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An averaging operator and non-separability of certain Banach spaces of holomorphic automorphic forms

By

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Abstract

In this paper, we consider a problem on non-separability of certain Banach spaces of holomorphic automorphic forms which appear in the recent development of theories of asymptotic Teichmüller spaces.

§ 1. Introduction

The Teichmüller space of a Riemann surface $R = \Delta/\Gamma$ is embedded (via the Bers embedding) in a certain Banach space $B(\Gamma)$ of holomorphic automorphic forms of weight -4 on the unit disk Δ invariant under the action of the Fuchsian group Γ. For Fuchsian groups Γ and G, if $\Gamma \subset G$, then the corresponding Banach spaces have the inclusion relation $B(\Gamma) \supset B(G)$. On the other hand, since any Möbius transformation of Δ acts on $B = B(1)$ as an isometric linear automorphism, we can define a bounded linear operator on $B(\Gamma) \subset B$ by averaging a finite number of isometric linear automorphisms induced by Möbius transformations. In particular, when Γ is a finite index subgroup of G, the average taken over all representatives of the coset of G modulo Γ gives a bounded linear operator from $B(\Gamma)$ to $B(G)$. This can be defined independent of the choice of the representatives.

We extend this averaging operator to a larger Banach space containing $B(\Gamma)$. Two holomorphic functions on Δ are defined to be asymptotically equivalent if their difference vanishes at the unit circle $\partial\Delta$ with respect to the hyperbolic supremum norm. We consider a Banach subspace $\widetilde{AB}(\Gamma)$ of B consisting of all holomorphic automorphic forms that are invariant under Γ modulo asymptotic equivalence. Then the averaging

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operator $\widetilde{L}$ maps $\widetilde{AB}(\Gamma)$ to $\widetilde{AB}(G)$, but this depends on the choice of the representatives of the coset G/Γ.

We are interested in a problem whether non-separability of $\widetilde{AB}(\Gamma)$ implies that of $\widetilde{AB}(G)$ when Γ is a Fuchsian group of cofinite hyperbolic area. Consider the kernel $\operatorname{Ker} \widetilde{L}$ of the averaging operator $\widetilde{L}$. If $\operatorname{Ker} \widetilde{L}$ is separable, then the quotient Banach space $\widetilde{AB}(\Gamma)/\operatorname{Ker} \widetilde{L}$ is non-separable, and it is mapped injectively into $\widetilde{AB}(G)$ by a bounded linear operator induced by $\widetilde{L}$. However, we encounter two problems here. The first problem is that this kernel also depends on the representatives defining $\widetilde{L}$ and we can only verify that the intersection of the kernels taken over various $\widetilde{L}$ is separable. The second problem is that the image of $\widetilde{AB}(\Gamma)/\operatorname{Ker} \widetilde{L}$ in $\widetilde{AB}(G)$ is not known to be closed and without this fact we cannot see that $\widetilde{AB}(G)$ is non-separable even if $\widetilde{AB}(\Gamma)/\operatorname{Ker} \widetilde{L}$ is non-separable. In this paper, we investigate these problems from a viewpoint of functional analysis.

A motivation of this work lies in a study of symmetric structures introduced in [3]. This concept has been extended to the asymptotic Teichmüller space of a Riemann surface Δ/Γ and it has been realized in a certain quotient Banach space of $B(\Gamma)$ as the asymptotic Bers embedding. See [1] and [2]. In his previous paper [6], the author had an idea of the problem concerning averaging operators, and in his forthcoming paper, it will be proved that, for *almost* every Fuchsian group Γ of cofinite hyperbolic area, the Banach space $\widetilde{AB}(\Gamma)$ and hence its quotient $AB(\Gamma)$ by asymptotic equivalence is non-separable. Then, one may ask how about the rest is. The present paper records the author's attempt at this question.

