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Asymptotics of degenerating Eisenstein series

Dedicated to Takahide Kurokawa and Kimio Miyajima
on the occasion of their 60th birthdays

By

Kunio Obitsu*

Abstract

We give some estimates for the asymptotic orders of degenerating Eisenstein series for certain families of degenerating punctured Riemann surfaces, motivated by the question of identifying L_2-cohomology of the Takhtajan-Zograf metric that is originally asked by To and Weng.

§ 1. Introduction

We consider the Teichmüller space $T_{g,n}$ and the associated Teichmüller curve $\mathcal{T}_{g,n}$ of Riemann surfaces of type (g, n) (i.e., Riemann surfaces of genus g and with $n > 0$ punctures). We will assume that $2g - 2 + n > 0$, so that each fiber of the holomorphic projection map $\pi : \mathcal{T}_{g,n} \to T_{g,n}$ is stable or equivalently, it admits the complete hyperbolic metric of constant sectional curvature -1. The kernel of the differential $T\mathcal{T}_{g,n} \to TT_{g,n}$ forms the so-called vertical tangent bundle over $\mathcal{T}_{g,n}$, which is denoted by $T^V\mathcal{T}_{g,n}$. The hyperbolic metrics on the fibers induce naturally a Hermitian metric on $T^V\mathcal{T}_{g,n}$.

In the study of the family of $\bar{\partial}_k$-operators acting on the k-differentials on Riemann surfaces (i.e., cross-sections of $\left(T^V\mathcal{T}_{g,n}\right)^{-k}\big|_{\pi^{-1}(s)} \to \pi^{-1}(s),\ s \in T_{g,n}$), Takhtajan and Zograf introduced in [11] a Kähler metric on $T_{g,n}$, which is known as the Takhtajan-Zograf metric. In [11], they showed that the Takhtajan-Zograf metric is invariant under the natural action of the Teichmüller modular group $\mathrm{Mod}_{g,n}$ and it satisfies the following

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*Faculty of Science, Kagoshima University, Kagoshima 890-0065, Japan.

remarkable identity on $T_{g,n}$:

$$c_1(\lambda_k, \|\cdot\|_{Q,k}) = \frac{6k^2-6k+1}{12\pi^2}\,\omega_{\mathrm{WP}} - \frac{1}{9}\omega_{\mathrm{TZ}}.$$

Here $\lambda_k = \det(\operatorname{ind}\bar{\partial}_k) = \bigwedge^{\max}\operatorname{Ker}\bar{\partial}_k \otimes (\bigwedge^{\max}\operatorname{Coker}\bar{\partial}_k)^{-1}$ denotes the determinant line bundle on $T_{g,n}$, $\|\cdot\|_{Q,k}$ denotes the Quillen metric on λ_k, and ω_{WP}, ω_{TZ} denote the Kähler form of the Weil-Petersson metric, the Takhtajan-Zograf metric on $T_{g,n}$ respectively. In [13], Weng studied the Takhtajan-Zograf metric in terms of Arakelov intersection, and he proved that $\frac{4}{3}\omega_{\mathrm{TZ}}$ coincides with the first Chern form of an associated metrized Takhtajan-Zograf line bundle over the moduli space $\mathcal{M}_{g,n} = T_{g,n}/\mathrm{Mod}_{g,n}$. Recently, Wolpert [16] gave a natural definition of a Hermitian metric on the Takhtajan-Zograf line bundle whose first Chern form gives $\frac{4}{3}\omega_{\mathrm{TZ}}$. Furthermore, we can observe that in the second term of the asymptotic expansion of the Weil-Petersson metric near the boundary of $\mathcal{M}_{g,n}$, the Takhtajan-Zograf metrics on the boundary moduli spaces could appear (see [7]).

We propose a program of identifying L_2-cohomology of $\mathcal{M}_{g,n}$ with respect to the Takhtajan-Zograf metric $H^*(\mathcal{M}_{g,n}, \omega_{TZ})$. Originally, Saper ([8]) applied Masur's formula ([4]) to show that L_2-cohomology of $\overline{\mathcal{M}}_{g,0}$ with respect to the Weil-Petersson metric $H^*(\mathcal{M}_{g,0}, \omega_{WP})$ is naturally isomorphic to $H^*(\overline{\mathcal{M}}_{g,0}, \mathbf{R})$. However, it is disappointing that the results for the asymptotics of the Takhtajan-Zograf metrics in [6] are not sufficient for us to determine $H^*(\mathcal{M}_{g,n}, \omega_{TZ})$.

In the present paper, we prove some estimates for the degenerating orders of Eisenstein series for certain families of degenerating punctured Riemann surfaces, which may be an important step for calculating $H^*(\mathcal{M}_{g,n}, \omega_{TZ})$. It should be noted that there are already some results for the behaviors of degenerating Eisenstein series ([2], [3], [5], [9]).

