An application of Penner’s coordinates of Teichmüller space of punctured surfaces

By

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Abstract

We apply R. C. Penner’s coordinates for the Teichmüller space of once punctured surfaces and also his rational representation of the mapping class group to obtain certain Diophantine equations which have infinitely many integer solutions. The family of these equations can be thought as a generalization of the classical Markoff equation.

§1. Introduction

The Markoff equation

\[ m_1^2 + m_2^2 + m_3^2 = 3m_1m_2m_3 \]

is preserved by the Markoff maps, which are iterative composites of

\[ (m_1, m_2, m_3) \rightarrow (m_1, m_3, 3m_1m_3 - m_2), \]
\[ (m_1, m_2, m_3) \rightarrow (m_3, m_2, 3m_2m_3 - m_1) \]

and their inverses. Since a Markoff map is a polynomial map with positive integer coefficients, it sends the solution \((1, 1, 1)\) to a positive integer solution. It is known that all positive integer solutions of (1.1) are found in the orbits of \((1, 1, 1)\) under the group generated by Markoff maps.

Let \(\{A, B\}\) be a canonical generator system of a once punctured torus subgroup of \(SL(2, \mathbb{R})\). Since \(\text{tr}ABA^{-1}B^{-1} = -2, (\text{tr}A, \text{tr}B, \text{tr}AB)\) is a solution of (1.1). The changes

\[ 2000 \text{ Mathematics Subject Classification(s): } 30F40, 57M05. \]

\[ \text{Key Words: Teichmüller space, hyperbolic geometry, mapping class group.} \]

Supported by Grand-in-Aid for Scientific Research (No. 18540179), Ministry of Education, Science and Culture of Japan.

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of canonical generators \( \{A, B\} \rightarrow \{A, BA\} \) and \( \{A, B\} \rightarrow \{AB, B\} \) induce the maps (1.2) and (1.3), respectively. Hence the group of Markoff maps acting on the solutions of the Markoff equation (1.1) can be understood to be the mapping class group \( \mathcal{M}_{1,1} \) acting on the Teichmüller space \( T_{1,1} \) of once punctured torus. In [4, §7], R. C. Penner treated Markoff maps as mapping classes in \( \mathcal{M}_{1,1} \) acting on the \( \lambda \) length coordinates of \( T_{1,1} \).

In this paper we pursue an analogy of the Markoff equation and present some Diophantine equations which admit infinitely many positive integer solutions. Each of these Diophantine equations is a model of the Teichmüller space of a once punctured surface represented in Penner’s \( \lambda \) length coordinate space. We will show that the images of a special positive integer solution under the mapping class group acting on the Teichmüller space are infinitely many positive integer solutions of the Diophantine equation.

The author would like to thank the referee for her or his careful reading of the manuscript and for many suggestions. He also thanks Ege Fujikawa for valuable comments.

§ 2. Distance between horocycles

Let \( \mathbb{H} = \{ z : \text{Im}[z] > 0 \} \) denote the hyperbolic plane equipped with the metric

\[
\frac{dx^2 + dy^2}{y^2}.
\]

The distance defined by (2.1) is denoted by \( d(\cdot, \cdot) \). The circle at infinity \( \partial \mathbb{H} \) is the boundary of \( \mathbb{H} \) in the Riemann sphere \( \mathbb{C} \cup \{\infty\} \). For two distinct points \( p, q \) of \( \partial \mathbb{H} \), \( l(p, q) \) denotes the hyperbolic geodesic line between \( p \) and \( q \).

Let \( p \) be a point of \( \partial \mathbb{H} \). A horocycle \( h \) at \( p \) is a Euclidean circle in \( \mathbb{H} \) tangent at \( p \) to \( \partial \mathbb{H} \) if \( p \neq \infty \) or a horizontal line in \( \mathbb{H} \) if \( p = \infty \). The point \( p \) is called the base point of \( h \). Let \( h_1 \) and \( h_2 \) be horocycles based at distinct points \( p_1 \) and \( p_2 \). Let

\[
\lambda(h_1, h_2) = e^{\delta/2}
\]

where \( \delta \) is the signed length of the portion of the geodesic \( l(p_1, p_2) \) intercepted between the two horocycles \( h_1 \) and \( h_2 \), \( \delta > 0 \) if \( h_1 \) and \( h_2 \) are disjoint and \( \delta < 0 \) otherwise.
We quote two important lemmas from [4]. For the sake of completeness we also give proofs of the lemmas. But our proofs do not involve calculus in the three dimensional Minkowski space as in employed in the first two sections of [4].

