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Elliptic modular transformations on Teichmüller and asymptotic Teichmüller spaces

By

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Abstract

We consider the actions of Teichmüller modular groups on infinite dimensional Teichmüller spaces. In particular, we focus our attention on elliptic Teichmüller modular transformations which have fixed points in the Teichmüller space, and observe periodicity and discreteness of the orbits. We also consider the actions of asymptotic Teichmüller modular groups on asymptotic Teichmüller spaces, and investigate elliptic elements which have fixed points in the asymptotic Teichmüller space.

§ 1. Introduction

The Teichmüller space is a deformation space of the complex structure of a Riemann surface. The quasiconformal mapping class group acts on the Teichmüller space biholomorphically, which induces the Teichmüller modular group. If a Riemann surface is analytically finite, then the Teichmüller space is finite-dimensional and the action of the Teichmüller modular group is well investigated. Indeed, the action is always properly discontinuous and a topological classification of the quasiconformal mapping classes by Thurston completely corresponds to an analytic classification of the Teichmüller modular transformations by Bers. On the other hand, for an analytically infinite Riemann surface, the Teichmüller space is infinite-dimensional and the orbits in the Teichmüller space under the actions of Teichmüller modular transformations are complicated and could have accumulation points. Thus it is difficult to classify all those elements. We focus our attention on elliptic Teichmüller modular transformations which have fixed points in the Teichmüller space. In the first half of this paper, we review recent results

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on elliptic Teichmüller modular transformations. In particular, we give a condition for an elliptic Teichmüller modular transformation to be of finite order. Moreover, we classify quasiconformal mapping classes by their orbits in the Teichmüller space, and explain that a quasiconformal mapping class that is of bounded type induces an elliptic Teichmüller modular transformation.

The asymptotic Teichmüller space is a certain quotient space of the Teichmüller space. The quasiconformal mapping class group also acts on the asymptotic Teichmüller space biholomorphically, which induces the asymptotic Teichmüller modular group. We consider elliptic elements of the asymptotic Teichmüller modular group which have fixed points in the asymptotic Teichmüller space. Note that a non-trivial quasiconformal mapping class can induce a trivial asymptotic Teichmüller modular transformation. Moreover there is an elliptic element of the asymptotic Teichmüller modular group that is not induced by an elliptic Teichmüller modular transformation. In the second half of this paper, we observe properties of elliptic elements of the asymptotic Teichmüller modular group on the basis of properties of elliptic Teichmüller modular transformations on the Teichmüller space, and propose several problems.

§ 2. Action of Teichmüller modular groups on Teichmüller spaces

In this section, we consider the action of the Teichmüller modular group on the Teichmüller space, which is induced by the quasiconformal mapping class group of a Riemann surface.

§ 2.1. Teichmüller spaces and Teichmüller modular groups

Throughout this paper, we assume that a Riemann surface R admits a hyperbolic structure. Furthermore, we also assume that R has a non-abelian fundamental group. Let d denote the hyperbolic distance on a Riemann surface R and let $\ell(c)$ denote the hyperbolic length of a curve c on R. For a non-trivial and non-cuspidal simple closed curve c on R, let c_* be the unique simple closed geodesic that is freely homotopic to c.

The *Teichmüller space* $T(R)$ of R is the set of all equivalence classes $[f]$ of quasiconformal homeomorphisms f of R. Here we say that two quasiconformal homeomorphisms f_1 and f_2 of R are *equivalent* if there exists a conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity. The homotopy is considered to be relative to the ideal boundary at infinity. A distance between two points $[f_1]$ and $[f_2]$ in $T(R)$ is defined by $d_T([f_1], [f_2]) = (1/2) \log K(f)$, where f is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then d_T is a complete distance on $T(R)$ which is called the Teichmüller distance. The Teichmüller space $T(R)$ can be embedded in the complex Banach space of all bounded holomorphic quadratic differentials on R', where R' is the complex conjugate of R. In this way, $T(R)$ is endowed with the complex structure. For details, see [26] and [34].

A *quasiconformal mapping class* is the homotopy equivalence class $[g]$ of quasiconformal automorphisms g of a Riemann surface, and the *quasiconformal mapping class group* $\mathrm{MCG}(R)$ of R is the group of all quasiconformal mapping classes of R. Here the homotopy is again considered to be relative to the ideal boundary at infinity. Every element $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism $[g]_*$ of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$, which is also isometric with respect to the Teichmüller distance. Let $\mathrm{Aut}(T(R))$ be the group of all biholomorphic automorphisms of $T(R)$. Then we have a homomorphism

$$\iota_T : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$$

given by $[g] \mapsto [g]_*$, and we define the *Teichmüller modular group* of R by

$$\mathrm{Mod}(R) = \iota_T(\mathrm{MCG}(R)).$$

We call an element of $\mathrm{Mod}(R)$ a *Teichmüller modular transformation*. It is proved in [4] that the homomorphism ι_T is injective (faithful) for all Riemann surfaces R of non-exceptional type. See also [8] and [29] for other proofs. Here we say that a Riemann surface R is of *exceptional type* if R has finite hyperbolic area and satisfies $2g + n \leq 4$, where g is the genus of R and n is the number of punctures of R. The homomorphism ι_T is also surjective for every Riemann surface R of non-exceptional type. In this case, $\mathrm{Mod}(R) = \mathrm{Aut}(T(R))$. The proof is a combination of the results of [3] and [27]. See [13] for a survey of the proof.

We define a condition on hyperbolic geometry of Riemann surfaces.

Definition 2.1. We say that a Riemann surface R satisfies the *bounded geometry condition* if R satisfies the following two conditions:

(i) (m-) *lower bound condition*: there exists a constant $m > 0$ such that, for every point $x \in R^\circ$, every homotopically non-trivial curve that starts from x and terminates at x has hyperbolic length greater than or equal to m. Here R° is the non-cuspidal part of R obtained by removing all horocyclic cusp neighborhoods whose areas are 1:

(ii) (M-) *upper bound condition*: there exists a constant $M > 0$ such that, for every point $x \in R$, there exists a homotopically non-trivial simple closed curve that starts from x and terminates at x and whose hyperbolic length is less than or equal to M.

If R satisfies the lower bound condition for a constant m and the upper bound condition for a constant M, we say that R satisfies (m, M)-bounded geometry condition.

Every normal cover of a compact Riemann surface that is not the universal cover satisfies the bounded geometry condition. See [9, Proposition 3]. Moreover, if a Riemann surface R admits such pants decomposition that the diameter of each pair of pants is uniformly bounded, then R satisfies the bounded geometry condition. However, we note that a Riemann surface that admits a uniform pants decomposition does not necessarily satisfy the bounded geometry condition. See [15, Proposition 2.6].

§ 2.2. Elliptic elements of Teichmüller modular groups

2.2.1. Analytically finite Riemann surfaces First we give an analytic classification of the Teichmüller modular transformations for general Riemann surfaces.

Definition 2.2. We say that a Teichmüller modular transformation of $\mathrm{Mod}(R)$ is *elliptic* if it has a fixed point in the Teichmüller space $T(R)$.

Moreover, we say that a Teichmüller modular transformation $[g]_* \in \mathrm{Mod}(R)$ is *parabolic* if $\inf_{p \in T(R)} d_T([g]_*(p), p) = 0$ but if $[g]_*$ has no fixed point in $T(R)$, and $[g]_* \in \mathrm{Mod}(R)$ is *hyperbolic* if $\inf_{p \in T(R)} d_T([g]_*(p), p) > 0$. See [1]. There is also a topological classification of the quasiconformal mapping classes due to Thurston such as periodic, reducible and pseudo-Anosov.

