RIMS Kôkyûroku Bessatsu
B16 (2010), 135–140

The spectrum of Schrödinger operators with periodic Aharonov-Bohm magnetic fields

By

Takuya MINE* and Yuji NOMURA**

Abstract

We shall consider the magnetic Schrödinger operators on $\mathbf{R}^2$. The magnetic field is the sum of a non-zero uniform magnetic field and periodic pointlike magnetic field on a lattice. In [1], we gave a sufficient condition for each Landau level to be a infinitely degenerated eigenvalue. This condition is also necessary for the lowest Landau level. Moreover, in the threshold case, the spectrum near the lowest Landau level is purely absolutely continuous. In this paper, we shall characterize the eigenfunctions corresponding to the second Landau level and show the absolutely continuity of the spectrum near the second Landau level in the threshold case.

§ 1. Introduction

We identify a vector $z = (x, y) \in \mathbf{R}^2$ with a complex number $z = x + iy$ in the sequel. We consider a magnetic Schrödinger operator on $\mathbf{R}^2$

$$\mathcal{L} = \left(\frac{1}{i}\nabla + \mathbf{a}\right)^2,$$

where $\mathbf{a} = (a_x, a_y)$ is the magnetic vector potential. We assume the magnetic field $\operatorname{rot}\mathbf{a} = \partial_x a_y - \partial_y a_x$ satisfies

$$\operatorname{rot}\mathbf{a}(z) = B + \sum_{\gamma\in\Gamma} 2\pi\alpha\delta_\gamma(z) \tag{1.1}$$

Received April 1, 2009. Revised July 23, 2009.
2000 Mathematics Subject Classification(s): 81Q10, 35P15, 35Q40, 47F05, 47N50
Key Words: *Schrödinger operator; periodic magnetic field; Aharonov Bohm magnetic field; Landau level*
Supported by JSPS grant Wakate B-20740093 and JSPS grant Kiban C-20540177
*Department of Comprehensive Sciences, Kyoto Institute of Technology, Matsugasaki, Sakyo-ku, Kyoto 606-8585, Japan.
e-mail: `mine@kit.ac.jp`
**Department of Computer Science, Graduate School of Science and Engineering, Ehime University, 3 Bunkyo-cho, Matsuyama, Ehime 790-8577, Japan.
e-mail: `nomura@cs.ehime-u.ac.jp`

in the distribution sense, where B is a positive constant, α is a real number satisfying $0 < \alpha < 1$, Γ is a lattice of rank 2 in $\mathbf{R}^2$ and δ_γ is the Dirac measure concentrated at $\gamma \in \Gamma$. The vector potential $\mathbf{a}$ satisfying (1.1) is constructed as follows:

$$
\begin{aligned}
&\mathbf{a}(z) = (\operatorname{Im}\phi(z), \operatorname{Re}\phi(z)), \\
&\phi(z) = \frac{B}{2}\overline{z} + \alpha\zeta(z),
\end{aligned} \tag{1.2}
$$

where $\zeta(z)$ is the Weierstrass ζ function associated with Γ:

$$
\zeta(z) = \frac{1}{z} + \sum_{\gamma \in \Gamma \setminus \{0\}} \left(\frac{1}{z-\gamma} + \frac{1}{\gamma} + \frac{z}{\gamma^2} \right).
$$

We define a operator L by

$$
Lu = \mathcal{L}u, \qquad u \in D(L) = C_0^\infty(\mathbf{R}^2 \setminus \Gamma).
$$

L is a positive symmetric operator on $L^2(\mathbf{R}^2)$. We denote the Friedrichs extention of L by H. It is shown that

$$
\begin{aligned}
D(H) = \{&u \in L^2(\mathbf{R}^2) \cap H^2_{\mathrm{loc}}(\mathbf{R}^2 \setminus \Gamma) \,|\, \mathcal{L}u \in L^2(\mathbf{R}^2), \\
&\lim_{z \to \gamma} |u(z)| = 0 \text{ for any } \gamma \in \Gamma\}.
\end{aligned} \tag{1.3}
$$

From the inequality $H \geq B$, it follows that $\sigma(H) \subset [B, \infty)$, where $\sigma(H)$ is the spectrum of H.