§ 2. A problem on averaging operators

Let B be the Banach space of all bounded holomorphic functions φ on the unit disk Δ with respect to the hyperbolic supremum norm $\rho^{-2}(z)|\varphi(z)|$, where $\rho(z)$ is the hyperbolic density on Δ. Let B_0 be a Banach subspace of B consisting of all elements φ that vanish at the boundary $\partial\Delta$, that is, $\rho^{-2}(z)|\varphi(z)| \to 0$ as $|z| \to 1$. The quotient Banach space is denoted by $AB = B/B_0$ and the projection by $\alpha : B \to AB$. An equivalence class represented by $\varphi \in B$ is denoted by $[\varphi]$, that is, $[\varphi] = \alpha(\varphi) \in AB$. The Banach space B_0 is separable. Indeed, its dual space B_0^* is isometric to the Banach space Q consisting of all integrable holomorphic functions on Δ. Since polynomials are dense in Q, it is separable and hence so is B_0 (p.71 in [7]).

For a holomorphic function φ on Δ, which is regarded as an automorphic form of weight -4, we define the pull-back $g^*\varphi$ of φ by a Möbius transformation g of Δ as $g^*\varphi(z) := \varphi(g(z))g'(z)^2$. Then every Möbius transformation g of Δ acts on B and AB by $\varphi \mapsto g^*\varphi$ and $[\varphi] \mapsto [g^*\varphi]$ respectively, which define isometric linear operators on B

and AB. For a Fuchsian group G, we define subspaces of B and AB consisting of all automorphic forms that are fixed by every element of G as follows:

$$B(G) = \{\varphi \in B \mid g^*\varphi = \varphi \ \text{ for } \ \forall\, g \in G\};$$
$$AB(G) = \{[\varphi] \in AB \mid [g^*\varphi] = [\varphi] \ \text{ for } \ \forall\, g \in G\}.$$

We also define the inverse image of $AB(G)$ under the projection $\alpha : B \to AB$, that is,

$$\widetilde{AB}(G) = \alpha^{-1}(AB(G)) = \{\varphi \in B \mid [g^*\varphi] = [\varphi] \ \text{ for} \forall\, g \in G\}.$$

Our problem is to consider non-separability of $AB(G) = \widetilde{AB}(G)/B_0$. Since B_0 is separable, this is equivalent to non-separability of $\widetilde{AB}(G)$. Let G contain a Fuchsian group Γ of cofinite hyperbolic area as a finite index subgroup, and assume that $AB(\Gamma)$ is non-separable. Note that such a Fuchsian group Γ actually exists though we will not show this fact here. Under these circumstances, we want to know that $AB(G)$ is also non-separable. A claim we can obtain so far is the following.

Theorem 2.1. *Let Γ be a Fuchsian group of cofinite hyperbolic area and G a Fuchsian group that contains Γ. Assume that $AB(\Gamma)$ is non-separable. Then $AB(G)$ is non-separable, or otherwise, there exists an averaging operator $\widetilde{L}$ on $\widetilde{AB}(\Gamma)$ whose image is not closed.*

We define an averaging operator precisely here. For a system of the representatives $\{g_1, g_2, \ldots, g_m\}$ of the coset of G modulo Γ, we consider a bounded linear operator

$$\widetilde{L}(\varphi) = \frac{1}{m}\sum_{k=1}^{m} g_k^*\varphi$$

for $\varphi \in B$. It is clear that $\widetilde{L}(B_0) \subset B_0$. We call $\widetilde{L}$ an *averaging operator* by restricting it to $\widetilde{AB}(\Gamma)$. Note that this depends on the choice of the representatives.

Proposition 2.2. *For any averaging operator $\widetilde{L}$, the image $\widetilde{L}(\widetilde{AB}(\Gamma))$ is contained in $\widetilde{AB}(G)$.*

Proof. An element $\psi \in \widetilde{L}(\widetilde{AB}(\Gamma))$ is written as $\psi = \frac{1}{m}\sum_{k=1}^{m} g_k^*\varphi$ for some $\varphi \in \widetilde{AB}(\Gamma)$. We will show that $[g^*\psi] = [\psi]$ for every $g \in G$. Fix $g \in G$. Then there is a permutation σ on $\{1, 2, \ldots, m\}$ and some $\gamma_{\sigma(k)} \in \Gamma$ for each k such that $gg_k = g_{\sigma(k)}\gamma_{\sigma(k)}$ $(1 \le k \le m)$. Hence

$$g^*\psi = g^*\left\{\frac{1}{m}\sum_{k=1}^{m} g_k^*\varphi\right\} = \frac{1}{m}\sum_{k=1}^{m} g_{\sigma(k)}^*\gamma_{\sigma(k)}^*\varphi = \frac{1}{m}\sum_{k=1}^{m} g_k^*\gamma_k^*\varphi.$$

