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# Long time asymptotics of heat kernels for one dimensional elliptic operators with periodic coefficients

By

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## § 1. Results

This is a summary of the paper [11]. Let $L$ be an elliptic differential operator on $\mathbf{R}$:

$$L = -\frac{d}{dx}\big(a(x)\frac{d}{dx}\big) + c(x),$$

where $a(x)$ and $c(x)$ are real-valued periodic functions with period 1. We assume that $a(x)$ is absolutely continuous, and $a(x) \geq \alpha$ for some positive constant $\alpha$, and that $c \in L^1_{loc}(\mathbf{R})$. For each $\gamma \in \mathbf{R}$, let $L_\gamma$ be the operator on the torus $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ defined by

$$L_\gamma = e^{\gamma x} L e^{-\gamma x} = -(\frac{d}{dx} - \gamma)a(x)(\frac{d}{dx} - \gamma) + c(x).$$

Regard $L_\gamma$ as a closed operator on $L^2(\mathbf{T})$ with the domain $D(L_\gamma) = \{u \in H^1(\mathbf{T}); L_\gamma u \in L^2(\mathbf{T})\}$. $\{L_\gamma\}_{\gamma \in \mathbf{R}}$ is a holomorphic family of type (B). In [1], [5], and [10], the following results were studied. $L_\gamma$ has an eigenvalue $E(\gamma) \in \mathbf{R}$ of multiplicity one such that the corresponding eigenspace is generated by a positive function $u_\gamma(x)$ on $\mathbf{T}$. Since $(L_\gamma)^* = L_{-\gamma}$, we have $E(\gamma) = E(-\gamma)$. We call $E(\gamma)$ the principal eigenvalue of $L_\gamma$.
**Facts.** *The function $E(\gamma)$ is real analytic. $E''(\gamma) < 0$ for any $\gamma \in \mathbf{R}$.*

$$E''(\gamma) = -2/d_1^2 + O(|\gamma|^{-1}) \ \text{as } |\gamma| \to \infty.$$
$$\sup_{\gamma \in \mathbf{R}} E(\gamma) = E(0) = \inf \sigma(L).$$

For each $k \in \mathbf{R}$, we consider the equation with respect to $\gamma$

$$E'(\gamma) = -k.$$

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From the facts above we see that the equation has a unique solution $\gamma = \gamma_k$. For $\gamma \in \mathbf{R}$, let $p_\gamma(x, x')$ be the positive continuous function on $\mathbf{R} \times \mathbf{R}$ defined by

$$p_\gamma(x, x') = p_\gamma(x', x) = \frac{u_\gamma(x) u_{-\gamma}(x')}{\int_0^1 u_\gamma(x) u_{-\gamma}(x) dx}, \quad x' \leq x.$$

Put

$$d(x, x') := \int_{x'}^{x} a(s)^{-1/2} ds \quad \text{and} \quad d_1 := d(1, 0).$$

Let $e^{-tL}(x, x')$ be the integral kernel of the semigroup $e^{-tL}$ of $L$, where $L$ is regarded as the selfadjoint operator on $L^2(\mathbf{R})$ with the domain $D(L) = \{u \in H^1(\mathbf{R}); Lu \in L^2(\mathbf{R})\}$. Our main theorem is the following.

**Theorem 1.1.** *$e^{-tL}(x, x')$ admits the following asymptotics as $t \to \infty$:*

$$e^{-tL}(x, x') = \exp[-tE(\gamma_k) - (x - x')\gamma_k] \frac{p_{\gamma_k}(x, x')}{(-2\pi t E''(\gamma_k))^{1/2}} (1 + O(t^{-1})),$$

*where $k = d(x, x')/d_1 t$. Here the term $O(t^{-1})$ satisfies the estimate $|O(t^{-1})| \leq C/t$ for some constant $C > 0$ independent of $t > 1$, and $x$, $x' \in \mathbf{R}$.*

Let $\lambda_0 = \inf \sigma(L)$.