We focus our attention on elliptic Teichmüller modular transformations. Every elliptic Teichmüller modular transformation $[g]_* \in \mathrm{Mod}(R)$ is realized as a conformal automorphism of the Riemann surface $f(R)$ corresponding to its fixed point $p = [f] \in T(R)$, that is $f \circ g \circ f^{-1}$ is homotopic to a conformal automorphism of $f(R)$ relative to the ideal boundary at infinity. Such a mapping class $[g] \in \mathrm{MCG}(R)$ is called a *conformal mapping class*.

Now we assume that a Riemann surface R is analytically finite. It is known that a Teichmüller modular transformation $[g]_* \in \mathrm{Mod}(R)$ is elliptic if and only if $[g] \in \mathrm{MCG}(R)$ is periodic. The sufficiency follows from the fact that the order of a conformal automorphism of an analytically finite Riemann surface R is finite. In fact, if R is a compact Riemann surface of genus $g \geq 2$, then the order of a conformal automorphism of R is not greater than $2(2g+1)$. See [24]. The necessity is a consequence of the theorem due to Nielsen. In fact, Kerckhoff [25] extended his result to the statement that every finite subgroup of $\mathrm{Mod}(R)$ has a common fixed point in $T(R)$, which is the answer to the Nielsen realization problem. Note that, since the homomorphism ι_T is bijective, $[g] \in \mathrm{MCG}(R)$ is periodic if and only if $[g]_* \in \mathrm{Mod}(R)$ is of finite order.

We also know that parabolic and hyperbolic Teichmüller modular transformations are induced by reducible and pseudo-Anosov mapping classes, respectively.

2.2.2. Analytically infinite Riemann surfaces We consider the case where a Riemann surface is analytically infinite. Then the variety of Teichmüller modular transformations become vast and the behavior of the orbit becomes complicated. First of all, there exists an elliptic Teichmüller modular transformation of infinite order, which is induced by a conformal automorphism of a Riemann surface of infinite order. Thus the periodicity of the orbit in the Teichmüller space does not necessarily imply the periodicity of a quasiconformal mapping class.

The following proposition gives a necessary and sufficient condition for an elliptic Teichmüller modular transformation to be of finite order.

Proposition 2.3 ([10]). *A conformal automorphism of a Riemann surface R is of finite order if and only if it fixes either a simple closed geodesic, a puncture, or a point on R.*

Sketch of proof. First we suppose that a conformal automorphism g fixes either a simple closed geodesic, a puncture, or a point on R. Since the group of conformal automorphisms of R acts properly discontinuously on R (see [38]), we conclude that g is of finite order.

Conversely, suppose that a conformal automorphism g of R has a finite order n. Let $R = \mathbb{H}/\Gamma$ for a Fuchsian group Γ. We take a lift $\tilde{g}$ of g to $\mathbb{H}$ which is an element of $\mathrm{PSL}_2(\mathbb{R})$. Then $\tilde{g}^n$ belongs to Γ and $\tilde{g}^m$ $(1 \leq m < n)$ does not belong to Γ. If $\tilde{g}^n$ is parabolic, then $\tilde{g}$ is parabolic. Hence g fixes a puncture of R. If $\tilde{g}^n$ is the identity, then $\tilde{g}$ is elliptic with a fixed point $\tilde{p} \in \mathbb{H}$. Hence g fixes the point on R that is the projection of $\tilde{p}$. Moreover, if $\tilde{g}^n$ is hyperbolic, then $\tilde{g}$ is hyperbolic. Hence g fixes a closed geodesic on R. In this case, we see that g fixes either a simple closed geodesic, a puncture or a point on R. □

In each case in Proposition 2.3, we have an estimate of the order of g concretely by the injectivity radius. The *injectivity radius* at a point $p \in R$ is the supremum of radii of embedded hyperbolic discs centered at p. For a compact Riemann surface R of genus $g \geq 2$, the hyperbolic area of R is $4\pi(g-1)$. Thus the injectivity radius at any point in R is not greater than a constant depending only on g. Since the order of a conformal automorphism is not greater than $2(2g+1)$, this means that the order of a conformal automorphism is estimated by the injectivity radius. We extend this result to conformal automorphisms of general Riemann surfaces. It is clear that if a Riemann surface R satisfies the M-upper bound condition, then the injectivity radius at every point in R is less than or equal to $M/2$.

Proposition 2.4 ([10]). *Let R be a Riemann surface satisfying M-upper bound condition, g a conformal automorphism of R, and n the order of g.* (i) *If $g(c) = c$ for a simple closed geodesic c on R whose hyperbolic length is ℓ, then $n \leq (e^M - 1)\cosh(\ell/2)$.*

(ii) *If* $g(p) = p$ *for a puncture* p *of* R*, then* $n \leq e^M - 1$. (iii) *If* $g(p) = p$ *for a point* p *in* R *at which the injectivity radius is* $M > 0$*, then* $n < 2\pi \cosh M$.

Sketch of proof. We prove only statement (i). We consider the quotient surface $\hat{R} = R/\langle g \rangle$ by the cyclic group $\langle g \rangle$. Then $\hat{c} = c/\langle g \rangle$ is a simple closed geodesic on $\hat{R}$ whose hyperbolic length is ℓ/n. By the collar lemma (see [2]), we take a collar $\{\hat{x} \in \hat{R} \mid d(\hat{c}, \hat{x}) < \omega(\ell/n)\}$ of $\hat{c}$, where $\sinh \omega(\ell/n) = (2\sinh(\ell/(2n)))^{-1}$. Then we can take a tubular neighborhood $C(c) = \{x \in R \mid d(c, x) < \omega(\ell/n)\}$ of c.

Let $\partial C(c)$ be the boundary of $C(c)$. We may assume that $d(c, \partial C(c)) = \omega(\ell/n) > M/2$. Indeed, if $d(c, \partial C(c)) = \omega(\ell/n) \leq M/2$, then this inequality easily yields the conclusion. We take a point $p \in C(c)$ satisfying $d(p, \partial C(c)) = M/2$. By M-upper bound condition, the hyperbolic length $\ell(\alpha)$ of the shortest non-trivial simple closed curve α that starts from p and terminates at p is less than or equal to M. Since $d(p, \partial C(c)) = M/2$, the curve α is in $C(c)$. By calculations on hyperbolic geometry, we see that

$$\sinh(\ell(\alpha)/2) \geq \frac{n}{2e^{M/2}\cosh(\ell/2)}.$$

Since $\ell(\alpha) \leq M$, this implies that

$$n \leq 2\, e^{M/2} \sinh(M/2) \cosh(\ell/2) = (e^M - 1)\cosh(\ell/2),$$

and we have the desired estimate. □

We mention an extension of Proposition 2.4 to a quasiconformal automorphism f. In this case, the Teichmüller modular transformation $[f]_* \in \mathrm{Mod}(R)$ induced by f need not have a fixed point in $T(R)$. However, if the maximal dilatation of f is smaller than some constant, then $[f]_*$ is of finite order.

Proposition 2.5 ([11]). *Let* R *be a Riemann surface satisfying* (m, M)*-bounded geometry condition. Then, for a given constant* $\ell > 0$*, there exists a constant* $K_0 = K_0(m, M, \ell) \geq 1$ *depending only on* m*,* M *and* ℓ *that satisfies the following: Let* g *be a quasiconformal automorphism of* R *such that* $g(c)$ *is freely homotopic to* c *for a simple closed geodesic* c *on* R *with* $\ell(c) \leq \ell$*. Suppose that* $K(g) \leq K_0$*. Then* $[g] \in \mathrm{MCG}(R)$ *is periodic, and the order of* $[g]$ *depends only on* M *and* ℓ.