We prepare some notations. Let H_0 be the Schrödinger operator with constant magnetic field B. It is well-known that

$$
\sigma(H_0) = \{E_n \mid n = 1, 2, \cdots\},
$$

where

$$
E_n = (2n-1)B \tag{1.4}
$$

which is an infinitely degenerated eigenvalue and is called the nth Landau level. Let $\omega_1, \omega_2 \in \mathbf{R}^2$ be a basis of the lattice Γ, that is $\Gamma = \omega_1\mathbf{Z} \oplus \omega_2\mathbf{Z}$ and we assume $\operatorname{Im}(\omega_2/\omega_1) > 0$. The set Ω denotes a fundamental domain of Γ defined by

$$
\Omega = \left\{ z = s\omega_1 + t\omega_2 \mid -\frac{1}{2} \leq s < \frac{1}{2},\ -\frac{1}{2} \leq t < \frac{1}{2} \right\}.
$$

The condition

$$
\frac{B}{2\pi}|\Omega| + \alpha \in \mathbf{Q} \tag{1.5}
$$

is called the rational flux condition. The number of the left-hand side is the magnetic flux in a fundamental domain divided by 2π. For an interval $I \subset \mathbf{R}$, $P_I(H)$ denotes the spectral projection of H corresponding to I. We define an entire function σ by

$$\sigma(z) = z \prod_{\gamma \in \Gamma \setminus \{0\}} \left(\left(1 - \frac{z}{\gamma}\right) e^{\frac{z}{\gamma} + \frac{z^2}{2\gamma^2}} \right),$$

which satisfies the equation

$$\sigma'(z) = \zeta(z)\sigma(z). \tag{1.6}$$

Our results in [1] are the followings.

Theorem 1.1 ([1]). *Assume $B > 0$ and $0 < \alpha < 1$.*

(i) If $\frac{B}{2\pi}|\Omega| + \alpha > 1$, then $E_1 = B$ is an infinitely degenerated eigenvalue of H, and the eigenspace of H associated with the eigenvalue E_1 is

$$\left\{ e^{-\frac{B}{4}|z|^2} |\sigma(z)|^{-\alpha} \overline{\sigma(z) f(z)} \in L^2(\mathbf{R}^2) \mid f \text{ is an entire function} \right\}. \tag{1.7}$$

(ii) If $\frac{B}{2\pi}|\Omega| + \alpha < 1$, then E_1 is not an eigenvalue of H. If we additionally assume the rational flux condition (1.5), then there exists a positive number ε such that $\sigma(H) \subset [B + \varepsilon, \infty)$.

(iii) If $\frac{B}{2\pi}|\Omega| + \alpha = 1$, then E_1 is not an eigenvalue of H, and E_1 is the edge of the purely absolutely continuous spectrum, i.e. there exists a constant E such that $B < E \leq 3B$, $[B, E) \subset \sigma(H)$ and $\operatorname{Ran} P_{[B,E)}(H) \subset \mathcal{H}_{\mathrm{ac}}$, where $\mathcal{H}_{\mathrm{ac}}$ denotes the absolutely continuous subspace for the operator H.

Theorem 1.2 ([1]). *Assume $B > 0$ and $0 < \alpha < 1$. If $\frac{B}{2\pi}|\Omega| + \alpha > n$ for some positive integer n, then E_n is an infinitely degenerated eigenvalue of H.*

Following theorem is our main result in this paper.