**Corollary 1.2.** *There exists a positive constant $C$ such that*

$$\sup_{x, x' \in \mathbf{R}} \left| e^{-t(L - \lambda_0)}(x, x') - \frac{p_0(x, x')}{(4\pi m_a t)^{1/2}} \exp\left(-\frac{(x - x')^2}{4 m_a t}\right) \right| \leq \frac{C}{t},$$

*where*

$$m_a := -\frac{E''(0)}{2} = \left( \int_0^1 u_0^2(x) dx \int_0^1 u_0^{-2}(x) a^{-1}(x) dx \right)^{-1}.$$

*In particular, if $c(x) \equiv 0$, then*

$$\sup_{x, x' \in \mathbf{R}} \left| e^{-tL}(x, x') - \frac{1}{(4\pi \bar{a} t)^{1/2}} \exp\left(-\frac{(x - x')^2}{4 \bar{a} t}\right) \right| \leq \frac{C}{t},$$

*where $\bar{a} := (\int_0^1 a^{-1}(x) dx)^{-1}$.*

If $c(x) \equiv 0$, the corollary is known in [2].

**Corollary 1.3.** *There exist positive constants $C$ and $T$ such that for any $t \geq T$ and $x, x' \in \mathbf{R}$,*

$$\frac{e^{-C(x - x')^2/t}}{C\sqrt{t}} \leq e^{-t(L - \lambda_0)}(x, x') \leq C \frac{e^{-(x - x')^2/Ct}}{\sqrt{t}}.$$

There are some other results for long-time behaviors of the heat kernel for operators with periodic coefficients (e.g. [9], [8], [4]).

## § 2. Outline of the proof of Theorem

Corresponding to the equation $(L-\lambda)\phi = 0$, we consider the equation

$$\frac{d}{dx}\begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix} = \begin{pmatrix}0 & a(x)^{-1}\\ c(x)-\lambda & 0\end{pmatrix}\begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix} \tag{2.1}$$

for $\lambda \in \mathbf{C}$. By the standard iteration method of ordinary differential equations, we can find unique solutions $(\varphi_1(x,\lambda), \varphi_2(x,\lambda))$ and $(\psi_1(x,\lambda), \psi_2(x,\lambda))$ to (2.1) with the initial conditions

$$\begin{pmatrix}\varphi_1(0,\lambda)\\ \varphi_2(0,\lambda)\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} \text{ and } \begin{pmatrix}\psi_1(0,\lambda)\\ \psi_2(0,\lambda)\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix},$$

respectively, in the space of $\mathbf{C}^2$-valued absolutely continuous functions. We can also see that $\varphi_j(x,\lambda)$ and $\psi_j(x,\lambda)$ are $C([-R,R])$-valued entire functions of $\lambda$ for any $R$.

Since $L$ is of limit-point type at $\infty$ and at $-\infty$, for $\lambda$ non-real, the equation

$$(L-\lambda)\chi = 0 \tag{2.2}$$

has unique solutions in $L^2(0,\infty)$ and in $L^2(-\infty,0)$ up to a constant multiple, and these solutions can be written as

$$\chi_+(x,\lambda) = \varphi_1(x,\lambda) + m_+(\lambda)\psi_1(x,\lambda) \in L^2(0,\infty),$$
$$\chi_-(x,\lambda) = \varphi_1(x,\lambda) + m_-(\lambda)\psi_1(x,\lambda) \in L^2(-\infty,0),$$

where $m_\pm(\lambda)$ are analytic functions on $\mathbf{C}_+ \cup \mathbf{C}_-$ such that $\pm \mathrm{Im}\,\lambda\,\mathrm{Im}\,m_\pm(\lambda) > 0$. The functions $\chi_\pm(x,\lambda)$ and $m_\pm(\lambda)$ depend on the coefficients $a(x)$ and $c(x)$, and we sometimes denote these by $\chi_\pm(x,\lambda;a,c)$ and $m_\pm(\lambda;a,c)$. Let $w(\lambda)$ be the analytic function on $\mathbf{C}_+ \cup \mathbf{C}_-$ defined by