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§ 2. Main Theorems

§ 2.1. Settings and notation

For simplicity of exposition, we consider a degenerating family $\{S_l\}$ of Riemann surfaces of type $(g,1)$ with two zero-homologous pinching geodesics γ_1 and γ_2 which divide the surface S_l into three components S_l^1, S_l^2, S_l^3: the geodesic γ_1 divides S_l^1 from S_l^2, the geodesic γ_2 divides S_l^2 from S_l^3, and S_l^1 has the unique puncture. (It should be noted that all claims in propositions, theorems, etc. are easily generalized to the case of any degenerating family of hyperbolic surfaces of finite type with at least one puncture. In some of the statements, we will give remarks for the general case.) The vector-valued parameter l varies around the origin in the Euclidian space $\mathbf{R}^{6g-4}$, where $l = 0$ represents the unique degenerate surface S_0 in the family. The limit surface S_0 consists of three components S_0^1, S_0^2, S_0^3 which are the limits of S_l^1, S_l^2, S_l^3 respectively as $l \to 0$. Let q_j be the node shared by S_0^j and S_0^{j+1} $(j = 1, 2)$. A puncture the smooth surface in the degenerate family originally has will be called an *old* puncture, for simplicity. It should be noted that the degenerating family can be described by the modified infinite-energy harmonic maps $f^l : S_0 \longrightarrow S_l \backslash \{\gamma_1, \gamma_2\}$, which are introduced by S. Wolpert ([15]).

Let $L_l(\gamma)$ be the hyperbolic length of a simple closed geodesic γ on S_l. For $0 \le k \le 1$ and $j = 1, 2$, set

$$N_{\gamma_j}(k) = \left\{ p \in S_l \;\middle|\; d_l(p, \gamma_j) \le k \,\sinh^{-1}\left(1 \Big/ \sinh \frac{L_l(\gamma_j)}{2}\right) \right\},$$

the collar neighborhood around γ_j in S_l, where $d_l(\cdot, \cdot)$ denotes the hyperbolic distance on S_l. Here we remark that

$$\sinh^{-1}\left(1/\sinh \frac{x}{2}\right) = -\log x + 2\log 2 + O(x^2), \quad x \to 0, \tag{2.1}$$

which will be essentially used in the proofs of Lemma 2.3 and Lemma 2.6.

For $a \ge 1$, the a-cusp region $C_j(a)$ $(\subset S_0^j \cup S_0^{j+1})$ around the node q_j is the union of two copies of $\langle z \mapsto z + 1 \rangle \backslash \{z \in H \mid \operatorname{Im} z \ge a\}$, equipped with the metric $ds^2 = (dy^2 + dx^2)/y^2$, where $H := \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$ is the upper half plane.

Let $(f^l)^* \Delta_l$ denote the pull-back of the negative hyperbolic Laplacian Δ_l on S_l by f^l, that is, for a C^2-function h on S_0,

$$(f^l)^* \Delta_l \,(h) = \Delta_l \left(h \circ (f^l)^{-1} \right) \circ f^l.$$

Let Δ_0 denote the negative hyperbolic Laplacian on S_0. Then, it is known that $(f^l)^* \Delta_l$ converges to Δ_0 uniformly on any compact subset of S_0 in the C^3-norm (see [15]). And, for a function g on S_l, the pull-back of g by f^l is defined as

$$(f^l)^* g = g \circ f^l.$$

It should be noted that a C^2-function g on S_l satisfies

$$(f^l)^* \Delta_l \left((f^l)^* g \right) = \Delta_l(g) \circ f^l, \tag{2.2}$$

which will be used in the proof of Lemma 2.7.

§ 2.2. The counting function of orbits

Let Γ_l be a Fuchsian group uniformizing S_l such that $S_l \simeq H/\Gamma_l$. We normalize it such that Γ_l contains a parabolic element $z \mapsto z+1$. A cyclic group generated by the parabolic element is denoted by Γ_∞. Then the Eisenstein series for Γ_l associated to the unique puncture is expressed as

$$E^l(z,s) = \sum_{\delta \in \Gamma_\infty \backslash \Gamma_l} (\operatorname{Im} \delta z)^s, \quad z \in H,\ \operatorname{Re} s > 1.$$

Here for any z in H and any equivalent class $[\delta]$ in $\Gamma_\infty \backslash \Gamma_l$, we can select the unique representative $\hat{\delta}$ for $[\delta]$ such that $-\frac{1}{2} \leq \operatorname{Re} \hat{\delta} z < \frac{1}{2}$. Such $\hat{\delta} = \hat{\delta}(z, [\delta])$ will be called the *canonical* representative.