§ 2.3. Classification of orbits in Teichmüller spaces

We observe the orbit in the Teichmüller space under the action of an elliptic Teichmüller modular transformation of infinite order.

2.3.1. Bounded and divergent type We classify quasiconformal mapping classes according to orbits in the Teichmüller space.

Definition 2.6. We say that a quasiconformal mapping class $[g] \in \mathrm{MCG}(R)$ is of *bounded type* if the orbit $\{[g^n]_*(p)\}_{n\in\mathbb{Z}}$ of each point $p \in T(R)$ is bounded.

Note that, for an analytically finite Riemann surface R, a quasiconformal mapping class $[g] \in \mathrm{MCG}(R)$ is of bounded type if and only if the Teichmüller modular transformation $[g]_* \in \mathrm{Mod}(R)$ is elliptic. Indeed, $T(R)$ is locally compact and $\mathrm{Mod}(R)$ acts on $T(R)$ discontinuously. Thus a quasiconformal mapping class $[g]$ is bounded type if and only if the orbit $\{[g^n]_*(p)\}_{n\in\mathbb{Z}}$ is a finite set for each point $p \in T(R)$. This is equivalent to saying that $[g]$ is periodic, namely it is elliptic as we have seen in Section 2.2.1.

Also for an analytically infinite Riemann surface, we have the following characterization of elliptic Teichmüller modular transformations.

Theorem 2.7. *Let R be a Riemann surface in general. A quasiconformal mapping class $[g] \in \mathrm{MCG}(R)$ is of bounded type if and only if the Teichmüller modular transformation $[g]_* \in \mathrm{Mod}(R)$ is elliptic. In particular, if $[g]$ is periodic, then $[g]_*$ is elliptic.*

This is an extension of the Nielsen realization theorem to analytically infinite Riemann surfaces. The proof is based on quasisymmetric conjugacy of a uniformly quasisymmetric group as follows. Let $\mathbb{D} \to R$ be the universal cover of a Riemann surface R and let H be the corresponding Fuchsian group acting on the unit disk model $\mathbb{D}$ of the hyperbolic plane. Let G be a subgroup of $\mathrm{Mod}(R)$ and assume that the orbit $G(p)$ is bounded for every $p \in T(R)$. We lift a quasiconformal automorphism g of R representing each $[g]_* \in G$ to $\mathbb{D}$ as a quasiconformal automorphism and extend it to a quasisymmetric automorphism of the boundary $\partial\mathbb{D}$. In this way, we have a group H_* of quasisymmetric automorphisms that contains the Fuchsian group H as a normal subgroup. Since the orbit $G(p)$ is bounded, we see that there exists a uniform bound for the quasisymmetric constants of all elements of H_*, namely H_* is a quasisymmetric group. Then Theorem 2.7 is a consequence of the following proposition.

Proposition 2.8 ([28])**.** *For a quasisymmetric group H_* acting on the unit circle $\partial\mathbb{D}$, there exists a quasisymmetric automorphism f of $\partial\mathbb{D}$ such that fH_*f^{-1} is the restriction of a Fuchsian group.*

We note a remarkable property on the orbit under an elliptic Teichmüller modular transformation.

Theorem 2.9 ([31])**.** *For every elliptic Teichmüller modular transformation $[g]_*$ of $\mathrm{Mod}(R)$ of infinite order, there exists a point $p \in T(R)$ whose orbit $\{[g^n]_*(p)\}_{n\in\mathbb{Z}}$ is not a discrete set.*

Remark. In [21, Example 5], we constructed a point $p \in T(R)$ concretely whose orbit $\{[g^n]_*(p)\}_{n\in\mathbb{Z}}$ is not discrete for a Riemann surface R that is a normal cover of a compact Riemann surface of genus 2 with a covering transformation group generated by a conformal automorphism g of R of infinite order. See also Proposition 3.10 in this paper.

We define another property of quasiconformal mapping classes.

Definition 2.10. We say that a quasiconformal mapping class $[g] \in \mathrm{MCG}(R)$ is of *divergent type* if the orbit $\{[g^n]_*(p)\}_{n\in\mathbb{Z}}$ of each point $p \in T(R)$ diverges to the point at infinity of $T(R)$ as $n \to \pm\infty$.

For an analytically finite Riemann surface R, a quasiconformal mapping class $[g] \in \mathrm{MCG}(R)$ is of divergent type if and only if the Teichmüller modular transformation $[g]_* \in \mathrm{Mod}(R)$ is either of hyperbolic type or of parabolic type. However, it was pointed out in [31] that there is an analytically infinite Riemann surface R and a quasiconformal mapping class $[g] \in \mathrm{MCG}(R)$ of non-divergent type such that $[g]_*$ is non-elliptic. Moreover, a sufficient condition for a non-elliptic Teichmüller modular transformation to be induced by a quasiconformal mapping class of divergent type was given.

Definition 2.11. We say that a quasiconformal mapping class $[g] \in \mathrm{MCG}(R)$ is *stationary* if there exists a compact subsurface W of R such that $g_n(W) \cap W \neq \emptyset$ for every representative g_n of $[g^n]$ and for every $n \in \mathbb{Z}$.

The stationary property is a generalization of the property that the quasiconformal mapping class group $\mathrm{MCG}(R)$ of an analytically finite Riemann surface R has.

Theorem 2.12 ([31]). *Let $[g] \in \mathrm{MCG}(R)$ be a stationary quasiconformal mapping class. If it is not of divergent type, then $[g]_*$ is of finite order.*

By combining Theorems 2.7 and 2.12, we immediately have the following.

Corollary 2.13. *Let $[g] \in \mathrm{MCG}(R)$ be a stationary quasiconformal mapping class. If it is not of divergent type, then $[g]_*$ is elliptic.*

2.3.2. Pure and essentially trivial mapping classes We define pure mapping classes and essentially trivial mapping classes and see that they are stationary.

Let R^* be the compactification of a Riemann surface R by topological ends. For the definition of topological ends, see [36, Chapter IV, 5D]. Every homeomorphic automorphism of R extends to R^* and moreover every mapping class determines a map on the ends $R^* - R$. We say that a quasiconformal mapping class $[g] \in \mathrm{MCG}(R)$ is *pure* if g fixes all non-cuspidal ends of R. We know that $[g]$ is pure if and only if, for all dividing simple closed oriented curves c on $\dot{R}$, the image $g(c)$ is homologous to c in $\dot{R}$. Here $\dot{R}$ is the Riemann surface obtained by filling all the punctures of R. See [14].

Proposition 2.14 ([14]). *Let R be a Riemann surface having more than two non-cuspidal ends. Then every pure mapping class of* $\mathrm{MCG}(R)$ *is stationary.*

Proof. Since a Riemann surface R has more than two non-cuspidal ends, there exists a pair of pants Y in R with geodesic boundary such that $R-Y$ has three connected components and that each of the connected components has a distinct non-cuspidal end of R. Since g fixes all non-cuspidal ends, g^n also fixes all non-cuspidal ends for all n. Then $g^n(Y) \cap Y \neq \emptyset$ for all n. $\square$

In Proposition 2.3, we have given a condition for an elliptic Teichmüller modular transformation to be of finite order. By using the two results above, we have another condition.