$$w(\lambda) := \int_0^1 \frac{m_+(\lambda; a_s, c_s)}{a(s)}\,ds,$$

where $a_s(x) = a(x+s)$ and $c_s(x) = c(x+s)$. Let $D(\lambda)$ be the discriminant of (2.2): $D(\lambda) := \varphi_1(1,\lambda) + \psi_2(1,\lambda)$. It is known that there exists a sequence of real numbers

$$-\infty < \lambda_0 < \mu_0 \le \mu_1 < \lambda_1 \le \lambda_2 < \cdots$$

such that it tends to infinity, $D(\lambda_n) = 2$, $D(\mu_n) = -2$, and the spectrum of $L$ is written as

$$\sigma(L) = \{\lambda \in \mathbf{R}; |D(\lambda)| \le 2\} = \cup_{n=0}^\infty([\lambda_{2n}, \mu_{2n}] \cup [\mu_{2n+1}, \lambda_{2n+1}]).$$

We summarize some basic facts. The proofs are based on the theories of Johnson and Morser [3] and Marchenko [6].

**Lemma 2.1.** *(i) $\chi_\pm(x,\lambda)$ are expressed as*

$$\chi_\pm(x,\lambda)=\exp\int_0^x \frac{m_\pm(\lambda;a_s,c_s)}{a(s)}ds=e^{\pm w(\lambda)x}u_\pm(x,\lambda)=e^{\pm w(\lambda)d(x,0)/d_1}v_\pm(x,\lambda),$$

*where $u_\pm(x,\lambda)$ and $v_\pm(x,\lambda)$ are some periodic functions of $x$ with period 1.*

*(ii) $w(\lambda)$, $\chi_\pm(x,\lambda)$, $u_\pm(x,\lambda)$ and $v_\pm(x,\lambda)$ extend analytically from $\mathbf{C}_+$ to $\mathbf{C}_-$ through the interval $(-\infty,\lambda_0)$.*

*(iii) $\operatorname{Im}\lambda\operatorname{Im}w(\lambda)>0$ for $\lambda$ non-real, and $\operatorname{Re}w(\lambda)<0$ for $\lambda\in\mathbf{C}\setminus[\lambda_0,\infty)$.*

*(iv) $w(\lambda)<0$ for $\lambda<\lambda_0$, and $\lim_{\lambda\uparrow\lambda_0}w(\lambda)=0$.*

*(v) For any $\varepsilon>0$ and $\delta>0$ there exists a positive constant $T$ such that for any $\kappa=\sigma+i\tau\in\mathbf{C}_+$ with $\delta|\sigma|\le\tau$ and $\tau\ge T$,*

$$\sup_{x\in\mathbf{R}}|v_\pm(x,\kappa^2)-(a(0)/a(x))^{1/4}|\le\varepsilon.$$

*(vi) The function $w(\lambda)$ admits the following expressions:*

$$w(\lambda)=\alpha+\frac{1}{\pi}\int_{\lambda_0}^\infty\Big(\frac{1}{t-\lambda}-\frac{t}{1+t^2}\Big)n(t)dt,\quad \lambda\in\mathbf{C}\setminus[\lambda_0,\infty),$$

*where $\alpha\in\mathbf{R}$ and $n(t):=\operatorname{Im}w(t+i0)$ is a continuous nondecreasing function such that* $\operatorname{supp}dn=\sigma(L)$.