**Lemma 2.1** (the half horocyclic length [4]). Let \((p_1, p_2, p_3)\) be a sequence of three distinct points of \(\partial \mathbb{H}\) which agrees with the positive orientation with respect to \(\mathbb{H}\). Let \(h_i, i = 1, 2, 3\), be a horocycle based at \(p_i\). If \(a = \lambda(h_1, h_2), b = \lambda(h_2, h_3)\) and \(c = \lambda(h_3, h_1)\), then the length of the portion of \(h_1\) intercepted between the two geodesic lines \(l(p_1, p_2)\) and \(l(p_1, p_3)\) is \(b/(ac)\).

**Proof.** We may assume that \(p_1 = \infty, p_2 = 0\) and \(h_1 = \{z : \text{Im}[z] = 1\}\). Let \(t = p_3 > 0\). Then we need to show that \(t = b/(ac)\). The horocycles \(h_2\) and \(h_3\) are the circles
\[
x^2 + \left(y - \frac{1}{2a^2}\right)^2 = \frac{1}{4a^4}, \quad (x - t)^2 + \left(y - \frac{1}{2c^2}\right)^2 = \frac{1}{4c^4},
\]
and meet \(l(p_2, p_3)\), the upper semicircle defined by \(x^2 - tx + y^2 = 0\), at
\[
P = \frac{t}{1 + at^2} + i\frac{a^2t^2}{1 + at^2}, \quad Q = \frac{t^3c^4}{1 + ct^2} + i\frac{c^2t^2}{1 + ct^2},
\]
respectively. By definition \(b = \exp(d(P, Q)/2)\) if \(\text{Re}[P] \leq \text{Re}[Q]\) or \(b = \exp(-d(P, Q)/2)\) if \(\text{Re}[P] > \text{Re}[Q]\). Note that \(\text{Re}[P] \leq \text{Re}[Q]\) if and only if \(act \geq 1\). If \(b = \exp(d(P, Q)/2)\), then by [1, Theorem 7.2.1]
\[
b = \sinh(d(P, Q)/2) + \cosh(d(P, Q)/2) = \frac{|P - Q| + |P - \overline{Q}|}{2|\text{Im}[P]\text{Im}[Q]|^{1/2}} = act.
\]
We obtain the same result for the case where \(b = \exp(-d(P, Q)/2)\). \(\square\)

We remark that the quantity \(\lambda(h_3, h_1)\lambda(h_1, h_2)^{-1}\lambda(h_2, h_3)^{-1}\) is twice the \(h\)-length defined in [4, p.313].

**Lemma 2.2** (Proposition 2.6 in [4]). Let \((p_1, p_2, p_3, p_4)\) be a sequence of four distinct points of \(\partial \mathbb{H}\) which agrees with the positive orientation with respect to \(\mathbb{H}\). Let \(h_i, i = 1, 2, 3, 4\), be a horocycle based at \(p_i\). If \(a = \lambda(h_1, h_2), b = \lambda(h_2, h_3), c = \lambda(h_3, h_4)\) and \(d = \lambda(h_4, h_1)\), then the length of the portion of \(h_1\) intercepted between the two geodesic lines \(l(p_1, p_2)\) and \(l(p_1, p_3)\) is \(b/(ad)\).
i = 1, 2, 3, 4, be a horocycle based at $p_i$. If $\lambda_a = \lambda(h_1, h_2)$, $\lambda_b = \lambda(h_2, h_3)$, $\lambda_c = \lambda(h_3, h_4)$, $\lambda_d = \lambda(h_4, h_1)$, $\lambda_e = \lambda(h_1, h_3)$ and $\lambda_f = \lambda(h_2, h_4)$, then

\begin{equation}
\lambda_e \lambda_f = \lambda_a \lambda_c + \lambda_b \lambda_d.
\end{equation}