Theorem 2.15 ([18]). *Let R be a Riemann surface having more than two non-cuspidal ends, and $[g] \in \mathrm{MCG}(R)$ a pure conformal mapping class. Then $[g]$ is of finite order.*

Proof. Since the mapping class $[g]$ is pure, it is stationary by Proposition 2.14. Since $[g]$ is a conformal mapping class, the Teichmüller modular transformation $[g]_* \in \mathrm{Mod}(R)$ is elliptic and thus it is not of divergent type. Hence by Theorem 2.12, it is of finite order. $\square$

In Theorem 2.15, we cannot replace the conclusion with the statement that $[g]$ is the identity. On the other hand, for an essentially trivial mapping class defined below, we have a strong conclusion.

Definition 2.16. A quasiconformal mapping class $[g] \in \mathrm{MCG}(R)$ is said to be *essentially trivial* if there exists a topologically finite geodesic subsurface V_g of R such that, for each connected component W of $R - V_g$, the restriction $g|_W : W \to R$ is homotopic to the inclusion map $\mathrm{id}|_W : W \hookrightarrow R$ relative to the ideal boundary at infinity.

It is clear that every essentially trivial mapping class is pure.

Proposition 2.17 ([18]). *Let R be an analytically infinite Riemann surface. Then every essentially trivial conformal mapping class $[g] \in \mathrm{MCG}(R)$ is the identity.*

Proof. Since $[g]$ is essentially trivial, there exists a topologically finite geodesic subsurface V_g of R such that, for each connected component W of $R-V_g$, the restriction

$g|_W : W \to R$ is homotopic to the inclusion map $\mathrm{id}|_W : W \hookrightarrow R$ relative to the ideal boundary at infinity. We take such a connected component W that is not relatively compact. If W is doubly connected, then the statement is easily proved. Thus we may assume that W is not doubly connected. Let Γ be a Fuchsian group such that $R = \mathbb{H}/\Gamma$, and let $\tilde{g}$ be a lift of g to $\mathbb{H}$. Let Γ_W be a subgroup of Γ such that it corresponds to W. Then we may assume that $\tilde{g}$ is the identity on the limit set $\Lambda(\Gamma_W)$ of the Fuchsian group Γ_W. Since $\Lambda(\Gamma_W)$ contains more than two points and $\tilde{g}$ is conformal, we conclude that $\tilde{g}$ is the identity. Thus we have the assertion. $\square$

§ 3. Action of asymptotic Teichmüller modular groups on asymptotic Teichmüller spaces

In this section, we consider the action of asymptotic Teichmüller modular groups on asymptotic Teichmüller spaces. First, we investigate conditions for quasiconformal mapping classes to induce non-trivial actions on asymptotic Teichmüller spaces. Then we see that non-trivial conformal mapping classes act on the asymptotic Teichmüller space non-trivially. Furthermore, we consider elliptic elements of the asymptotic Teichmüller modular group and observe discreteness of the orbits of elliptic elements. Finally, we give a necessary and sufficient condition for elliptic elements to be of finite order.

§ 3.1. Asymptotic Teichmüller spaces and asymptotic Teichmüller modular groups

The asymptotic Teichmüller space has been introduced in [23] for the hyperbolic plane and in [5] and [6] for an arbitrary Riemann surface. We say that a quasiconformal homeomorphism f of R is *asymptotically conformal* if, for every $\epsilon > 0$, there exists a compact subset V of R such that the maximal dilatation $K(f|_{R-V})$ of the restriction of f to $R - V$ is less than $1 + \epsilon$. We say that two quasiconformal homeomorphisms f_1 and f_2 of R are *asymptotically equivalent* if there exists an asymptotically conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity on R relative to the ideal boundary at infinity. The *asymptotic Teichmüller space* $AT(R)$ of R is the set of all asymptotic equivalence classes $[[f]]$ of quasiconformal homeomorphisms f of R. The asymptotic Teichmüller space $AT(R)$ is of interest only when R is analytically infinite. Otherwise $AT(R)$ is trivial, that is, it consists of just one point. Conversely, if R is analytically infinite, then $AT(R)$ is not trivial. Since a conformal homeomorphism is asymptotically conformal, there is a projection $\pi : T(R) \to AT(R)$ that maps each Teichmüller equivalence class $[f] \in T(R)$ to the asymptotic Teichmüller equivalence class $[[f]] \in AT(R)$. The asymptotic Teichmüller space $AT(R)$ has a complex structure such that π is holomorphic. See also [7] and [22].

For a quasiconformal homeomorphism f of R, the boundary dilatation of f is defined by $H^*(f) = \inf K(f|_{R-V})$, where the infimum is taken over all compact subsets V of R. Furthermore, for a Teichmüller equivalence class $[f] \in T(R)$, the boundary dilatation of $[f]$ is defined by $H([f]) = \inf H^*(f')$, where the infimum is taken over all elements $f' \in [f]$. A distance between two points $[[f_1]]$ and $[[f_2]]$ in $AT(R)$ is defined by $d_{AT}([[f_1]], [[f_2]]) = (1/2) \log H([f_2 \circ f_1^{-1}])$, where $[f_2 \circ f_1^{-1}]$ is the Teichmüller equivalence class of $f_2 \circ f_1^{-1}$ in $T(f_1(R))$. Then d_{AT} is a complete distance on $AT(R)$, which is called the asymptotic Teichmüller distance. For every point $[[f]] \in AT(R)$, there exists an asymptotically extremal element $f_0 \in [[f]]$ satisfying $H([f]) = H^*(f_0)$.

Every element $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism $[g]_{**}$ of $AT(R)$ by $[[f]] \mapsto [[f \circ g^{-1}]]$, which is also isometric with respect to d_{AT}. See [6]. Let $\mathrm{Aut}(AT(R))$ be the group of all biholomorphic automorphisms of $AT(R)$. Then we have a homomorphism

$$\iota_{AT} : \mathrm{MCG}(R) \to \mathrm{Aut}(AT(R))$$

given by $[g] \mapsto [g]_{**}$, and we define the *asymptotic Teichmüller modular group* for R (the geometric automorphism group of $AT(R)$) by

$$\mathrm{Mod}_{AT}(R) = \iota_{AT}(\mathrm{MCG}(R)).$$

We call an element of $\mathrm{Mod}_{AT}(R)$ an asymptotic Teichmüller modular transformation. It is different from the case of the representation $\iota_T : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$ that the homomorphism ι_{AT} is not injective, namely $\mathrm{Ker}\, \iota_{AT} \neq \{[\mathrm{id}]\}$ unless R is either the unit disc or the once-punctured disc (see [4]). We call an element of $\mathrm{Ker}\, \iota_{AT}$ asymptotically trivial and call $\mathrm{Ker}\, \iota_{AT}$ the *asymptotically trivial mapping class group.*

§ 3.2. Asymptotically trivial mapping class groups

We give conditions of quasiconformal mapping classes to induce non-trivial actions on the asymptotic Teichmüller space $AT(R)$. First we note the following lemma, which gives an estimate of the ratio of the hyperbolic length of a simple closed geodesic to that of the image under a quasiconformal homeomorphism. This is an improvement of the well-known result given in [37] and [39].

Lemma 3.1 ([12]). *Let c be a simple closed geodesic on a Riemann surface R. For a subset V of R, let $d = d(c, V)$ be the hyperbolic distance between c and V. If f is a K-quasiconformal homeomorphism of R onto another Riemann surface such that the restriction of f to $R - V$ is $(1 + \epsilon)$-quasiconformal for some $\epsilon \geq 0$, then an inequality*

$$\frac{1}{\alpha} \cdot \ell(c) \leq \ell(f(c)_*) \leq \alpha \cdot \ell(c)$$

is satisfied for a constant

$$\alpha = \alpha(K, \epsilon, d) = K + (1 + \epsilon - K)\frac{2\arctan(\sinh d)}{\pi}$$

with $1 \leq \alpha \leq K$ *and* $\lim_{d\to\infty} \alpha = 1 + \epsilon$.