*(vii) Let $\theta(\zeta)$ be the analytic function on $\mathbf{C}_+$ defined by*

$$\theta(\zeta):=-iw(\zeta^2+\lambda_0),\quad \zeta\in\mathbf{C}_+.$$

*Then $\theta(\zeta)$ admits the expression:*

$$\theta(\zeta)=d_1\zeta+\frac{1}{\pi}\int_{-\infty}^\infty\Big(\frac{1}{t-\zeta}-\frac{t}{1+t^2}\Big)\tilde\gamma(t)dt,\quad \zeta\in\mathbf{C}_+,$$

*where $\tilde\gamma(t)=\operatorname{Im}\theta(t+i0)$ is a continuous, bounded, even, and non-negative function such that* $\operatorname{supp}\tilde\gamma(t)=\overline{\tilde\rho}$, *where*

$$\tilde\rho:=\cup_{n=0}^\infty\big[(\nu_{2n},\nu_{2n+1})\cup(\nu'_{2n+1},\nu'_{2n+2})\cup(-\nu_{2n+1},-\nu_{2n})\cup(-\nu'_{2n+2},-\nu'_{2n+1})\big]$$

*with $\nu_n:=\sqrt{\mu_n-\lambda_0}$ and $\nu'_n:=\sqrt{\lambda_n-\lambda_0}$. Furthermore, $\theta(\zeta)$ extends analytically through the interval $(-\nu_0,\nu_0)$ from $\mathbf{C}_+$ to a domain in $\mathbf{C}_-$.*

We write

$$\theta(\zeta)=\theta_1(\xi,\eta)+i\theta_2(\xi,\eta),\quad \zeta=\xi+i\eta.$$

By Lemma 2.1(vii) we can see that $\theta_1(\xi,\eta)$ and $\theta_2(\xi,\eta)$ are odd and even in $\xi$, respectively. We have $w(\lambda_0-\eta^2)=-\theta_2(0,\eta)$ for $\eta\ge0$. By Lemma 2.1(i), $(L-$

$\lambda)e^{\pm w(\lambda)x}u_\pm(x,\lambda)=0$, and so $(L-\lambda_0+\eta^2)\exp(\mp\theta_2(0,\eta)x)u_\pm(x,\lambda_0-\eta^2)=0$. Since $m_\pm(\lambda_0-\eta^2)$ are real, $u_\pm(x,\lambda_0-\eta^2)$ are positive. Thus $\lambda_0-\eta^2$ is the principal eigenvalue of $L(\pm i\theta_2(0,\eta))$, i.e.,

$$E(\theta_2(0,\eta))=E(-\theta_2(0,\eta))=\lambda_0-\eta^2, \tag{2.3}$$

and the corresponding eigenspace is generated by $u_\pm(x,\lambda_0-\eta^2)$.

By Lemma 2.1(iv), for $\lambda\in\mathbf{C}\setminus[\lambda_0,\infty)$, $\chi_+(x,\lambda)\in L^2(0,\infty)$ and $\chi_-(x,\lambda)\in L^2(-\infty,0)$. Note that for $\lambda\in\mathbf{C}\setminus[\lambda_0,\infty)$, the relation

$$[\chi_+(x,\lambda),\chi_-(x,\lambda)]w'(\lambda)=\int_0^1 u_+(x,\lambda)u_-(x,\lambda)dx$$

holds (see [7]), and the left-hand side does not vanish. Therefore, the integral kernel $R_\lambda(x,y)$ of the resolvent $(L-\lambda)^{-1}$ is expressed as

$$\begin{aligned}R_\lambda(x,x')=R_\lambda(x',x)&=\frac{\chi_+(x,\lambda)\chi_-(x',\lambda)}{[\chi_+(x,\lambda),\chi_-(x,\lambda)]}\\&=w'(\lambda)e^{w(\lambda)(x-x')}\frac{u_+(x,\lambda)u_-(x',\lambda)}{\int_0^1 u_+(x,\lambda)u_-(x,\lambda)dx}\\&=w'(\lambda)e^{w(\lambda)d(x,x')/d_1}\frac{v_+(x,\lambda)v_-(x',\lambda)}{\int_0^1 v_+(x,\lambda)v_-(x,\lambda)dx},\quad x'\le x,\end{aligned}\tag{2.4}$$

for $\lambda\in\mathbf{C}\setminus[\lambda_0,\infty)$.