\textbf{Proof.} We assume again that $p_1 = \infty$, $p_2 = 0$ and $h_1 = \{z : \text{Im}[z] = 1\}$. Then the segment $s$ in $h_1$ between $l(p_1, p_2)$ and $l(p_1, p_4)$ is divided into the subsegment $s_1$ between $l(p_1, p_2)$ and $l(p_1, p_3)$ and the one $s_2$ between $l(p_1, p_3)$ and $l(p_1, p_4)$. The length of $s$ is the sum of the lengths of $s_1$ and $s_2$. This is by Lemma 2.1

\[ \frac{\lambda_f}{\lambda_a \lambda_d} = \frac{\lambda_b}{\lambda_a \lambda_e} + \frac{\lambda_c}{\lambda_e \lambda_d}, \]

which is (2.3). \qed

A collection of pairwise disjoint geodesic lines in $\mathbb{H}$ is called a \textit{geodesic ideal triangulation} if they divide $\mathbb{H}$ into ideal triangles. Let $\tilde{\Delta}$ be a geodesic ideal triangulation and $\mathcal{P}$ the set of end points of geodesic lines in $\tilde{\Delta}$. Suppose that for each point $p$ of $\mathcal{P}$ a horocycle $h_p$ based at $p$ is given. For a geodesic line $c$ such that both of its end points $p$ and $q$ are in $\mathcal{P}$, we define $\lambda(c) = \lambda(h_p, h_q)$.

\textbf{Lemma 2.3.} Let $c$ be a geodesic line with both end points in $\mathcal{P}$. Suppose that $c$ meets the arcs $a_1, a_2, ..., a_n$ of $\tilde{\Delta}$. Then

\begin{equation}
\lambda(e) = \frac{P_e}{\lambda(a_1) \lambda(a_2) \cdots \lambda(a_n)},
\end{equation}

where $P_e$ is a homogeneous polynomial of degree $n + 1$ in $\{\lambda(a) : a \in \tilde{\Delta}\}$ with positive integer coefficients.

\textbf{Proof.} We prove the lemma by the induction on $n$. Let $p$ and $q$ be the end points of $c$. We regard $e$ as a directed line from $p$ to $q$ and suppose that $c$ meets $a_1, a_2, ..., a_n$ in this order. See Figure 2.2. Let $q_L$ and $q_R$ be the end points of $a_n$, chosen so that $q_L$ lies on the left of $e$. Since $a_n$ is the last arc in $\tilde{\Delta}$ which meets $e$, $b = l(q_R, q)$ and $c = l(q_L, q)$ are arcs of $\tilde{\Delta}$.

If $n = 1$, then $a = l(p, q_R)$ and $d = l(p, q_L)$ are arcs of $\tilde{\Delta}$, too. Then by Lemma 2.2

\[ \lambda(e) = \frac{\lambda(a) \lambda(c) + \lambda(b) \lambda(d)}{\lambda(a_1)}. \]

So the lemma is true for this case.

If $n > 1$, then there exists an $m < n - 1$ such that $a_{m+1}, ..., a_n$ have $q_L$ or $q_R$ as a common end point. Without loss of generality we assume that the common end point
is $q_L$. See Figure 2.2. Then $a = l(p, q_R)$ meets $a_1, ..., a_{n-1}$ and $d = l(p, q_L)$ meets $a_1, ..., a_m$. Therefore, assuming that

$$\lambda(a) = \frac{P_a}{\lambda(a_1)\lambda(a_2) \cdots \lambda(a_{n-1})} \quad \text{and} \quad \lambda(d) = \frac{P_d}{\lambda(a_1)\lambda(a_2) \cdots \lambda(a_m)}$$

with homogeneous polynomials $P_a$ of degree $n$ and $P_d$ of degree $m+1$, we have

$$\lambda(e) = \frac{\lambda(a)\lambda(c) + \lambda(b)\lambda(d)}{\lambda(a_n)} = \frac{P_a\lambda(c) + P_d\lambda(b)\lambda(a_{m+1}) \cdots \lambda(a_n)}{\lambda(a_1)\lambda(a_2) \cdots \lambda(a_{n-1})}.$$ 