Since $\gamma_k^*\varphi - \varphi \in B_0$ for every k, we conclude that $g^*\psi - \psi \in B_0$, namely, $[g^*\psi] = [\psi]$. $\square$

The averaging operator $\widetilde{L}$ induces a bounded linear operator $L : AB(\Gamma) \to AB(G)$ via the projection $\alpha : B \to AB$. Indeed, since $\widetilde{L}(B_0) \subset B_0$, the map L is well-defined for $\hat{\varphi} \in AB(\Gamma)$ independent of the choice of an element in $\alpha^{-1}(\hat{\varphi})$. Also the restriction of $\widetilde{L}$ to $B(\Gamma)$ gives $\widetilde{L}|_{B(\Gamma)} : B(\Gamma) \to B(G)$. Summing up, we have the following commutative diagram:

$$\begin{array}{ccc} B(\Gamma) & \xrightarrow{\widetilde{L}|_{B(\Gamma)}} & B(G) \\ \iota\downarrow & & \downarrow\iota \\ \widetilde{AB}(\Gamma) & \xrightarrow{\widetilde{L}} & \widetilde{AB}(G) \\ \alpha\downarrow & & \downarrow\alpha \\ AB(\Gamma) & \xrightarrow{L} & AB(G) \end{array}$$

Here we remark that both $\widetilde{L}|_{B(\Gamma)}$ and L are determined independently of the choice of representative of the coset G/Γ. Moreover, both $\widetilde{L}|_{B(\Gamma)}$ and L are surjective. In fact, each element of $B(G)$ is fixed by $\widetilde{L}$ and each element of $AB(G)$ is fixed by L. However, we do not know the surjectivity of $\widetilde{L}$. Since $\alpha \circ \widetilde{L} = L \circ \alpha$ is surjective, the surjectivity of $\widetilde{L}$ is equivalent to saying that $B_0 = \alpha^{-1}(0)$ is contained in the image of $\widetilde{L}$.

Since the averaging operator $\widetilde{L}$ depends on the representatives of G/Γ, we should denote it by taking the dependence into account. Assume that g_1 represents Γ and fix the other representatives $\{g_2, \ldots, g_m\}$. Let $\{h_1, h_2, \ldots, h_n\}$ be a system of generators of Γ and set $h_0 = \mathrm{id}$. For each i $(0 \le i \le n)$, we consider an averaging operator

$$\widetilde{L}_i(\varphi) = \frac{1}{m}\left\{h_i^*\varphi + \sum_{k=2}^{m} g_k^*\varphi\right\}$$

for $\varphi \in \widetilde{AB}(\Gamma)$. Concerning the kernels of the operators $\widetilde{L}_i$, we have the following.

Proposition 2.3. *The intersection $\bigcap_{i=0}^n \operatorname{Ker} \widetilde{L}_i$ is contained in the finite dimensional Banach space $B(\Gamma)$.*

Proof. Let φ be an element of $\bigcap_{i=0}^n \operatorname{Ker} \widetilde{L}_i$. Then

$$h_i^*\varphi + \sum_{k=2}^{m} g_k^*\varphi = 0 = h_0^*\varphi + \sum_{k=2}^{m} g_k^*\varphi$$

for every i $(1 \le i \le n)$. This in particular implies that $h_i^*\varphi = \varphi$. Since $\{h_1, h_2, \ldots, h_n\}$ generates Γ, we see that $\gamma^*\varphi = \varphi$ for every $\gamma \in \Gamma$. This shows that φ belongs to $B(\Gamma)$. $\square$

Proof of Theorem 2.1. For the averaging operator $\widetilde{L}_0$, consider the composition $\alpha \circ \widetilde{L}_0 : \widetilde{AB}(\Gamma) \to AB(G)$, which is coincident with $L \circ \alpha$, and thus surjective. Set