By the inverse Laplace transformation we have

$$e^{-t(L-\lambda_0)}=\frac{1}{2\pi i}\int_C e^{-t\lambda}(L-\lambda_0-\lambda)^{-1}d\lambda,$$

where $C=C_1\cup C_2\cup C_3$ and

$$C_1=\{re^{-i\theta_0};r:\infty\to r_0\},\quad C_2=\{r_0e^{i\theta};\theta:2\pi-\theta_0\to\theta_0\},\quad C_3=\{re^{i\theta_0};r:r_0\to\infty\},$$

for $0<\theta_0<\pi/2$ and $r_0>0$. Changing the integral variable so that $\lambda=\zeta^2$ and taking account of the estimate $\|(L-\lambda_0-\lambda)^{-1}\|\le C/|\lambda|$ for $\lambda\in\mathbf{C}$ with $0<\theta_0\le\arg\lambda\le 2\pi-\theta_0$, we have by Cauchy's integral theorem

$$e^{-t(L-\lambda_0)}=\frac{1}{2\pi i}\int_\Gamma e^{-t\zeta^2}(L-\lambda_0-\zeta^2)^{-1}2\zeta d\zeta,$$

where $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3$ and

$$\begin{aligned}&\Gamma_1=\{re^{i(\pi-\theta_1)}-a_1+ia_2;r:\infty\to 0\},\quad \Gamma_2=\{\xi+ia_2;\xi:-a_1\to a_1\},\\&\Gamma_3=\{re^{i\theta_1}+a_1+ia_2;r:0\to\infty\},\end{aligned}$$

for any $a_1$, $a_2 > 0$ and $0 < \theta_1 < \pi/4$. We consider the function

$$e(t,x,x') := \frac{1}{2\pi i}\int_\Gamma e^{-t\zeta^2} R_{\zeta^2+\lambda_0}(x,x')\, 2\zeta d\zeta$$

in the case $x' \le x$. Let $q_\zeta(x,x')$ be the $C(\mathbf{T}\times\mathbf{T})$-valued analytic function of $\zeta \in \mathbf{C}_+$ defined by

$$q_\zeta(x,x') = q_\zeta(x',x) = \frac{v_+(x,\zeta^2+\lambda_0)v_-(x',\zeta^2+\lambda_0)}{\int_0^1 v_+(x,\zeta^2+\lambda_0)v_-(x,\zeta^2+\lambda_0)dx}, \quad x' \le x. \tag{2.5}$$

By (2.4) and (2.5) we have

$$\begin{aligned} e(t,x,x') &= \frac{1}{2\pi i}\int_\Gamma e^{-t\zeta^2} e^{w(\zeta^2+\lambda_0)d(x,x')/d_1} q_\zeta(x,x') w'(\zeta^2+\lambda_0)\, 2\zeta d\zeta \\ &= \frac{1}{2\pi}\int_\Gamma e^{-t\phi_k(\zeta)} q_\zeta(x,x')\theta'(\zeta)\, d\zeta, \end{aligned} \tag{2.6}$$

where

$$\phi_k(\zeta) := \zeta^2 - ik\theta(\zeta), \quad k := d(x,x')/d_1 t \ge 0.$$

In the following we regard $k$ as a nonnegative parameter.