The numerator of the last expression is a homogeneous polynomial of degree $n + 1$ with positive integer coefficients. \hfill \square

![Figure 2.2](image)

### § 3. Coordinates for the Teichmüller space of a once punctured surface

#### § 3.1. Teichmüller space of a once punctured surface

Let $F_g$ denote the oriented closed surface of genus $g \geq 1$ and $p$ a point of $F_g$. Let $\hat{F}$ denote the punctured surface $F_g - \{p\}$. The fundamental group $G_{g,1}$ of $\hat{F}$ has the following presentation:

$$G_{g,1} = \langle a_1, b_1, ..., a_g, b_g, d : (\prod_{k=1}^{g} a_kb_k a_k^{-1} b_k^{-1})d = 1 \rangle.$$ 

A point of the *Teichmüller space* $T = T_{g,1}$ is a class of faithful and finite covolume Fuchsian representations of $G_{g,1}$ into $SL(2, \mathbb{R})$. Points of $T$ are represented by *marked* groups $\Gamma_m$, where $\Gamma$ is a Fuchsian group and $m : G_{g,1} \rightarrow \Gamma$ is an isomorphism. Let $\mathcal{P}(\Gamma)$ denote the set of all parabolic fixed points of $\Gamma$. Then $(\mathbb{H} \cup \mathcal{P}(\Gamma))/\Gamma$ is a closed surface. We denote this surface by $\mathbb{H}/\Gamma$. 
§3.2. λ-length of an ideal arc

An ideal arc \( c \) of the pointed surface \((F,p)\) is a homotopically nontrivial path joining \( p \) to itself in \( \hat{F} \). An ideal arc \( c \) is simple if \( c \cap \hat{F} \) is a simple arc.

We fix a positive number \( \alpha \). For each point \( \Gamma_m \) of \( \mathcal{T}_{g,1} \), \( D=m(d) \) is a parabolic transformation in \( \Gamma \). Let \( p \) be the fixed point of \( D \). We choose a horocycle \( h \) based at \( p \) so that \( D \) acts on \( h \) by the translation of distance \( \alpha \). Let \( \mathcal{H}(\Gamma)=\{ \gamma(h) : \gamma \in \Gamma \} \), a \( \Gamma \)-invariant set of horocycles.

By Nielsen’s theorem [5, Satz V.9], the marking \( m \) of \( \Gamma_m \) is induced by an orientation preserving homeomorphism

\[
f_{m} : \hat{F} \to \mathbb{H}/\Gamma,
\]

which extends to a homeomorphism of \( F \) onto \( \mathbb{H}/\Gamma \). We denote this map again by \( f_{m} \).

Let \( c \) be an ideal arc of \((F,p)\) and send it by \( f_{m} \) to an arc connecting the puncture on \( \mathbb{H}/\Gamma \) to itself. A lift of this arc to \( \mathbb{H} \) connects two parabolic fixed points \( p_1 \) and \( p_2 \) of \( \Gamma \). Let \( h_1, h_2 \) be horocycles of \( \mathcal{H}(\Gamma) \) based at \( p_1 \) and \( p_2 \). We define

\[
\lambda(c, \Gamma_m) = \lambda(h_1, h_2)
\]

and call it the λ length of \( c \) with respect to \( \Gamma_m \). The value \( \lambda(c, \Gamma_m) \) does not depend on the choice of a lift of \( f(c) \).