$E^l(z,s)$ is invariant under the action of Γ_l. Thus it can be considered as a function on S_l. Moreover, it is well known that the Eisenstein seires satisfies

$$(\Delta - s(s-1))\, E^l(z,s) = 0, \quad z \in H,\ \operatorname{Re} s > 1, \tag{2.3}$$

which will play a crucial role in the proof of Lemma 2.7. Here $\Delta := 4\, (\operatorname{Im} z)^2 \frac{\partial^2}{\partial z \partial \bar{z}}$ is the negative hyperbolic Laplacian on H , invariant under Γ_l, and thus it naturally descends to Δ_l on S_l.

Now we are ready to present a new way to study the asymptotics of the Eisenstein series. When $\operatorname{Im} z < 1$ and z is not equivalent to any point of $\{w \in H |\ \operatorname{Im} w > 1\}$ under the action of $\Gamma_\infty \backslash \Gamma_l$, it is easy to see that for $[\delta]$ in $\Gamma_\infty \backslash \Gamma_l$, $\operatorname{Im} \hat{\delta}(z) = e^{-d(h, \hat{\delta} z)}$, where $d(\cdot,\cdot)$ denotes the hyperbolic distance in H and $h = \{w \in H | -\frac{1}{2} \leq \operatorname{Re} w < \frac{1}{2}, \operatorname{Im} w = 1\}$.

We introduce two counting functions of orbits of z with $\operatorname{Im} z < 1$,

$$\Pi_l(h,z,t) := \sharp\{[\delta] \in \Gamma_\infty \backslash \Gamma_l \mid d(h, \hat{\delta} z) \leq t\},$$
$$\Pi_l(z,t) := \sharp\{[\delta] \in \Gamma_\infty \backslash \Gamma_l \mid d(i, \hat{\delta} z) \leq t\},$$

where $\hat{\delta}$ is the *canonical* representative. Here we should remark that $d(i, \hat{\delta} z) = \min_{\delta \in [\delta]} d(i, \delta z)$.

For z with $\operatorname{Im} z < 1$ not equivalent to any point of $\{w \in H |\ \operatorname{Im} w > 1\}$ under the action of $\Gamma_\infty \backslash \Gamma_l$, we can observe

$$E^l(z,s) = \int_0^\infty e^{-st}\ d\Pi_l(h,z,t).$$

We will state a famous property of $\Pi_l(z,t)$ as in the form suited to our purpose.

Proposition 2.1. *There exists an absolute constant U such that for $z \in H$ with $\operatorname{Im} z < 1$, the following estimate holds:*

$$\Pi_l(z,t) \leq U e^t \qquad \text{for any } t \geq 0 \text{ and any } \Gamma_l.$$

Proof. Our proof is based on the discussion in [12] p.516. Let $B(p,r)$ denote a hyperbolic ball centered at p with radius r in H. Now the collar lemma assures us that in any hyperbolic surface with at least one puncture, each puncture has a horocyclic neighborhood with area 2 (see [10]). Then we can find a universal constant $\varepsilon > 0$ such that orbits $B(\delta i, \varepsilon)$ for $\delta \in \Gamma_l$ are mutually disjoint for any Γ_l. Because if $d(\delta i, z) \le t$ then $B(\delta i, \varepsilon) \subset B(z, t+\varepsilon)$, we have

$$\begin{aligned}\Pi_l(z,t) &= \sharp\{[\delta] \in \Gamma_\infty \backslash \Gamma_l \mid d((\hat{\delta})^{-1} i, z) \le t\} \\ &\le \sharp\{[\delta] \in \Gamma_\infty \backslash \Gamma_l \mid (\hat{\delta})^{-1}(B(i,\varepsilon)) \subset B(z, t+\varepsilon)\} \\ &\le \frac{|B(z,t+\varepsilon)|}{|B(i,\varepsilon)|} = \sinh^2\Big(\frac{t+\varepsilon}{2}\Big)\Big/\sinh^2\frac{\varepsilon}{2} \\ &\le \frac{e^\varepsilon}{2\sinh^2\frac{\varepsilon}{2}} e^t \quad \text{for } t \ge 0.\end{aligned}$$

Here $|\cdot|$ denotes the hyperbolic area in H. □

Proposition 2.2. *Let $s > 1$. Let $z \in H$ with $\operatorname{Im} z < 1$ be not equivalent to any point of $\{w \in H|\ \operatorname{Im} w > 1\}$ under the action of $\Gamma_\infty \backslash \Gamma_l$. Then we obtain*

$$\Pi_l(z,t) \le \Pi_l(h,z,t) \le \Pi_l(z,t+1),$$

$$E^l(z,s) = s\int_0^\infty e^{-st}\Pi_l(h,z,t)\ dt,$$

$$s\int_0^\infty e^{-st}\ \Pi_l(z,t)\ dt \le E^l(z,s) \le s\int_0^\infty e^{-st}\ \Pi_l(z,t+1)\ dt.$$

