrix* $\hat{S}(\rho)$ *belongs to* $I_\mu^0(\hat{M}_{\mathrm{fr},+}, \hat{M}_{\mathrm{fr},-}; \mathcal{D}_\partial)$ *with*

$$\mathcal{D}_\partial = \{(\theta, \omega; \theta', \omega') \in (T^*\partial M \setminus 0) \times (T^*\partial M \setminus 0);\ (\theta, \omega) = \exp \pi H_{\sqrt{K_\partial}}(\theta', \omega')\}$$

for a.e. $\rho \in \mathbb{R}_+$.

Remarks. 1. Using the Legendrian distributions, Melrose-Zworski [7] proved the above corollary.
2. If g^{sc} is short-range in the sense of (4.4), then $\hat{S}(\rho) \in I_1^0(\hat{M}_{\mathrm{fr},+}, \hat{M}_{\mathrm{fr},-}; \mathcal{D}_\partial)$.

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