§3.3. Penner’s coordinates for the Teichmüller space

An ideal triangulation \( \Delta = (c_1, c_2, ..., c_q) \) of \( \hat{F} \) is a maximal system of simple ideal arcs of \((F,p)\) such that

1. \( c_i \) and \( c_j \) are not homotopic in \( F \) relative to \( p \), and
2. \( c_i \) and \( c_j \) do not intersect in \( \hat{F} \),

if \( i \neq j \). Since \( \Delta \) is a maximal system, each complementary component of arcs in \( \Delta \) is bounded by three ideal arcs. We call the component a triangle in \( \Delta \). The number \( q \) of ideal arcs in \( \Delta \) necessarily equals \( 6g-3 \) and the number of ideal triangles is \( 4g-2 \).

Let \( \Gamma_m \in \mathcal{T}_{g,1} \). Then \( f_m(\Delta) = (f_m(c_1),...,f_m(c_q)) \) is an ideal triangulation of \((\mathbb{H}/\Gamma, f_m(p))\). We deform \( f(c_i \cap \hat{F}) \) in its homotopy class into a geodesic arc in \( \mathbb{H}/\Gamma \). Then we obtain a geodesic ideal triangulation \( \Delta(\Gamma_m) \) of \( \hat{F} \).

We define a map \( \lambda_{\Delta} : \mathcal{T}_{g,1} \to \mathbb{R}^q_+ \) by

\[
(\lambda_1, ..., \lambda_q) = \lambda_{\Delta}(\Gamma_m) = (\lambda(e_1, \Gamma_m), ..., \lambda(e_q, \Gamma_m)).
\]
Let \( \{T_{1}, T_{2}, ..., T_{4g-2}\} \) be the set of ideal triangles in \( \Delta \) and \( (c_{i1}, c_{i2}, c_{i3}) \) be the sides of \( T_{i}, i = 1, ..., 4g-2 \). Then \( \{c_{i1}, c_{i2}, c_{i3}\} \) is a subset of \( \Delta \). Let \( \lambda_{ik} = \lambda(c_{ik}, \Gamma_{m}), k = 1, 2, 3 \). The following theorem is an immediate consequence of Lemma 2.1 and it shows that the image of \( \lambda_{\Delta} \) is a real algebraic variety determined by a zero locus of an algebraic equation. See also [3, Section 5.1], where the equation (3.1) in the theorem is obtained as a limit of real algebraic representations for Teichmüller spaces of surfaces with cone points.

**Theorem 3.1.** For all \( \Gamma_{m} \in \mathcal{T}_{g,1} \),

\[
(3.1) \quad \sum_{i=1}^{4g-2} \left( \frac{\lambda_{i1}}{\lambda_{i2}\lambda_{i3}} + \frac{\lambda_{i2}}{\lambda_{i1}\lambda_{i3}} + \frac{\lambda_{i3}}{\lambda_{i1}\lambda_{i2}} \right) = \alpha.
\]

**Proof.** To simplify the notation, we identify \( \dot{F} \) with \( \mathbb{H}/\Gamma \) and \( \Delta \) with \( \Delta(\Gamma_{m}) \). We consider a small circle \( \beta \) around the puncture, positively directed with respect to the orientation of \( F_{g} \), and let \( S_{1}, S_{2}, ..., S_{r}, r = 3(4g-2) \), be the triangles in \( \Delta \) which \( \beta \) meets in this order. We may assume that \( \Gamma \) contains the matrix

\[
P = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}.
\]

\( P \) acts on the horocycle \( h = \{z : \text{Im}[z] = 1\} \) as the translation \( z \rightarrow z + \alpha \). Let \( \tilde{S}_{i} \) be a lift of \( S_{i} \), chosen so that \( \tilde{S}_{i} \) has vertices \( \infty, p_{i-1} \) and \( p_{i} \) with \( 0 = p_{0} < p_{1} < p_{2} < \cdots < p_{r} = P(0) \).