$K_\# = \operatorname{Ker}(\alpha \circ \widetilde{L}_0)$, which is a Banach subspace $\widetilde{L}_0^{-1}(B_0)$ of $\widetilde{AB}(\Gamma)$. The operator $\alpha \circ \widetilde{L}_0$ induces a bijective linear map between the quotient Banach space $\widetilde{AB}(\Gamma)/K_\#$ and $AB(G)$. By the open mapping theorem (p.78 in [7], p.75 in [8]), this bijection is actually an isomorphism between the Banach spaces, that is, both directions are bounded linear maps. Hence, in order to show that $AB(G)$ is non-separable, we have only to see that $K_\#$ is separable. Assuming on the contrary that $K_\#$ is non-separable, we will derive a contradiction.

Set $K_0 := \operatorname{Ker} \widetilde{L}_0|_{K_\#} = K_\# \cap \operatorname{Ker} \widetilde{L}_0$. We restrict $\widetilde{L}_1$ to K_0 and set $K_1 := \operatorname{Ker} \widetilde{L}_1|_{K_0}$, which is coincident with $K_\# \cap \operatorname{Ker} \widetilde{L}_0 \cap \operatorname{Ker} \widetilde{L}_1$. Inductively, after K_{j-1} has been defined, we set $K_j := \operatorname{Ker} \widetilde{L}_j|_{K_{j-1}}$, which is coincident with $K_\# \cap \bigcap_{i=0}^{j} \operatorname{Ker} \widetilde{L}_i$.

By assumption, $K_\#$ is non-separable, whereas $K_n = K_\# \cap \bigcap_{i=0}^{n} \operatorname{Ker} \widetilde{L}_i$ is separable by Proposition 2.3. Hence there is some j $(0 \le j \le n)$ such that K_{j-1} is non-separable but K_j is separable, where we regard $K_{-1} = K_\#$. Then, setting $K := K_{j-1}$ and $\widetilde{L} := \widetilde{L}_j$, we consider the operator $\widetilde{L}|_K$ restricted to K. By construction, K is non-separable but $\operatorname{Ker} \widetilde{L}|_K$ is separable. Moreover, the image $\widetilde{L}(K)$ is contained in B_0. Indeed, since $\alpha \circ \widetilde{L} = \alpha \circ \widetilde{L}_0$ and $K \subset K_\#$, we have $\alpha \circ \widetilde{L}(K) = \{0\}$ and hence $\widetilde{L}(K) \subset \alpha^{-1}(0) = B_0$.

We are assuming that an arbitrary averaging operator $\widetilde{L}$ has a closed range. By the open mapping theorem again, this assumption implies that the image of every closed subspace of $\widetilde{AB}(\Gamma)$ under $\widetilde{L}$ is also closed. Hence the operator $\widetilde{L}|_K$ induces an isomorphism between the quotient Banach space $K/\operatorname{Ker} \widetilde{L}|_K$ and the Banach subspace $\widetilde{L}(K)$. Since K is non-separable whereas $\operatorname{Ker} \widetilde{L}|_K$ is separable, $\widetilde{L}(K)$ should be non-separable. However, since $\widetilde{L}(K)$ is contained in the separable Banach space B_0, this is impossible. □

Remark. The last line of the above proof is based on a fact that every subspace of a separable Banach space is separable. In general, every subset of a separable metric space is separable in the induced metric. See p.9 in [4] for instance.

§ 3. Non-separable Banach spaces mapped into separable Banach spaces

In this section, we exhibit examples of injective bounded linear operators that map non-separable Banach spaces into separable Banach spaces. This means that, without some extra properties of the Banach spaces B and B_0 of the holomorphic automorphic forms, we cannot remove the second alternative conclusion in the statement of Theorem 2.1, that is to say, we will fail to obtain the desired result.

Let $\ell^\infty(\mathbb{Z})$ be the Banach space of all bounded bilateral sequences $\xi = \{x_n\}_{n \in \mathbb{Z}}$ of real numbers equipped with the supremum norm and $c_0(\mathbb{Z})$ the subspace consisting of all elements $\xi \in \ell^\infty(\mathbb{Z})$ vanishing at $\pm\infty$, namely $\xi(n) := x_n \to 0$ as $n \to \pm\infty$. Note that $\ell^\infty(\mathbb{Z})$ is non-separable whereas $c_0(\mathbb{Z})$ is separable (p.34 in [7]).