We can write

$$\phi_k(\zeta) = \mathrm{Re}\,\phi_k(\xi,\eta) + i\mathrm{Im}\,\phi_k(\xi,\eta) = (\xi^2 - \eta^2 + k\theta_2(\xi,\eta)) + i(2\xi\eta - k\theta_1(\xi,\eta)).$$

Put $l_0 := \nu_0/2$. In the region $\{(\xi,\eta); |\xi| \le l_0, \eta \ge 0\}$, we shall find a critical point of $\phi_k(\zeta)$, that is a solution to

$$2\xi + k\partial_\xi\theta_2(\xi,\eta) = 0, \quad -2\eta + k\partial_\eta\theta_2(\xi,\eta) = 0. \tag{2.7}$$

**Lemma 2.2.** *There exists a constant $C > 0$ such that for $|\xi| \le l_0$ and $\eta \ge 0$,*

$$C^{-1} \le \partial_\eta\theta_2(\xi,\eta) \le C, \quad C^{-1} \le \partial_\eta[\eta/\partial_\eta\theta_2(\xi,\eta)] \le C.$$

Let $k \ge 0$. Since $\theta_2(\xi,\eta)$ is an even function in $\xi$, the first equation of (2.7) is satisfied for $\xi = 0, \eta \ge 0$. By Lemma 2.2 there exists a unique solution $\eta = \eta(\xi,k)$ to the second equation of (2.7), i.e.

$$\frac{\eta}{\partial_\eta\theta_2(\xi,\eta)} = \frac{k}{2}.$$

Thus $(0,\eta(0,k))$ is a critical point of $\phi_k$.

We replace $\Gamma$ with $\Gamma_1' \cup \Gamma_2'$, where

$$\begin{aligned} \Gamma_1' &:= \{\xi + i\eta(\xi,k); \xi : -l_0 \to l_0\}, \\ \Gamma_2' &:= \{\xi + i[-(\xi+l_0)/2 + \eta(-l_0,k)]; \xi : -\infty \to -l_0\} \\ &\qquad \cup \{\xi + i[(\xi-l_0)/2 + \eta(l_0,k)]; \xi : l_0 \to \infty\}. \end{aligned}$$

We can see that the amplitude function $q_\zeta(x,x')\theta'(\zeta)$ in the integrand in (2.6) satisfies $|q_\zeta(x,x')\theta'(\zeta)| \le C$ for $\zeta \in \Gamma_1' \cup \Gamma_2'$, for some $C > 0$ independent of $k \ge 0$ and $x, x' \in \mathbf{R}$. For the integral on $\Gamma_1'$ we have

$$\begin{aligned}(2.8)\qquad &\frac{1}{2\pi}\int_{\Gamma_1'} e^{-t\phi_k(\zeta)} q_\zeta(x,x')\theta'(\zeta)d\zeta \\ &= \frac{1}{2\pi}\int_{-l_0}^{l_0} \exp[-t\phi_k(\xi+i\eta(\xi,k))] q_{\xi+i\eta(\xi,k)}(x,x')\theta'(\xi+i\eta(\xi,k))(1+i\partial_\xi\eta(\xi,k))\,d\xi.\end{aligned}$$

The following holds.

**Lemma 2.3.** *(i) There exists a constant $C > 0$ independent of $k \ge 0$ such that*

$$C^{-1} \le \partial_\xi^2[\phi_k(\xi+i\eta(\xi,k))]|_{\xi=0} = \partial_\xi^2[\mathrm{Re}\phi_k(\xi,\eta(\xi,k))]|_{\xi=0} \le C.$$

*(ii) For any integer $n \ge 1$ there exists a constant $C_n > 0$ such that for any $\xi$, $|\xi| \le l_0$, and $k \ge 0$,*

$$|\partial_\xi^n[\phi_k(\xi+i\eta(\xi,k))]| \le C_n.$$

*(iii) For any integer $n \ge 0$ there exists a constant $C_n > 0$ such that for any $\xi$, $|\xi| \le l_0$, and $k \ge 0$ and $x$, $x' \in \mathbf{R}$,*

$$|\partial_\xi^n[q_{\xi+i\eta(\xi,k)}(x,x')\theta'(\xi+i\eta(\xi,k))(1+i\partial_\xi\eta(\xi,k))]| \le C_n.$$