Theorem 1.3. *Assume $B > 0$ and $0 < \alpha < 1$.*

(i) If $\frac{B}{2\pi}|\Omega| + \alpha > 2$, then E_2 is an infinitely degenerated eigenvalue of H, and the eigenspace of H associated with the eigenvalue E_2 is

$$\left\{ \mathcal{A}^\dagger \left(e^{-\frac{B}{4}|z|^2} |\sigma(z)|^{-\alpha} \overline{\sigma(z)^2 f(z)} \right) \in L^2(\mathbf{R}^2) \mid f \text{ is an entire function} \right\}, \tag{1.8}$$

where

$$\mathcal{A}^\dagger = -2\partial_{\overline{z}} + \overline{\phi(z)}.$$

(ii) If $\frac{B}{2\pi}|\Omega| + \alpha = 2$, then E_2 is not an eigenvalue of H and

$$E_2 \in \sigma_{\mathrm{ac}}(H),$$

where $\sigma_{\mathrm{ac}}(H)$ is the absolutely continuous spectrum of H.

§ 2. Outline of the proof of Theorem 1.1

Proposition 2.1. *We have*

$$Hu = E_1 u$$

for $u \in D(H)$ if and only if

$$u(z) = e^{-\frac{B}{4}|z|^2} |\sigma(z)|^{-\alpha} \overline{\sigma(z) f(z)} \tag{2.1}$$

for some entire function $f(z)$ and $u(z) \in L^2(\mathbf{R}^2)$.

Proof. We define an operator $\mathcal{A}$ by

$$\mathcal{A} = 2\partial_z + \phi(z),$$

where $\partial_z = \dfrac{1}{2}(\partial_x - i\partial_y)$. From (1.6), we have

$$\mathcal{A} = e^{-\frac{B}{4}|z|^2} |\sigma(z)|^{-\alpha} (2\partial_z) e^{\frac{B}{4}|z|^2} |\sigma(z)|^{\alpha}. \tag{2.2}$$

Then three operators $\mathcal{A}, \mathcal{A}^\dagger$ and $\mathcal{L}$ satisfy the following relations:

$$\begin{aligned} \mathcal{L} &= \mathcal{A}\mathcal{A}^\dagger + B \\ &= \mathcal{A}^\dagger \mathcal{A} - B. \end{aligned} \tag{2.3}$$

Since H is the Friedrichs extension of L, (2.3) implies that

$$((H - B)u, u) = (\mathcal{A}^\dagger \mathcal{A} u, u) = \|\mathcal{A} u\|^2 \tag{2.4}$$

for $u \in D(H)$. Therefore we have that $Hu = Bu$ if and only if

$$\mathcal{A} u = 0 \text{ in } \mathbf{C} \setminus \Gamma \tag{2.5}$$

for $u \in D(H)$. By (2.2), any solution of (2.5) is written as

$$u(z) = e^{-\frac{B}{4}|z|^2} |\sigma(z)|^{-\alpha} \overline{h(z)},$$

where $h(z)$ is a holomorphic function on $\mathbf{C} \setminus \Gamma$. From the boundary conditions $\lim_{z \to \gamma} |u(z)| = 0$ for any $\gamma \in \Gamma$, the function $h(z)$ must be factorized as $h(z) = \sigma(z) f(z)$ with an entire function $f(z)$. □

We denote $\eta_j = \zeta\left(\dfrac{\omega_j}{2}\right)$ for $j = 1, 2$. Then we have the Legendre relation:

$$\eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi i, \tag{2.6}$$

and

$$\sigma(z+\gamma) = (-1)^{m+n+mn} e^{\eta\left(z+\frac{\gamma}{2}\right)}\sigma(z), \tag{2.7}$$

where $\gamma = m\omega_1 + n\omega_2$ and $\eta = m\eta_1 + n\eta_2$ for $m, n \in \mathbf{Z}$ (see, e.g. [3]).

Outline of proof of Theorem 1.1. (i) Let

$$f(z) = P(z)e^{(\alpha-1)\mu z^2},$$

where $P(z)$ is an arbitrary polynomial and $\mu = \dfrac{i}{4|\Omega|}(\eta_1\overline{\omega_2} - \eta_2\overline{\omega_1})$. Let u be the function given by (2.1) with the above function f. By (2.6) and (2.7), we have

$$|u(z)| \leq Q(z)e^{d|z|^2},$$

where $Q(z)$ is some function of polynomial order and $d = -\dfrac{B}{4} + \dfrac{\pi(1-\alpha)}{2|\Omega|}$. Since d is negative by the assumption, the solution u belongs to $L^2(\mathbf{R}^2)$. Thus E_1 is an infinitely degenerated eigenvalue of H.