Proof. Because $d(i,\delta z) \le d(h,\delta z) + 1$, it follows from Proposition 2.1 that

$$\Pi_l(h,z,t) \le \Pi_l(z,t+1) \le eUe^t.$$

Then, integrations by parts and Proposition 2.1 provide

$$\begin{aligned}E^l(z,s) &= \int_0^\infty e^{-st}\ d\Pi_l(h,z,t) = [e^{-st}\Pi_l(h,z,t)]_0^\infty + s\int_0^\infty e^{-st}\Pi_l(h,z,t)\ dt \\ &= s\int_0^\infty e^{-st}\Pi_l(h,z,t)\ dt \le s\int_0^\infty e^{-st}\Pi_l(z,t+1)\ dt.\end{aligned}$$

This is the right-hand inequality in the statement.

Next we will prove the left-hand inequality. Since $d(h,\delta z) \le d(i,\delta z)$, it is easy to see that

$$\begin{aligned}\Pi_l(h,z,t) &\ge \Pi_l(z,t), \\ E^l(z,s) &= s\int_0^\infty e^{-st}\Pi_l(h,z,t)\ dt \\ &\ge s\int_0^\infty e^{-st}\Pi_l(z,t)\ dt.\end{aligned}$$

This completes the proof. □

§ 2.3. Upper bounds for degenerating Eisenstein series

We are going to present upper bounds for Eisenstein series on the components S_l^2 and S_l^3.

Lemma 2.3. *Assume* $\mathrm{Re}\, s > 1$. *There exists an absolute constant* $M_2(\mathrm{Re}\, s)$ *depending only on* $\mathrm{Re}\, s$ *such that for* $L_l(\gamma_1), L_l(\gamma_2) < 2\sinh^{-1} 1$ *and* $0 \leq k \leq 1$, *then*

$$\begin{aligned} |E^l(z,s)| &\leq M_2(\mathrm{Re}\, s)\ L_l(\gamma_1)^{(1+k)(\mathrm{Re}\, s-1)} && \text{on } \partial N_{\gamma_1}(k) \cap S_l^2, \\ |E^l(z,s)| &\leq M_2(\mathrm{Re}\, s)\ L_l(\gamma_1)^{2(\mathrm{Re}\, s-1)} L_l(\gamma_2)^{(1+k)(\mathrm{Re}\, s-1)} && \text{on } \partial N_{\gamma_2}(k) \cap S_l^3. \end{aligned}$$

Proof. Because $|E^l(z,s)| \leq E^l(z, \mathrm{Re}\, s)$ holds, it is enough to show in the case $s > 1$. For z in H, $[z]$ denotes the corresponding point of S_l. (2.1) implies easily that the distance of any curve connecting $[z]$ on $\partial N_{\gamma_1}(k) \cap S_l^2$ and the horocycle $[h]$ is greater than $(1+k)$[the width of half collar]. Therefore we see

$$\Pi_l(h, z, -(1+k)\log L_l(\gamma_1)) = 0.$$

Then Proposition 2.2 yields

$$E^l(z,s) = s\int_{-(1+k)\log L_l(\gamma_1)}^{\infty} e^{-st}\Pi_l(h,z,t)\ dt.$$

By Propositions 2.1 and 2.2, it concludes that

$$\begin{aligned} E^l(z,s) &\leq s\int_{-(1+k)\log L_l(\gamma_1)}^{\infty} e^{-st}\Pi_l(z,t+1)\ dt \\ &\leq s\int_{-(1+k)\log L_l(\gamma_1)}^{\infty} e^{-st} eUe^t\ dt \\ &= eUs\int_{-(1+k)\log L_l(\gamma_1)}^{\infty} e^{-(s-1)t}\ dt \\ &= \frac{eUs}{s-1}\ L_l(\gamma_1)^{(1+k)(s-1)}. \end{aligned}$$

The second case is similar. Just replace $-(1+k)\log L_l(\gamma_1)$ with $-2\log L_l(\gamma_1) - (1+k)\log L_l(\gamma_2)$. □

Corollary 2.4. *Assume as in Lemma 2.3. Then for all* $L_l(\gamma_1), L_l(\gamma_2) < 2\sinh^{-1} 1$ *and all* k *with* $0 \leq k \leq 1$, *it holds that*

$$\begin{aligned} |E^l(z,s)| &\leq M_2(\mathrm{Re}\, s)\ L_l(\gamma_1)^{(1+k)(\mathrm{Re}\, s-1)} && \text{on } S_l^2 - N_{\gamma_1}(k), \\ |E^l(z,s)| &\leq M_2(\mathrm{Re}\, s)\ L_l(\gamma_1)^{2(\mathrm{Re}\, s-1)}\ L_l(\gamma_2)^{(1+k)(\mathrm{Re}\, s-1)} && \text{on } S_l^3 - N_{\gamma_2}(k). \end{aligned}$$