By using this lemma, we have the following condition for a quasiconformal automorphism of a Riemann surface R to induce a non-trivial action on $AT(R)$.

Lemma 3.2 ([12]). *Let g be a quasiconformal automorphism of a Riemann surface R. Suppose there exists a constant $\delta > 1$ such that, for every compact subset V of R, there is a simple closed geodesic c on R outside of V satisfying either*

$$\frac{\ell(g(c)_*)}{\ell(c)} \leq \frac{1}{\delta} \qquad or \qquad \frac{\ell(g(c)_*)}{\ell(c)} \geq \delta.$$

Then g is not homotopic to any asymptotically conformal automorphism of R. In particular, the action of $[g] \in \mathrm{MCG}(R)$ on $AT(R)$ is non-trivial, namely $[g] \notin \mathrm{Ker}\,\iota_{AT}$.

Proof. We take a constant $\varepsilon_0 > 0$ so that $1 + \varepsilon_0 < \delta$. Suppose to the contrary that g is homotopic to an asymptotically conformal automorphism h of R. Then there exists a compact subset V of R such that the restriction of h to $R - V$ is $(1 + \varepsilon_0)$-quasiconformal. Let $\alpha = \alpha(K, \varepsilon, d)$ be the constant obtained in Lemma 3.1, which tends to $1 + \varepsilon$ as $d \to \infty$. We take a constant $d_0 > 0$ so that $\alpha_0 = \alpha(K(h), \varepsilon_0, d_0) < \delta$. By the assumption, we can take a simple closed geodesic c on R so that $d(c, V) \geq d_0$ and that either $\ell(h(c)_*)/\ell(c) \leq 1/\delta$ or $\ell(h(c)_*)/\ell(c) \geq \delta$. On the other hand, we have $(1/\alpha_0) \cdot \ell(c) \leq \ell(h(c)_*) \leq \alpha_0 \cdot \ell(c)$ by Lemma 3.1. This is a contradiction. □

By Lemma 3.2, we have another condition for a quasiconformal mapping class to induce a non-trivial action on $AT(R)$.

Lemma 3.3 ([19]). *Let R be a Riemann surface satisfying the lower bound condition, and $[g] \in \mathrm{MCG}(R)$ a quasiconformal mapping class of R. Suppose that there exists a sequence of mutually disjoint simple closed geodesics $\{c_n\}_{n=1}^{\infty}$ such that the hyperbolic lengths of c_n are uniformly bounded and $g(c_n)_* \neq c_{n'}$ for any n and n'. Then $[g]$ is not asymptotically trivial, namely $[g] \notin \mathrm{Ker}\,\iota_{AT}$.*

Proof. Since R satisfies the lower bound condition, there exists a quasiconformal homeomorphism f of R such that $2\ell(f(c_n)_*) < \ell(f(c)_*)$ for every n and for every simple closed geodesic c other than $\{c_n\}_{n=1}^{\infty}$. See [19, Lemma 7.1]. Set $\tilde{g} = f \circ g \circ f^{-1}$. By the assumption $g(c_n)_* \neq c_{n'}$, we see that

$$\frac{\ell(\tilde{g}(f(c_n))_*)}{\ell(f(c_n)_*)} = \frac{\ell(f(g(c_n))_*)}{\ell(f(c_n)_*)} > 2$$

for every n. Since the sequence $\{f(c_n)_*\}_{n=1}^{\infty}$ exits from any compact subsurface in the Riemann surface $f(R)$ (see [30, Proposition 1]), we can apply Lemma 3.2. Then we conclude that $\tilde{g}$ is not homotopic to any asymptotically conformal automorphism of $f(R)$. This implies that $[g] \notin \mathrm{Ker}\, \iota_{AT}$. □

By using Lemma 3.3, we see that every conformal mapping class acts on $AT(R)$ non-trivially under the bounded geometry condition.

Proposition 3.4 ([19]). *Let R be a topologically infinite Riemann surface satisfying the bounded geometry condition. Then every non-trivial conformal mapping class $[g] \in \mathrm{MCG}(R)$ is not asymptotically trivial.*

Proof. Here we only prove for the case that $[g]$ is of infinite order. We take a conformal representative g in the mapping class $[g]$. Let c be a simple closed geodesic on R and set $c_n := g^n(c)$ for every $n \in \mathbb{Z}$. Then all hyperbolic lengths of c_n are the same. By replacing g with g^k for some $k \in \mathbb{Z}$ if necessary, we may assume that c_n and $c_{n'}$ are mutually disjoint for every n and n'. We apply Lemma 3.3 for the sequence $\{c_{2n}\}_{n\in\mathbb{Z}}$, and have a conclusion. □

In fact, for a conformal mapping class of infinite order, Proposition 3.4 is true under no assumption on a Riemann surface as the following theorem says.

Theorem 3.5. *Let R be a topologically infinite Riemann surface. Then every conformal mapping class $[g] \in \mathrm{MCG}(R)$ of infinite order is not asymptotically trivial.*

This was proved in [32]. See also [33]. In [18], we gave a simple proof under the assumption that a Riemann surface has more than two non-cuspidal ends. Moreover, in the proof of [12, Proposition 4.3], we constructed concretely a point in $AT(R)$ where the asymptotic Teichmüller modular transformation induced by a conformal automorphism of R acts non-trivially. For detail, see also Proposition 3.10.

On the other hand, we do not know yet whether a non-trivial conformal mapping class of finite order is not asymptotically trivial under no assumption on a Riemann surface.

Next, we consider a condition for a mapping class to be asymptotically trivial. It is easy to see that every essentially trivial mapping class is asymptotically trivial. The following theorem gives a complete characterization of asymptotically trivial mapping classes under the bounded geometry condition of Riemann surfaces.

Theorem 3.6 ([14], [19]). *For a topologically infinite Riemann surface, every asymptotically trivial mapping class is pure. In addition, if a Riemann surface satisfies the bounded geometry condition, then every asymptotically trivial mapping class is essentially trivial.*

Remark. (i) There exists a quasiconformal mapping class that is pure but is not asymptotically trivial. Indeed, let $\hat{R}$ be a compact Riemann surface, and R a normal covering surface of $\hat{R}$ whose covering transformation group is a cyclic group $\langle\phi\rangle$ generated by a conformal automorphism ϕ of R of infinite order. Then ϕ is pure. On the other hand, ϕ is not asymptotically trivial by Proposition 3.4.

(ii) The second statement of Theorem 3.6 is not true if R does not satisfy the bounded geometry condition. Indeed, we consider a Riemann surface R that does not satisfy the lower bound condition. Then there exists a sequence of mutually disjoint simple closed geodesics $\{c_n\}_{n=1}^{\infty}$ on R such that $\ell(c_n) \to 0$. Let $[g] \in \mathrm{MCG}(R)$ be a quasiconformal mapping class caused by infinitely many Dehn twists with respect to each c_n. Then $[g]$ is asymptotically trivial mapping classes but is not essentially trivial.

§ 3.3. Elliptic elements of infinite order of asymptotic Teichmüller modular groups

We define the ellipticity of asymptotic Teichmüller modular transformations, and observe the orbits of elliptic elements in the asymptotic Teichmüller space.

Definition 3.7. We say that an asymptotic Teichmüller modular transformation of $\mathrm{Mod}_{AT}(R)$ is *elliptic* if it has a fixed point in $AT(R)$.