Let \( h_{i} \) be the horocycle of \( \mathcal{H}(\Gamma) \) based at \( p_{i}, i = 0, 1, ..., r \). Then by Lemma 2.2,

\[
(3.2) \quad \alpha = \sum_{i=1}^{r} (p_{i} - p_{i-1}) = \sum_{i=1}^{r} \frac{\lambda(h_{i-1}, h_{i})}{\lambda(h, h_{i-1})\lambda(h, h_{i})}.
\]

Since each triangle \( T_{i} \) meets the circle \( \beta \) at three different ends, \( T_{i} \) appears three times in the sequence \( S_{1}, ..., S_{r} \), and then contributes the term

\[
\frac{\lambda_{i1}}{\lambda_{i2}\lambda_{i3}} + \frac{\lambda_{i2}}{\lambda_{i1}\lambda_{i3}} + \frac{\lambda_{i3}}{\lambda_{i1}\lambda_{i2}}
\]

to the right-hand side of (3.2). Therefore we obtain (3.1). \( \square \)

The equation (3.1) can be written as

\[
(3.3) \quad P_{\Delta}(\lambda_{1}, \lambda_{2}, ..., \lambda_{q}) - \alpha \lambda_{1}\lambda_{2} \cdots \lambda_{q} = 0,
\]
where \( P_{\Delta}(\lambda_1, \lambda_2, ..., \lambda_q) \) is a sum of \( 12g - 6 \) monomials of degree \( q - 1 \). We define
\[
\mathcal{A}_{\Delta} = \{ (\lambda_1, ..., \lambda_q) \in \mathbb{R}_+^q : P_{\Delta}(\lambda_1, \lambda_2, ..., \lambda_q) - \alpha \lambda_1 \lambda_2 \cdots \lambda_q = 0 \}.
\]
The map \( \lambda_{\Delta} \) above is the restriction to \( T_{g,1} \) of the real analytic diffeomorphism from the decorated Teichmüller space \( \tilde{T}_{g,1} \) to \( \mathbb{R}_+^q \) in [4, Theorem 3.1]. Hence
\[
(3.4) \quad \lambda_{\Delta} : T_{g,1} \rightarrow \mathcal{A}_{\Delta}
\]
is also a real analytic diffeomorphism.

**Theorem 3.2.** Let \( \Delta = (c_1, c_2, ..., c_q) \) be an ideal triangulation of \( (F,p) \). Let \( \Gamma_m \) be an arbitrary point of \( T_{g,1} \) and define \( \lambda_i = \lambda(c_i, \Gamma_m) \), \( i = 1, ..., q \). Then for any ideal arc \( c \) in \( \hat{F} \),
\[
(3.5) \quad \lambda(c, \Gamma_m) = \frac{P_c(\lambda_1, \lambda_2, ..., \lambda_q)}{\lambda_1^{m_1} \lambda_2^{m_2} \cdots \lambda_q^{m_q}},
\]
where \( m_i \) is the geometric intersection number of \( c \) and \( c_i \) in \( \hat{F} \). \( P_c \) is a homogeneous polynomial of degree \( m_1 + m_2 + \cdots + m_q + 1 \) with positive integer coefficients.

**Proof.** Let \( \hat{\Delta}(\Gamma_m) \) be the lift of \( \Delta(\Gamma_m) \). Take a lift of \( f(c) \) and let \( e \) be the geodesic line which connects the end points of the lift. Then Lemma 2.3 applied to \( e \) and \( \hat{\Delta} \) shows that \( \lambda(e, \Gamma_m) \) has the form (3.5). \( \square \)

\section*{§ 4. Integer solutions of a Diophantine equation}

Let \( M_{g,1} \) denote the mapping class group of \( \hat{F} \). Each element \( \varphi \) of \( M_{g,1} \) acts on \( T_{g,1} \) by changing the marking \( m \) to \( m \circ \varphi_*^{-1} \), where \( \varphi_* \) is the automorphism of the surface group \( G_{g,1} \) induced by \( \varphi \).