Here $M_2(\mathrm{Re}\, s)$ *is the constant appearing in Lemma 2.3.*

Proof. Because $|E^l(z,s)| \le E^l(z, \operatorname{Re} s)$ holds, it is enough to show the statements for $s > 1$. By (2.3), it is easy to see that $E^l(z,s)$ is subharmonic. The maximal principle for subharmonic functions provides

$$\begin{aligned}\sup_{z \in S_l^2 - N_{\gamma_1}(k)} E^l(z,s) &\le \sup_{z \in S_l^3 \cup S_l^2 - N_{\gamma_1}(k)} E^l(z,s) \\ &= \sup_{z \in \partial N_{\gamma_1}(k) \cap S_l^2} E^l(z,s) \\ &\le M_2(s)\ L_l(\gamma_1)^{(1+k)(s-1)}.\end{aligned}$$

(Remark: even in the case $S_l^3 \cup S_l^2$ has other *old* punctures, our discussion remains valid because $E^l(z,s)$ assumes 0 at the *old* punctures.) The second case is similar. Just use the second inequality in Lemma 2.3. □

We will summarize the special case for $k = 0, 1$ in Corollary 2.4 as follows.

Theorem 2.5. *Assume* $\operatorname{Re} s > 1$. *Then for all* $L_l(\gamma_1), L_l(\gamma_2) < 2\sinh^{-1} 1$, *it holds that*

$$\begin{aligned}
&|E^l(z,s)| \le M_2(\operatorname{Re} s)\ L_l(\gamma_1)^{(\operatorname{Re} s - 1)} && \text{on } S_l^2,\\
&|E^l(z,s)| \le M_2(\operatorname{Re} s)\ L_l(\gamma_1)^{2(\operatorname{Re} s - 1)} && \text{on } S_l^2 - N_{\gamma_1}(1),\\
&|E^l(z,s)| \le M_2(\operatorname{Re} s)\ L_l(\gamma_1)^{2(\operatorname{Re} s - 1)}\ L_l(\gamma_2)^{(\operatorname{Re} s - 1)} && \text{on } S_l^3,\\
&|E^l(z,s)| \le M_2(\operatorname{Re} s)\ L_l(\gamma_1)^{2(\operatorname{Re} s - 1)}\ L_l(\gamma_2)^{2(\operatorname{Re} s - 1)} && \text{on } S_l^3 - N_{\gamma_2}(1).
\end{aligned}$$

Here $M_2(\operatorname{Re} s)$ *is the constant appearing in Lemma 2.3.*

Remark. Corollary 2.4 and Theorem 2.5 have essentially improved the order estimates for the degenerating Eisenstein series in [5] Theorem 1 (2).

§ 2.4. Lower bounds for degenerating Eisenstein series

Now we are ready to present lower bounds for Eisenstein series on the components S_l^2 and S_l^3. Henceforth, the set of points in S_l the injectivity radii of which are greater than $\sinh^{-1} 1$ will be called the *thick part* of S_l.

Lemma 2.6. *Let* $s > 1$. *There exist positive constants* $K_i = K_i(s, \{S_l\})$ $(i = 1, 2, 3)$ *depending only on* s *and the degenerating family* $\{S_l\}$ *such that for* $L_l(\gamma_1), L_l(\gamma_2) < 2\sinh^{-1} 1$ *and* $0 \le k \le 1$, *then*

$$\begin{aligned}
&E^l(z,s) \ge K_1\ L_l(\gamma_1)^{(1+k)s} && \text{on } \partial N_{\gamma_1}(k) \cap S_l^2,\\
&E^l(z,s) \ge K_2\ L_l(\gamma_1)^{2s} L_l(\gamma_2)^{(1-k)s} && \text{on } \partial N_{\gamma_2}(k) \cap S_l^2,\\
&E^l(z,s) \ge K_3\ L_l(\gamma_1)^{2s} L_l(\gamma_2)^{(1+k)s} && \text{on } \partial N_{\gamma_2}(k) \cap S_l^3.
\end{aligned}$$