Every elliptic element $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ is realized as an asymptotically conformal automorphism of the Riemann surface $f(R)$ corresponding to its fixed point $\hat{p} = [[f]] \in AT(R)$, that is $f \circ g \circ f^{-1}$ is homotopic to an asymptotically conformal automorphism of $f(R)$ relative to the ideal boundary at infinity. Such a mapping class $[g] \in \mathrm{MCG}(R)$ is called an *asymptotically conformal mapping class*. It is clear that, if $[g]_* \in \mathrm{Mod}(R)$ is elliptic, then $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ is also elliptic. However, the converse is not true. A trivial example is a Teichmüller modular transformation caused by a single Dehn twist. This is not elliptic as a Teichmüller modular transformation, but the quasiconformal mapping class acts trivially on $AT(R)$. In particular, it has a fixed point in $AT(R)$. In [35], a Riemann surface R and a quasiconformal mapping class $[g] \in \mathrm{MCG}(R)$ were constructed so that $[g]_* \in \mathrm{Mod}(R)$ is not elliptic but $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ is elliptic and non-trivial. We note a remarkable property of the orbit in the Teichmüller space by such a quasiconformal mapping class.

Proposition 3.8 ([31]). *For a quasiconformal mapping class $[g] \in \mathrm{MCG}(R)$, suppose that $[g]_* \in \mathrm{Mod}(R)$ is not elliptic but $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ is elliptic. Then $[g]$ is of divergent type.*

Moreover, in [32], a Riemann surface R, an elliptic Teichmüller modular transformation $[g]_* \in \mathrm{Mod}(R)$ and a point $p \in T(R)$ were constructed so that $[g]_*(p) \neq p$ but

$[g]_{**}(\pi(p)) = \pi(p)$ for the projection $\pi(p) \in AT(R)$ of p. We have another example of an elliptic element of $\mathrm{Mod}_{AT}(R)$ as follows.

Example 3.9. Let R be a Riemann surface constructed in [17, Section 3], which does not satisfy the lower bound condition. By modifying the construction slightly as in Remark 3.4 of that paper, we see that R admits an asymptotically conformal automorphism g of infinite order such that it is not asymptotically trivial. Then $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ is elliptic. On the other hand, we have proved that the orbit $\{[g^n]_*(p)\}_{n\in\mathbb{Z}}$ on $T(R)$ is discrete for every point $p \in T(R)$. Thus, by Theorem 2.9, it implies that $[g]_* \in \mathrm{Mod}(R)$ is not elliptic.

Now we consider the orbit in the asymptotic Teichmüller space under the action of an elliptic element of $\mathrm{Mod}_{AT}(R)$. First we observe the following phenomenon.

Proposition 3.10 ([12]). *Let R be a normal cover of a compact Riemann surface of genus 2 whose covering transformation group is a cyclic group $\langle g\rangle$ generated by a conformal automorphism g of R of infinite order. Then there exists a point $p \in T(R)$ such that the orbit $\{[g^n]_*(p)\}_{n\in\mathbb{Z}}$ is not discrete in $T(R)$, and the orbit $\{[g^n]_{**}(\hat{p})\}_{n\in\mathbb{Z}}$ is not discrete either in $AT(R)$ for the projection $\hat{p} = \pi(p) \in AT(R)$.*

Sketch of proof. Let $L^\infty(\mathbb{Z})$ be the Banach space of all bounded bilateral infinite sequence of real numbers, and let $(\xi_i)_{i\in\mathbb{Z}}$ $(0 < \xi_i \le 1)$ be a point of $L^\infty(\mathbb{Z})$ defined in [16, Definition 4.3] as follows: set $\xi_0 = 1$ and $\xi_1 = \xi_{-1} = (1/2)\xi_0 = 1/2$. We proceed as $\xi_i = \xi_{i-6} = (2/3)\xi_{i-3}$ for $i = 2, 3, 4$ and $\xi_i = \xi_{i-18} = (3/4)\xi_{i-9}$ for $i = 5, \ldots, 13$. Inductively, set

$$\xi_i = \xi_{i-2\cdot 3^k} = \frac{k+1}{k+2}\cdot \xi_{i-3^k}$$

for $\sum_{j=0}^{k-1} 3^j + 1 \le i \le \sum_{j=0}^{k} 3^j$ stratified with the indices $k \in \mathbb{N}$. This is equivalent to the following direct definition by using 3-adic expansion. Every integer $i \in \mathbb{Z}$ is uniquely written as $i = \sum_{j=0}^{\infty} \varepsilon_j(i)\cdot 3^j$, where $\varepsilon_j(i)$ is either -1, 0 or 1. Then ξ_i is defined by $\xi_i = \prod_{\varepsilon_j(i)\neq 0}(j+1)/(j+2)$, where the product is taken over all $j \in \mathbb{N}$ satisfying $\varepsilon_j(i) \neq 0$. Then it was proved that the point $\xi = (\xi_i)_{i\in\mathbb{Z}} \in L^\infty(\mathbb{Z})$ satisfies $\lim_{\ell\to\infty} \|\sigma^{3^\ell}(\xi) - \xi\|_\infty = 0$.

We take a sequence $\{c_n\}_{n\in\mathbb{Z}}$ of non-dividing simple closed geodesics on R such that $g(c_n) = c_{n+1}$. We also take a g-invariant pants decomposition $\mathcal{P}$ whose boundary contains $\{c_n\}_{n\in\mathbb{Z}}$. We can choose a quasiconformal homeomorphism f of R such that $\ell(f(c_n)_*) = 1 + \xi_n$ and that $\ell(f(c)_*) = \ell(c)$ for every c other than $\{c_n\}_{n\in\mathbb{Z}}$ that is a boundary component of some pair of pants in the pants decomposition $\mathcal{P}$. Set $p = [f] \in T(R)$ and $\hat{p} = [[f]] \in AT(R)$. Then we see that $d_T([g^{3^k}]_*(p), p) \to 0$ $(k \to 0)$ and thus $d_{AT}([g^{3^k}]_{**}(\hat{p}), \hat{p}) \to 0$ $(k \to 0)$. Moreover we see that $[g^{3^k}]_{**}(\hat{p}) \neq [g^{3^m}]_{**}(\hat{p})$ for every

$k \neq m$ by applying Lemma 3.2 to the quasiconformal automorphism $f \circ g^{-(3^m - 3^k)} \circ f^{-1}$ of $f(R)$. Hence we have the assertion. $\square$

We extend Proposition 3.10 as follows.

Theorem 3.11. *Let R be a Riemann surface, and g a quasiconformal automorphism of R of infinite order. Suppose that there exists a point $p \in T(R)$ whose orbit $\{[g^n]_*(p)\}_{n\in\mathbb{Z}}$ is not a discrete set. Then $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ is also of infinite order. Moreover the orbit $\{[g^n]_{**}(\hat{p})\}_{n\in\mathbb{Z}}$ of $\hat{p} = \pi(p) \in AT(R)$ is not a discrete set, either.*

Proof. Suppose to the contrary that $[g]_{**}$ has a finite order n, namely $[g^n] \in \mathrm{Ker}\,\iota_{AT}$. Since $[g^n]_{**}$ has a fixed point in $AT(R)$ in particular, either $[g^n]_* \in \mathrm{Mod}(R)$ is elliptic or $[g^n] \in \mathrm{MCG}(R)$ is of divergent type by Proposition 3.8. By the assumption, the orbit $\{[g^n]_*(p)\}_{n\in\mathbb{Z}}$ is not a discrete set for a point $p \in T(R)$, and thus $[g^n]$ is not of divergent type. Hence $[g^n]_*$ is elliptic of infinite order. Then we may regard g^n as a conformal automorphism of infinite order. However, since every conformal automorphism of infinite order is not asymptotically trivial by Theorem 3.5, this contradicts the assumption $[g^n] \in \mathrm{Ker}\,\iota_{AT}$. Thus we conclude that $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ is of infinite order.