We consider the shift operator $\sigma : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ that sends $\{x_n\}$ to $\{x_{n+1}\}$. More precisely, this is defined by $(\sigma(\xi))(n) = \xi(n+1)$. Set $L_- = \sigma - \mathrm{id}$, that is, $(L_-(\xi))(n) = \xi(n+1) - \xi(n)$. The kernel of the bounded linear operator L_- is clearly the subspace $\mathrm{const}(\mathbb{Z})$ consisting of all elements $\xi \in \ell^\infty(\mathbb{Z})$ such that $\xi(n) \equiv c$ for every $n \in \mathbb{Z}$, which is isometric to $\mathbb{R}$. Consider the inverse image $L_-^{-1}(c_0(\mathbb{Z}))$ and denote this Banach subspace by D_-. We restrict the bounded linear operator L_- to D_- and denote this by the same notation

$$L_- : D_- \to c_0(\mathbb{Z}).$$

It is clear that $\mathrm{Ker}\, L_- = \mathrm{const}(\mathbb{Z})$. Set the quotient Banach space $D_-/\mathrm{const}(\mathbb{Z})$ by $\check{D}_-$. Then L_- induces an injective bounded linear operator

$$\check{L}_- : \check{D}_- \to c_0(\mathbb{Z}).$$

We will see that D_- and hence $\check{D}_-$ is non-separable. Note that, from this fact, we also see that the image of L_- is not closed.

Lemma 3.1. *The Banach subspace $D_- = L_-^{-1}(c_0(\mathbb{Z}))$ of $\ell^\infty(\mathbb{Z})$ is non-separable.*

Proof. We will find a set of uncountably many elements in D_- any two of which are uniformly separated. For every integer $n \in \mathbb{Z}$, we can choose a pair of integers (m, k) $(m \geq 1,\ k \geq 0)$ uniquely satisfying

$$|n| = \sum_{i=1}^{m-1} 2^i + k \quad (0 \leq k \leq 2^m - 1).$$

For an arbitrary subset I of $\mathbb{N}$, we define an element $\xi_I \in \ell^\infty(\mathbb{Z})$ so that

$$\xi_I(n) = 1_I(m) \cdot \frac{\min\{k, 2^m - k\}}{2^{m-1}},$$

where m and k are uniquely determined integers by n and $1_I(m)$ is the characteristic function of I for the variable m. Then $(L_-(\xi_I))(n) \leq 1/2^{m-1}$, which implies that $\xi_I \in D_-$. On the other hand, for distinct subsets I and I' of $\mathbb{N}$, we have $\|\xi_I - \xi_{I'}\|_\infty = 1$. Since there are uncountably many subsets I of $\mathbb{N}$, we see that D_- is non-separable. □

In a similar way, we define $L_+ = \sigma + \mathrm{id}$. Then $\mathrm{Ker}\, L_+$ consists of all elements $\xi \in \ell^\infty(\mathbb{Z})$ such that $\xi(n) = (-1)^n c$ for every $n \in \mathbb{Z}$, which is again isometric to $\mathbb{R}$. Let D_+ be the inverse image $L_+^{-1}(c_0(\mathbb{Z}))$ and the restriction of L_+ to D_+ is denoted by $L_+ : D_+ \to c_0(\mathbb{Z})$. It induces an injective bounded linear operator

$$\check{L}_+ : \check{D}_+ \to c_0(\mathbb{Z})$$

for the quotient Banach space $\check{D}_+ = D_+/\mathrm{Ker}\, L_+$. We can also prove that D_+ is non-separable and so is $\check{D}_+$ as before.