Proof. We mimic the proof of Lemma 4.2 in [14] p.84. For $z \in H$ with $\operatorname{Im} z < 1$,

$$(\operatorname{Im}\, z)^s \geq e^{-sd(z,h)}.$$

Since $E^l(z,s) = \sum_{\delta \in \Gamma_\infty \backslash \Gamma_l} (\operatorname{Im} \delta z)^s$ is a sum of positive terms over $\Gamma_\infty \backslash \Gamma$-orbits of z, we obtain

$$E^l(z,s) \geq e^{-s\hat{d}(z,h)},$$

where $\hat{d}(z,h)$ denotes the distance from h to the Γ-orbits of z. We should recall two facts here. The first one is (2.1). The second one is that the diameters of the *thick parts* of S_l are bounded by a positive constant D for all small $L_l(\gamma_1), L_l(\gamma_2)$, where D depends only on the degenerating family $\{S_l\}$. (For example, by using the Bers constant we can easily see the second fact. Refer to Theorem 5.2.6 in [1] p.130.) Then, for $z \in \partial N_{\gamma_1}(k) \cap S_l^2$, we can observe that $\hat{d}(z,h) \leq -(1+k)\log L_l(\gamma_1) + D'$. Here D' is a constant depending only on the degenerating family. Then we have

$$E^l(z,s) \geq e^{-s\hat{d}(z,h)} \geq e^{-sD'} L_l(\gamma_1)^{(1+k)s}.$$

The remaining two cases are similar. $\square$

Lemma 2.7. *Let $s > 1$. For $i = 1, 2$, let Ω_i be any region ($\Subset S_0^{i+1}$) containing $\partial C_i(1) \cap S_0^{i+1}$. There exist positive constants $P_i = P_i(s, \Omega_i, \{S_l\})$ depending only on s and Ω_i and the degenerating family $\{S_l\}$ such that for any sufficiently small $L_l(\gamma_1)$, then*

$$\begin{aligned} (f^l)^* E^l(z,s) &\geq P_1\ L_l(\gamma_1)^{2s} && \text{on } \Omega_1, \\ (f^l)^* E^l(z,s) &\geq P_2\ L_l(\gamma_1)^{2s}\ L_l(\gamma_1)^{2s} && \text{on } \Omega_2. \end{aligned}$$

Proof. We will show only the first case. The second case is similar. We set

$$P_l = \inf_{z \in \Omega_1} L_l(\gamma_1)^{-2s} (f^l)^* E^l(z,s).$$

Suppose that there exists a subsequence $l_j \to 0$ such that $\lim_{j\to\infty} P_{l_j} = 0$. Consider the function $P_{l_j}^{-1} L_{l_j}(\gamma_1)^{-2s} (f^{l_j})^* E^{l_j}(z,s)$. By (2.2) and (2.3), we can observe that

$$((f^{l_j})^* \Delta_{l_j} - s(s-1))\ P_{l_j}^{-1} L_{l_j}(\gamma_1)^{-2s} (f^{l_j})^* E^{l_j}(z,s) = 0$$

and

$$\inf_{z \in \Omega_1} P_{l_j}^{-1} L_{l_j}(\gamma_1)^{-2s} (f^{l_j})^* E^{l_j}(z,s) = 1.$$

We choose another region Ω_1' such that $\Omega_1 \Subset \Omega_1' \Subset S_0^2$. Because $((f^{l_j})^*\Delta_{l_j} - s(s-1))$ are uniformly non-degenerate on Ω_1', the Harnack inequality provides

$$\begin{aligned}\sup_{z\in\Omega_1} P_{l_j}^{-1} L_{l_j}(\gamma_1)^{-2s}(f^{l_j})^* E^{l_j}(z,s) &\leq c(\Omega_1,\Omega_1') \inf_{z\in\Omega_1} P_{l_j}^{-1} L_{l_j}(\gamma_1)^{-2s}(f^{l_j})^* E^{l_j}(z,s) \\ &= c(\Omega_1,\Omega_1') < \infty.\end{aligned}$$

Then using the interior Schauder estimate and the diagonal method as in the proof of Theorem 1 in [5], we can have a further subsequence which will be denoted by the same notation such that $P_{l_j}^{-1} L_{l_j}(\gamma_1)^{-2s}(f^{l_j})^* E^{l_j}(z,s)$ and its first and second derivatives converge uniformly on any compact subset of Ω_1 to a nonnegative function $G(z,s)$ and its derivatives respectively. Then $G(z,s)$ satisfies

$$(\Delta_0 - s(s-1))\; G(z,s) = 0$$

and

$$\sup_{z\in\Omega_1} G(z,s) \leq \overline{\lim_{j\to\infty}} \sup_{z\in\Omega_1} P_{l_j}^{-1} L_{l_j}(\gamma_1)^{-2s}(f^{l_j})^* E^{l_j}(z,s) \leq c(\Omega_1,\Omega_1') < \infty.$$