Next we prove the second statement. Suppose to the contrary that the orbit $\{[g^n]_{**}(\hat{p})\}_{n\in\mathbb{Z}}$ is a discrete set. We may assume that $[g^n]_{**}(\hat{p}) = \hat{p}$ for every n. It was proved in [32] that, for every elliptic element $[g]_{**} \in \mathrm{Mod}_{AT}(R)$, the orbit $\{[g]_*(p)\}$ of any point $p \in T(R)$ over the fixed point on $AT(R)$ is a discrete set in the fiber in $T(R)$ containing p. This contradicts the assumption that the orbit $\{[g^n]_*(p)\}_{n\in\mathbb{Z}}$ is not a discrete set. Hence we conclude that the orbit $\{[g^n]_{**}(\hat{p})\}_{n\in\mathbb{Z}}$ of $\hat{p} = \pi(p) \in AT(R)$ is not a discrete set. $\square$

In the last of this subsection, we explore the following problem on the orbit in the asymptotic Teichmüller space by an elliptic element of $\mathrm{Mod}_{AT}(R)$. This corresponds to Theorem 2.9 for the orbit in the Teichmüller space by an elliptic Teichmüller modular transformation.

Problem. *For every elliptic element $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ of infinite order, there exists a point $\hat{p} \in AT(R)$ such that the orbit $\{[g^n]_{**}(\hat{p})\}_{n\in\mathbb{Z}}$ is not a discrete set.*

Note that this problem is true for an elliptic element $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ induced by a conformal automorphism g of R of infinite order. Indeed, by Theorem 2.9, we have a point $p \in T(R)$ whose orbit $\{[g^n]_*(p)\}_{n\in\mathbb{Z}}$ is not a discrete set. Thus by Theorem 3.11, we have the conclusion.

§ 3.4. Elliptic elements of finite order of asymptotic Teichmüller modular groups

Similar to an elliptic Teichmüller modular transformation of $\mathrm{Mod}(R)$, an elliptic element of $\mathrm{Mod}_{AT}(R)$ is not necessarily of finite order. In this subsection, we consider a necessary and sufficient condition for an elliptic element to be of finite order. First we give the following sufficient condition, which can be regarded as the asymptotic version of Proposition 2.4.

Theorem 3.12 ([19]). *Let R be a Riemann surface satisfying the bounded geometry condition. Let $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ be an elliptic element. Suppose that, for some constant $\ell > 0$ and in any topologically infinite neighborhood of each topological end of R, there exists a simple closed geodesic c with $\ell(c) \leq \ell$ such that $g(c)$ is freely homotopic to c. Then $[g]_{**}$ is of finite order.*

We give an example of a Riemann surface R and a quasiconformal automorphism of R satisfying the assumption in Theorem 3.12.

Example 3.13. Let R be a topologically infinite Riemann surface that is a normal cover of a compact Riemann surface of genus 3 and admits a conformal automorphism g_0 of order 2. We assume that there exists a sequence $\{c_n\}_{n\in\mathbb{Z}}$ of disjoint simple closed geodesics on R such that $\{c_n\}_{n\in\mathbb{Z}}$ and $\{g_0(c_n)\}_{n\in\mathbb{Z}}$ are mutually disjoint, and assume that there exists a sequence $\{c'_n\}_{n\in\mathbb{Z}}$ of disjoint simple closed geodesics on R such that $g_0(c'_n) = c'_n$ for every n. Let $[g_1]$ be a Dehn twist along c_0. Set $g = g_1 \circ g_0$. Then $[g]_* \in \mathrm{Mod}(R)$ is not elliptic. Indeed, $[g^2]$ is the Dehn twist along both c_0 and $g(c_0)$, and thus it is not elliptic. Since we can take a quasiconformal homeomorphism g so that it is conformal outside of the collar of c_0, the mapping class $[g] \in \mathrm{MCG}(R)$ is asymptotically conformal. Moreover, by Lemma 3.3, $[g]$ is not asymptotically trivial. Hence $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ is elliptic and non-trivial, and satisfies the assumption in Theorem 3.12.

Recall that the Nielsen theorem states that every Teichmüller modular transformation of finite order is elliptic. We have the corresponding result for asymptotic Teichmüller modular transformations under the bounded geometry condition.

Theorem 3.14 ([19]). *Let R be a Riemann surface satisfying the bounded geometry condition. If $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ is of finite order, then it is elliptic.*

Sketch of Proof. Let $[g] \in \mathrm{MCG}(R)$ be a quasiconformal mapping class such that $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ is of finite order n. This means that $[g^n] \in \mathrm{Ker}\, \iota_{AT}$. Since every asymptotically trivial mapping class is essentially trivial by Theorem 3.6, the quasiconformal mapping class $[g^n]$ is essentially trivial. Then we see that, outside some topologically finite geodesic subsurface, the mapping class $[g]$ is periodic. By usual arguments, there is a conformal structure such that $[g]$ can be realized as a conformal automorphism off the subsurface, that is, $[g]$ is asymptotically conformal. This is equivalent to saying that $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ has a fixed point in $AT(R)$. $\square$

By Theorems 3.12 and 3.14, we finally have a necessary and sufficient condition for an elliptic element to be of finite order.

Theorem 3.15 ([19]). *Let R be a Riemann surface satisfying the bounded geometry condition. An asymptotic Teichmüller modular transformation $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ is of finite order if and only if $[g]_{**}$ is elliptic and there exist an integer $s \geq 1$ and a constant $\ell > 0$ such that in any topologically infinite neighborhood of each topological end of R, there exists a simple closed geodesic c with $\ell(c) \leq \ell$ such that $g^s(c)$ is freely homotopic to c.*

Proof. Suppose that $[g]_{**}$ is of finite order. Then $[g]_{**}$ is elliptic by Theorem 3.14 and $[g^s] \in \mathrm{Ker}\,\iota_{AT}$ for some integer $s \geq 1$ as well. By Theorem 3.6, this implies that $[g^s]$ is essentially trivial, namely there exists a topologically finite geodesic subsurface V of R such that, for each connected component W of $R - V$, the restriction $g^s|_W : W \to R$ is homotopic to the inclusion map $\mathrm{id}|_W : W \hookrightarrow R$ relative to the ideal boundary at infinity. Thus $g^s(c)$ is freely homotopic to c for every simple closed geodesic c in W. This shows the sufficiency.

Conversely, suppose that $[g]_{**}$ is elliptic and there exist an integer $s \geq 1$ and a constant $\ell > 0$ such that, in any topologically infinite neighborhood of each topological end of R, there exists a simple closed geodesic c with $\ell(c) \leq \ell$ such that $g^s(c)$ is freely homotopic to c. Since $[g^s]_{**}$ is also elliptic, we apply Theorem 3.12 to $[g^s]$. Then we conclude that $[g^s]_{**}$ is of finite order, and hence so is $[g]_{**}$. This shows the necessity. $\square$

We explore a problem whether we can extend Theorem 3.12 to an element of $\mathrm{Mod}_{AT}(R)$ which need not have a fixed point in $AT(R)$ but is induced by a quasiconformal automorphism of R with sufficiently small boundary dilatation. This problem can be regarded as an asymptotic version of Proposition 2.5.

In [20], we extend Theorem 3.14 to the following statement, which can be regarded as an answer to the asymptotically conformal version of the Nielsen realization problem.