We fix an ideal triangulation \( \Delta = (c_1, c_2, ..., c_q) \) of \( (F,p) \) and consider Penner’s coordinate-system \( \lambda_{\Delta} : T_{g,1} \rightarrow \mathbb{R}_+^q \). Then, by definition,
\[
\lambda_{\Delta}(\varphi(\Gamma_m)) = (\lambda(\varphi^{-1}(c_1), \Gamma_m), ..., \lambda(\varphi^{-1}(c_q), \Gamma_m)).
\]
By [4, Corollary 7.4] each entry \( \lambda(\varphi^{-1}(c_i), \Gamma_m) \) is a rational function. Moreover, Theorem 3.2 shows that it is of degree 1 of the form as is described in (3.5). Therefore we obtain a rational map \( R_{\varphi} : \mathbb{R}^q \rightarrow \mathbb{R}^q \). Penner showed in [4] that the correspondence \( \varphi \rightarrow R_{\varphi} \) is a faithful representation of \( M_{g,1} \) to a group of rational transformations in \( \mathbb{R}^q \). Since \( R_{\varphi} \circ \lambda_{\Delta} = \lambda_{\Delta} \circ \varphi \), \( R_{\varphi} \) preserves the algebraic equation (3.3).
If $\alpha = 12g - 6$, then $(\lambda_1, \lambda_2, ..., \lambda_q) = (1, 1, ..., 1)$ is a solution of the equation (3.3). For all $\varphi \in \mathcal{M}_{g,1}$, the entries of $R_{\varphi}$ are of the form as in (3.5). Therefore $R_{\varphi}(1, 1, ..., 1)$ are positive integer solutions of (3.3).

If $\Lambda = (\lambda_1, ..., \lambda_q)$ is fixed by a $\varphi \in \mathcal{M}_{g,1}$, then $\varphi$ is the class of a conformal automorphism of the Riemann surface corresponding to $\Lambda$. By Wiman’s theorem the order of the group of conformal automorphisms of a Riemann surface of type $(g, 1)$, $g \geq 1$, does not exceed $2(2g + 1)$ if $g > 1$ or 3 if $g = 1$ [2]. This means that only a finite number of elements in $\mathcal{M}_{g,1}$ send $(1, 1, 1)$ to itself, and hence the orbit space $\{R_{\varphi}(1, 1, 1) : \varphi \in \mathcal{M}_{g,1}\}$ contains infinitely many points.

**Proposition 4.1.** If $\alpha = 12g - 6$, then there are infinitely many positive integer solutions of the equation (3.3).

Since $R_{\varphi}$ with $\varphi \in \mathcal{M}_{g,1}$ is a rational map, an integer solution of (3.3) may not necessarily be sent to an integer solution by $R_{\varphi}$. This part is different from the case of Markoff maps which are polynomial maps of positive integer coefficients and hence we cannot employ Markoff’s method which concludes that the set of all positive integer solutions of the Markoff equation coincides with the orbit of $(1, 1, 1)$ under all Markoff maps. The author does not know whether all positive integer solutions of (3.3) are in the orbit of $(1, 1, ..., 1)$ under $\mathcal{M}_{g,1}$. In [3] positive integer solutions of a Diophantine equation which arises from the Teichmüller space of twice punctured torus are considered. For this case there are positive integer solutions such that their orbits under the mapping class group $\mathcal{M}_{1,2}$ contain non integer points.

Let $\Delta_1$ and $\Delta_2$ be two ideal triangulations of $(F_g, p)$. Then $\lambda_{\Delta_2} \circ \lambda_{\Delta_1}^{-1}$ is a rational map whose entries are of the form (3.5). Let $(1, 1, ..., 1)$ be a solution of

$$P_{\Delta_1} - (12g - 6)\lambda_1\lambda_2 \cdots \lambda_q = 0,$$

which is the equation (3.3) with $\Delta = \Delta_1$ and $\alpha = 12g - 6$. Then $\lambda_{\Delta_2} \circ \lambda_{\Delta_1}^{-1}(1, 1, ..., 1)$ is a positive integer solution of

$$(4.1) \quad P_{\Delta_2} - (12g - 6)\lambda_1\lambda_2 \cdots \lambda_q = 0.$$

The author does not know whether this point belongs to the orbits of $(1, 1, ..., 1)$ as a solution of (4.1) under $\mathcal{M}_{g,1}$.

**References**