Now it should be noted that $\Omega_1 \supset (f^l)^{-1}(\partial N_{\gamma_1}(1) \cap S_l^2)$ for any sufficiently small l because $(f^l)^{-1}(\partial N_{\gamma_1}(1) \cap S_l^2)$ converges to $\partial C_1(1) \cap S_0^2$ as $l \to 0$. We choose another region $\Omega_1'' \Subset \Omega_1$ such that $\Omega_1'' \supset (f^l)^{-1}(\partial N_{\gamma_1}(1) \cap S_l^2)$ for any sufficiently small l. Then we have

$$\begin{aligned}\sup_{z\in\Omega_1} G(z,s) &\geq \sup_{z\in\Omega_1''} G(z,s) \\ &= \lim_{j\to\infty} \sup_{z\in\Omega_1''} P_{l_j}^{-1} L_{l_j}(\gamma_1)^{-2s}(f^{l_j})^* E^{l_j}(z,s) \\ &\geq \lim_{j\to\infty} \sup_{z\in(f^l)^{-1}(\partial N_{\gamma_1}(1)\cap S_l^2)} P_{l_j}^{-1} L_{l_j}(\gamma_1)^{-2s}(f^{l_j})^* E^{l_j}(z,s) \\ &\geq \lim_{j\to\infty} \inf_{z\in(f^l)^{-1}(\partial N_{\gamma_1}(1)\cap S_l^2)} P_{l_j}^{-1} L_{l_j}(\gamma_1)^{-2s}(f^{l_j})^* E^{l_j}(z,s) \\ &= \lim_{j\to\infty} \inf_{w\in\partial N_{\gamma_1}(1)\cap S_l^2} P_{l_j}^{-1} L_{l_j}(\gamma_1)^{-2s} E^{l_j}(w,s) \\ &\geq \lim_{j\to\infty} P_{l_j}^{-1} K_1 = +\infty.\end{aligned}$$

Here we used the first inequality in Lemma 2.6. This is a contradiction. □

Remark. All claims in Lemmas 2.6, 2.7 remain valid even in the case where the components S_0^2, S_0^3 have other *old* punctures. However, care for such additional *old* punctures is needed in the proof of the following theorem.

Theorem 2.8. *Let $s > 1$. For all sufficiently small $L_l(\gamma_1), L_l(\gamma_2)$, it holds that*

$$\begin{aligned} E^l(z,s) &\geq Q_1\ L_l(\gamma_1)^{2s} && \text{on } S_l^2 - f^l(C_2(a)),\\ E^l(z,s) &\geq Q_2\ L_l(\gamma_1)^{2s} L_l(\gamma_2)^s && \text{on } S_l^2 \cap N_{\gamma_2}(1),\\ E^l(z,s) &\geq Q_3\ L_l(\gamma_1)^{2s} L_l(\gamma_2)^{2s} && \text{on } S_l^3. \end{aligned}$$

Here $Q_1 = Q_1(s, a, \{S_l\})$ is a positive constant depending only on s, a and the degenerating family $\{S_l\}$. $Q_i = Q_i(s, \{S_l\})$ $(i = 2, 3)$ are positive constants depending only on s and the degenerating family $\{S_l\}$.

Remark. In the case where S_l^i has additional *old* punctures $(i = 2, 3)$, we have to replace S_l^i with $S_l^i - f^l$(the union of all neighborhoods of old punctures), and all Q_i's depend on all the removed neighborhoods.

Proof. First, we will show the first inequality. Set

$$Q_l = \inf_{z \in (f^l)^{-1}(N_{\gamma_1}(1)) \cap S_0^2} L_l(\gamma_1)^{-2s} (f^l)^* E^l(z,s) = \inf_{w \in N_{\gamma_1}(1) \cap \bar{S}_l^2} L_l(\gamma_1)^{-2s} E^l(w,s).$$

Due to the first inequality in Lemma 2.7, all we have to prove is that Q_l is larger than a positive constant for all small l. Assume that there exists a subsequence $l_j \to 0$ such that $\lim_{j\to\infty} Q_{l_j} = 0$. For each j, we can find a point $w_j \in N_{\gamma_1}(1) \cap \bar{S}_l^2$ such that

$$L_{l_j}(\gamma_1)^{-2s} E^{l_j}(w_j, s) = \inf_{w \in N_{\gamma_1}(1) \cap \bar{S}_{l_j}^2} L_{l_j}(\gamma_1)^{-2s} E^{l_j}(w,s).$$

If w_j is not on the geodesic γ_1, set $z_j = (f^{l_j})^{-1}(w_j)$. Divide our situation into three cases. (If necessary, we will take a subsequence which is denoted by the same symbol, for simplicity.)

I. infinitely many w_j are on the geodesic γ_1,

II. there exists $b \geq 1$ such that all but finitely many z_j are outside of $C_1(b) \cap S_0^2$,

III. there exists a subsequence such that $\lim_{j\to\infty} z_j = q_1$.

In case I, due to the first inequality with $k = 0$ in Lemma 2.6,

$$Q_{l_j} = L_{l_j}(\gamma_1)^{-2s} E^{l_j}(w_j, s) \geq K_1 L_{l_j}(\gamma_1)^{-s} \geq K_1 > 0 \quad \text{for all large } j.$$

This is a contradiction.