(ii) Let $f(z)$ be an arbitrary entire function which is not identically equal to 0 and let

$$u(z) = e^{-\frac{B}{4}|z|^2}|\sigma(z)|^{-\alpha}\overline{\sigma(z)f(z)}.$$

Putting $g(z) = f(z)e^{-(\alpha-1)\mu z^2}$, we have

$$|u(z)| \geq Ce^{d|z|^2}|g(z)| \tag{2.8}$$

for z satisfying $\mathrm{dist}(z, \Gamma) > \epsilon$ for some $\epsilon > 0$. Since d is positive by the assumption, the solution u dose not belong to $L^2(\mathbf{R}^2)$ by (2.8).

§ 3. Outline of the proof of Theorem 1.3

By using of the following Proposition 3.1 we can prove Theorem 1.3.

Proposition 3.1. *Assume* $\dfrac{B}{2\pi}|\Omega| + \alpha \geq 2$. *If* $Hu = E_2 u$ *for* $u \in D(H)$, *then we have*

$$\mathcal{A}^2 u = 0.$$

For $a > 0$ and $w \in \mathbf{C}$, let

$$B_a(w) = \{z \in \mathbf{C} \mid |z-w| < a\}$$

and

$$D_a = \mathbf{C} \setminus \cup_{\gamma\in\Gamma}\overline{B_a(\gamma)}.$$

For sufficiently small $\delta > \delta' > 0$ and $\epsilon > 0$, we define

$$D_\delta^n = D_\delta \cap Q_n \quad \text{and} \quad \widetilde{D_{\delta'}^n} = D_{\delta'} \cap Q_{n+\epsilon},$$

where

$$Q_r = \{z \in \mathbf{C} \mid |\operatorname{Re} z| < r + \frac{1}{2}, \quad |\operatorname{Im} z| < r + \frac{1}{2}\}.$$

Proposition 3.1 is derived from the following Lemmas.

Lemma 3.2. *If $\mathcal{L}f = E_2 f$ for $f \in H^2_{\mathrm{loc}}(\mathbf{R}^2 \setminus \Gamma)$, then we have*

$$\|\mathcal{A}^2 f\|_{L^2(D_\delta^n)} \leq C \|f\|_{L^2(\widetilde{D_{\delta'}^n})}$$

for some $C > 0$ which is independent of f and $n \in \mathbf{N}$.

Lemma 3.3. *Assume $\frac{B}{2\pi}|\Omega| + \alpha > 2$. If $Hu = E_2 u$ and $\mathcal{A}^2 u \neq 0$, then there exists a constant $c_1 > 0$ such that*

$$\|\mathcal{A}^2 u\|_{L^2(D_\delta^n)} \geq e^{c_1 n^2}$$

for any $n \in \mathbf{N}$.

Lemma 3.4. *Assume $\frac{B}{2\pi}|\Omega| + \alpha = 2$. If $Hu = E_2 u$ and $\mathcal{A}^2 u \neq 0$, then there exists a constant $c_2 > 0$ such that*

$$\|\mathcal{A}^2 u\|_{L^2(D_\delta^n)} \geq c_2 n$$

for any $n \in \mathbf{N}$.

References

[1] T. Mine and Y. Nomura, Periodic Aharonov-Bohm solenoids in a constant magnetic field, *Rev. Math. Phys.* **18**, no. 8, (2006) 913-934

[2] T. Mine and Y. Nomura, The spectrum of Schrödinger operators with periodic Aharonov-Bohm magnetic fields, in preparation.

[3] A. Hurwitz and R. Courant, *Vorlesungen uber Allgemeine Funktionentheorie und Elliptische Funktionen*, Springer, 1998.