Theorem 3.16. *Let R be an analytically infinite Riemann surface satisfying the bounded geometry condition. Then every finite subgroup of $\mathrm{Mod}_{AT}(R)$ has a common fixed point in $AT(R)$.*

The proof of Theorem 3.16 is also carried out by a similar argument as above relying on the fact that every asymptotically trivial mapping class is essentially trivial under the bounded geometry condition. In the light of Theorem 2.7, we further propose a problem of finding a common fixed point in $AT(R)$ when the orbit of a given subgroup of $\mathrm{Mod}_{AT}(R)$ is bounded.

References

[1] Bers, L., An extremal problem for quasiconformal mappings and a theorem by Thurston, *Acta Math.*, **141** (1978), 73–98.

[2] Buser, P., *Geometry and Spectra of Compact Riemann Surfaces*, Progress in Mathematics **106**, Birkhäuser Boston-Basel-Berlin, 1992.

[3] Earle, C. J. and Gardiner, F. P., Geometric isomorphisms between infinite dimensional Teichmüller spaces, *Trans. Amer. Math. Soc.*, **348** (1996), 1163–1190.

[4] Earle, C., Gardiner, F. P. and Lakic, N., Teichmüller spaces with asymptotic conformal equivalence, I.H.E.S., 1995, preprint.

[5] Earle, C. J., Gardiner, F. P. and Lakic, N., Asymptotic Teichmüller space, Part I: The complex structure, *In the tradition of Ahlfors and Bers*, *Contemporary Math.*, AMS, **256** (2000), 17–38.

[6] Earle, C. J., Gardiner, F. P. and Lakic, N., Asymptotic Teichmüller space, Part II: The metric structure, *In the tradition of Ahlfors and Bers, III*, *Contemporary Math.*, AMS, **355** (2004), 187–219.

[7] Earle, C. J., Markovic, V. and Šarić, D., Barycentric extension and the Bers embedding for asymptotic Teichmüller space, *Complex Manifolds and Hyperbolic Geometry*, *Contemporary Math.*, AMS, **311** (2002), 87–105.

[8] Epstein, A., Effectiveness of Teichmüller modular groups, *In the tradition of Ahlfors and Bers*, *Contemporary Math.*, AMS, **256** (2000), 69–74.

[9] Fujikawa, E., Limit sets and regions of discontinuity of Teichmüller modular groups, *Proc. Amer. Math. Soc.*, **132** (2004), 117–126.

[10] Fujikawa, E., The order of conformal automorphisms of Riemann surfaces of infinite type, *Kodai Math. J.*, **26** (2003), 16–25. (— supplement, *Perspectives of Hyperbolic Spaces*, RIMS Kôkyûroku, **1329** (2003), 62–68.)

[11] Fujikawa, E., The order of periodic elements of Teichmüller modular groups, *Tohoku Math. J.*, **57** (2005), 45–51.

[12] Fujikawa, E., The action of geometric automorphisms of asymptotic Teichmüller spaces, *Michigan Math. J.*, **54** (2006), 269–282.

[13] Fujikawa, E., Another approach to the automorphism theorem for Teichmüller spaces, *In the tradition of Ahlfors-Bers, IV*, *Contemporary Math.*, AMS, **432** (2007), 39–44.

[14] Fujikawa, E., Pure mapping class group acting on Teichmüller space, *Conform. Geom. Dyn.*, **12** (2008), 227–239.

[15] Fujikawa, E., Limit set of quasiconformal mapping class group on asymptotic Teichmüller space, *Teichmüller Theory and Moduli Problem,* Ramanujan Math. Soc. Lecture Notes Series, **10** (2010), 167–178.

[16] Fujikawa, E. and Matsuzaki, K., Recurrent and periodic points for isometries of L^∞ spaces, *Indiana Univ. Math. J.*, **55** (2006), 975–997.

[17] Fujikawa, E. and Matsuzaki, K., Non-stationary and discontinuous quasiconformal mapping class groups, *Osaka J. Math.*, **44** (2007), 173-185.

[18] Fujikawa, E. and Matsuzaki, K., Elliptic quasiconformal mapping classes acting on asymptotic Teichmüller spaces, *Proceedings of the 15th International Conference on Finite or Infinite Dimensional Complex Analysis and Applications*, Osaka Municipal Universities Press (2008), 169–173.

[19] Fujikawa, E. and Matsuzaki, K., Stable quasiconformal mapping class groups and asymptotic Teichmüller spaces, *Amer. J. Math.*, (to appear).

[20] Fujikawa, E. and Matsuzaki, K., The Nielsen realization problem for asymptotic Teichmüller modular groups, preprint.

[21] Fujikawa, E., Shiga, H. and Taniguchi, M., On the action of the mapping class group for Riemann surfaces of infinite type, *J. Math. Soc. Japan*, **56** (2004), 1069–1086.

[22] Gardiner, F. P. and Lakic, N., *Quasiconformal Teichmüller Theory,* Mathematical Surveys and Monographs **76**, Amer. Math. Soc., 2000.

[23] Gardiner, F. P. and Sullivan, D., Symmetric structure on a closed curve, *Amer. J. Math.*, **114** (1992), 683–736.

[24] Harvey, W., Cyclic groups of automorphisms of a compact Riemann surface, *Quart. J. Math.*, **17** (1966), 86–97.

[25] Kerckhoff, S. P., The Nielsen realization problem, *Ann. of Math.*, **117** (1983), 235–265.

[26] Lehto, O., *Univalent Functions and Teichmüller Spaces*, Graduate Text in Math., **109**, Springer.

[27] Markovic, V., Biholomorphic maps between Teichmüller spaces, *Duke Math. J.*, **120** (2003), 405–431.

[28] Markovic, V., Quasisymmetric groups, *J. Amer. Math. Soc.*, **19** (2006), 673–715.

[29] Matsuzaki, K., Inclusion relations between the Bers embeddings of Teichmüller spaces, *Israel J. Math.*, **140** (2004) 113–124.

[30] Matsuzaki, K., A countable Teichmüller modular group, *Trans. Amer. Math. Soc.*, **357** (2005), 3119–3131.

[31] Matsuzaki, K., A classification of the modular transformations on infinite dimensional Teichmüller spaces, *In the tradition of Ahlfors-Bers, IV*, *Contemporary Math.*, AMS, **432** (2007), 167–178.

[32] Matsuzaki, K., The action of elliptic modular transformations on asymptotic Teichmüller spaces, *Teichmüller Theory and Moduli Problem,* Ramanujan Math. Soc. Lecture Notes Series, **10** (2010), 481–488.

[33] Matsuzaki, K., Properties of asymptotically elliptic modular transformations of Teichmüller spaces, *Infinite dimensional Teichmüller space and moduli space*, RIMS Kôkyûroku Bessatsu, Research Institute for Mathematical Sciences, Kyoto University, **B17** (2010), 73–84.

[34] Nag, S., *The Complex Analytic Theory of Teichmüller Spaces*, John Wiley & Sons 1988.

[35] Petrovic, I., A Teichmüller model for period doubling, *In the tradition of Ahlfors and Bers, III*, *Contemporary Math.*, AMS, **355** (2004), 333–351.

[36] Sario, L. and Nakai, M., *Classification theory of Riemann surfaces*, Die Grundlehren der mathematischen Wissenschaften, **164**, Springer, 1970.

[37] Sorvali, T., The boundary mapping induced by an isomorphism of covering groups, *Ann. Acad. Sci. Fennicæ*, Series A **526** (1972).

[38] Tsuji, M., *Potential Theory in Modern Function Theory*, Chelsea, New York, 1959.

[39] Wolpert, S., The length spectra as moduli for compact Riemann surfaces, *Ann. of Math.*, **109** (1979), 323–351.