Finally, we remark that non-separability of D_+ implies non-separability of D_-. For the self-composition σ^2 of the shift operator, we consider $L_{2,-} = \sigma^2 - \mathrm{id}$ and $D_{2,-} = L_{2,-}^{-1}(c_0(\mathbb{Z}))$. If we divide a sequence $\{x_n\}_{n\in\mathbb{Z}}$ into the even subsequence $\{x_{2m}\}_{m\in\mathbb{Z}}$ and the odd subsequence $\{x_{2m+1}\}_{m\in\mathbb{Z}}$, then we have an identification of $\ell^\infty(\mathbb{Z})$ with the direct product $\ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z})$ of the Banach spaces. Under this correspondence, $L_{2,-}$ is conjugate to the product of the linear operators

$$L_- \times L_- : \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z}),$$

and $D_{2,-}$ is equivalent to $D_- \times D_-$. On the other hand, $D_{2,-}$ contains the subspace D_+. Indeed, for every $\xi \in D_+$, we have $\sigma\xi + \xi \in c_0(\mathbb{Z})$ and hence $\sigma^2\xi + \sigma\xi \in c_0(\mathbb{Z})$. Subtracting the first from the second, we see that $L_{2,-}(\xi) = \sigma^2\xi - \xi \in c_0(\mathbb{Z})$, which means that $\xi \in D_{2,-}$. Hence, non-separability of D_+ implies non-separability of D_- through $D_{2,-} \cong D_- \times D_-$.

§ 4. Dual spaces and dual operators

At this point, we have recognized that additional properties of B and B_0 should be necessary if the desired consequence is able to be obtained in our methods. In this section, we consider dual spaces and dual operators. The dual space Q of B_0 is also separable, which might give us a chance that a stronger condition can be merged in our arguments. For instance, the closed range theorem asserts that, if the range of the dual operator is closed, then so is the range of the original operator, and vice versa (p.169 in [7], p.205 in [8]).

First we introduce a general fact from functional analysis. Let $S : X \to Y$ be an injective bounded linear operator such that the range $\mathcal{R}(S) = S(X)$ is dense in Y. We consider the inverse operator $T : Y \to X$ of S. The domain $\mathcal{D}(T)$ of T is the range $\mathcal{R}(S)$ of S, which is dense in Y. It is clear that T is an injective closed operator because the graph of S is closed. The dual operator $S^* : Y^* \to X^*$ between the dual spaces Y^* and X^* is defined, which is a bounded linear operator. Also the dual operator $T^* : X^* \to Y^*$ is defined with a domain $\mathcal{D}(T^*)$ that is coincident with $\mathcal{R}(S^*) = S^*(Y^*)$. It is known that T^* is an injective closed operator and the inverse $(T^*)^{-1}$ is equal to S^* (Th.5.30 in [5]).

Proposition 4.1. *In circumstances as above, suppose that Y^* is separable. If the domain $\mathcal{D}(T^*)$ is dense in X^*, then X is separable. In particular, if X is reflexive, then X is separable.*

Proof. If X^* is separable then so is X (p.71 in [7]). Hence we show that X^* is separable. Since $\mathcal{D}(T^*)$ is dense in X^*, we have only to show that $\mathcal{D}(T^*)$ has a countable dense subset. But, since $(T^*)^{-1} : Y^* \to X^*$ is a bounded linear operator and Y^* is separable by assumption, we have done. The latter assertion follows from a claim that, if X is reflexive and $T : Y \to X$ is a closed operator with $\mathcal{D}(T)$ dense in Y, then $\mathcal{D}(T^*)$ is dense in X^*. (See Th.5.29 in [5]; the assumption that Y is reflexive is not necessary for this claim. Also cf. p.196 in [8] though the claim is stated for Hilbert spaces.) $\square$

Note that, for any subspace Y of a Banach space $\widetilde{Y}$ whose dual space $\widetilde{Y}^*$ is separable, the dual space Y^* is also separable. Indeed, the Hahn-Banach extension theorem (p.68 in [7], p.108 in [8]) implies that the bounded linear operator $\widetilde{Y}^* \to Y^*$ that is dual to the inclusion map $Y \to \widetilde{Y}$ is surjective.

In the situation of Theorem 2.1, we set $\widetilde{Y} = B_0$ and let Y be the closure of $\widetilde{L}(K)$ in B_0. Then Proposition 4.1 might be applicable to see that $X = K/\mathrm{Ker}\,\widetilde{L}|_K$ is separable. If this were the case, by the same argument as in the proof of Theorem 2.1 but without the assumption that $\widetilde{L}$ has closed range, we could obtain the desired conclusion that $AB(G)$ is non-separable.

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