Put $\eta_k := \eta(0,k)$. Taking account of this lemma, we apply a saddle point method to the right-hand side of (2.8). Then it is equal to

$$\begin{aligned}&\frac{\exp[-t\phi_k(i\eta_k)]}{(2\pi t\partial_\xi^2[\mathrm{Re}\,\phi(\xi,\eta(\xi,k))]|_{\xi=0})^{1/2}}[q_{i\eta_k}(x,x')\partial_\eta\theta_2(0,\eta_k) + O(t^{-1})] \\ &= \frac{\exp[-t(k\theta_2(0,\eta_k)-\eta_k^2)]}{(2\pi t(2-k\partial_\eta^2\theta_2(0,\eta_k)))^{1/2}} q_{i\eta_k}(x,x')\partial_\eta\theta_2(0,\eta_k)(1+O(t^{-1})),\end{aligned}$$

where $O(t^{-1})$ satisfies $|O(t^{-1})| \le Ct^{-1}$ with a constant $C > 0$ independent of $t > 1$ and $x, x' \in \mathbf{R}$.

On the other hand, for the integral on $\Gamma_2'$ we can show that

$$\left|\int_{\Gamma_2'} e^{-t\phi_k(\zeta)} q_\zeta(x,x')\theta'(\zeta)d\zeta\right| \le Ce^{-Ct}\exp[-t(k\theta_2(0,\eta_k)-\eta_k^2)],$$

for some positive constant $C$. Thus we have proved

$$(2.9)\qquad e(t,x,x') = \exp[-t(k\theta_2(0,\eta_k)-\eta_k^2)]\frac{\partial_\eta\theta_2(0,\eta_k)q_{i\eta_k}(x,x')}{(2\pi t(2-k\partial_\eta^2\theta_2(0,\eta_k)))^{1/2}}(1+O(t^{-1})).$$

By (2.3) and Lemma 2.2 we have

$$E'(\theta_2(0,\eta)) = -2\eta/\partial_\eta\theta_2(0,\eta), \quad \eta \ge 0.$$

This implies that $\eta(0,k)$ is the unique solution to $E'(\theta_2(0,\eta)) = -k$. By Lemma 2.2, the map $\eta \in [0,\infty) \mapsto \theta_2(0,\eta) \in [0,\infty)$ is a one-to-one correspondence, therefore the equation $E'(\gamma) = -k \le 0$ has a unique solution $\gamma = \gamma_k$, that is

$$\gamma_k = \theta_2(0, \eta(0,k)). \tag{2.10}$$

This together with (2.3) yields that

$$E(\gamma_k) = \lambda_0 - \eta(0,k)^2, \tag{2.11}$$

$$-E''(\gamma_k) = [2 - k\partial_\eta^2 \theta_2(0,\eta(0,k))]/\partial_\eta \theta_2(0,\eta(0,k))^2. \tag{2.12}$$

Furthermore, note that

$$u_{\pm\gamma_k}(x) = c_\pm u_\pm(x, E(\gamma_k)) = c_\pm e^{\pm\gamma_k(x - d(x,0)/d_1)} v_\pm(x, E(\gamma_k))$$

for some constants $c_\pm$. Hence

$$p_{\gamma_k}(x,x') = e^{\gamma_k[(x-x') - d(x,x')/d_1]} q_{i\eta_k}(x,x'). \tag{2.13}$$

By (2.9-13) we have the desired asymptotics

$$\begin{aligned} e^{-tL}(x,x') &= \exp[-t(E(\gamma_k) + k\gamma_k)] \frac{q_{i\eta_k}(x,x')}{(-2\pi t E''(\gamma_k))^{1/2}} (1 + O(t^{-1})) \\ &= \exp[-tE(\gamma_k) - (x-x')\gamma_k] \frac{p_{\gamma_k}(x,x')}{(-2\pi t E''(\gamma_k))^{1/2}} (1 + O(t^{-1})). \end{aligned}$$

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