In case II, we choose a region Ω_1 $(\Subset S_0^2)$ which contains $\partial C_1(1) \cap S_0^2$ and z_j. Due to the first inequality in Lemma 2.7,

$$Q_{l_j} = L_{l_j}(\gamma_1)^{-2s} (f^{l_j})^* E^{l_j}(z_j, s) \geq \inf_{z \in \Omega_1} L_{l_j}(\gamma_1)^{-2s} (f^{l_j})^* E^{l_j}(z,s) \geq P_1(\Omega_1) > 0.$$

This is a contradiction.

In case III, there exists $0 \le k_j \le 1$ such that $f^{l_j}(z_j) \in \partial N_{\gamma_1}(k_j) \cap S^2_{l_j}$. Then due to the first inequality in Lemma 2.6, we have

$$Q_{l_j} = L_{l_j}(\gamma_1)^{-2s}(f^{l_j})^* E^{l_j}(z_j, s) \ge \inf_{w \in \partial N_{\gamma_1}(k_j) \cap S^2_{l_j}} L_{l_j}(\gamma_1)^{-2s} E^{l_j}(w, s)$$
$$\ge K_1 L_{l_j}(\gamma_1)^{(k_j - 1)s} \ge K_1 > 0$$

for all large j. This is a contradiction. We have proved the first inequality.

Next, we will show the second inequality in a similar method. We set

$$Q'_l = \inf_{w \in N_{\gamma_2}(1) \cap \bar{S}^2_l} L_l(\gamma_1)^{-2s} L_l(\gamma_2)^{-s} E^l(w, s).$$

Assume that there exists a subsequence $l_j \to 0$ such that $\lim_{j\to\infty} Q'_{l_j} = 0$. For each j, we can find a point $w_j \in N_{\gamma_2}(1) \cap \bar{S}^2_l$ such that

$$L_{l_j}(\gamma_1)^{-2s} L_{l_j}(\gamma_2)^{-s} E^{l_j}(w_j, s) = \inf_{w \in N_{\gamma_2}(1) \cap \bar{S}^2_{l_j}} L_{l_j}(\gamma_1)^{-2s} L_{l_j}(\gamma_2)^{-s} E^{l_j}(w, s).$$

If w_j is not on the geodesic γ_2, set $z_j = (f^{l_j})^{-1}(w_j)$. Divide our situation into three cases. (If necessary, we will take a subsequence which is denoted by the same symbol, for simplicity.)

I$'$. infinitely many w_j are on the geodesic γ_2,

II$'$. there exists $b \ge 1$ such that all but finitely many z_j are outside of $C_2(b) \cap S^2_0$,

III$'$. there exists a subsequence such that $\lim_{j\to\infty} z_j = q_2$.

In case I$'$, due to the second inequality with $k = 0$ in Lemma 2.6,

$$Q'_{l_j} = L_{l_j}(\gamma_1)^{-2s} L_{l_j}(\gamma_2)^{-s} E^{l_j}(w_j, s) \ge K_2 > 0 \quad \text{for all large } j.$$

This is a contradiction.

In case II$'$, we choose a region Ω'_1 ($\Subset S^2_0$) which contains $\partial C_1(1) \cap S^2_0$ and z_j. Due to the first inequality in Lemma 2.7,

$$Q'_{l_j} = L_{l_j}(\gamma_1)^{-2s} L_{l_j}(\gamma_2)^{-s}(f^{l_j})^* E^{l_j}(z_j, s) \ge \inf_{z \in \Omega'_1} L_{l_j}(\gamma_1)^{-2s} L_{l_j}(\gamma_2)^{-s}(f^{l_j})^* E^{l_j}(z, s)$$
$$\ge P_1(\Omega'_1) L_{l_j}(\gamma_2)^{-s} \ge P_1(\Omega'_1) > 0$$

for all large j. This is a contradiction.

In case III$'$, there exists $0 \le k_j \le 1$ such that $f^{l_j}(z_j) \in \partial N_{\gamma_2}(k_j) \cap S^2_{l_j}$. Then due to the second inequality in Lemma 2.6, we have

$$Q'_{l_j} = L_{l_j}(\gamma_1)^{-2s} L_{l_j}(\gamma_2)^{-s} E^{l_j}(f^{l_j}(z_j), s) \ge \inf_{w \in \partial N_{\gamma_2}(k_j) \cap S^2_{l_j}} L_{l_j}(\gamma_1)^{-2s} L_{l_j}(\gamma_2)^{-s} E^{l_j}(w, s)$$
$$\ge K_2 L_{l_j}(\gamma_1)^{-k_j s} \ge K_2 > 0$$

for all large j. This is a contradiction. We have proved the second inequality. We can prove the third inequality in the same way as the first inequality, using the third inequality in Lemma 2.6 and the second inequality in Lemma 2.7. $\square$

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