RIMS Kôkyûroku Bessatsu **B17**
(2010), 127–133

Modulus of continuity, a Hardy-Littlewood theorem and its application

By

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§ 1. Introduction

Let D be a simply connected proper domain in $\mathbb{C}$ and $\varphi : \Delta \to D$ a Riemann mapping from the unit disk $\Delta = \{|z| < 1\}$ onto D. The geometric function theory gives us various informations of the mapping φ. For example, if D is a quasi-disk, then we have

$$|\varphi'(z)| = O((1 - |z|)^{-\kappa}) \tag{1.1}$$

for some $\kappa \in [0, 1)$ as $|z| \to 1$ (cf. [8]). On the other hand, if a simply connected domain D is an invariant component of a finitely generated Kleinian group G, we can say much more on the Riemann mapping φ. In fact, if D is a Jordan domain, then G must be a quasi-Fuchsian group by a theorem of Maskit ([3]). Hence, D is a quasi-disk, and the inequality (1.1) holds. Recently ([9]), we have shown that the converse is also true. Namely, we have shown the following;

Theorem 1.1. *Let $D \ni \infty$ be a simply connected invariant component of a finitely generated non-elementary Kleinian group G and φ a Riemann mapping from the unit disk onto D. Then the following are equivalent.*

1. *G is a quasi-Fuchsian group and D is a quasi-disk.*
2. *(1.1) holds for some $\kappa \in [0, 1)$ as $|z| \to 1$.*

In other words, the growth rate of the derivatives of the Riemann mappings characterizes quasi-Fuchsian groups.

2000 Mathematics Subject Classification(s): 30C22, 30C99.
Key Words: *modulus of continuity, Hardy-Littlewood theorem.*
This research was partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research (A), 2005–2009, 17204010.
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Remark. The condition (2) implies that D is a Hölder domain. It is known that every quasi-disk is a Hölder domain. Thus (1) implies (2). But the converse is not true in general.

It is a natural question what happens for φ if D is a simply connected invariant component of G other than a quasi-Fuchsian group. In fact, we have obtained the growth rate of $|\varphi'(z)|$ of Riemann mappings φ for regular b-groups and Kleinian groups with bounded geometry. Particularly, when G is a regular b-group, we have estimated the modulus of continuity of φ on the unit circle and we have shown the local connectivity of the limit set of G.

In this note, we will show a Hardy-Littlewood theorem to estimate the growth rate of $|\varphi'(z)|$ from the modulus of continuity and as a corollary, the growth rate of $|\varphi'(z)|$ for Kleinian groups with bounded geometry. It is an alternative proof of a result obtained in our previous paper [9].

§ 2. A Hardy-Littlewood theorem

Let f be a continuous function on the unit circle. The modulus of continuity of f is the function

$$\omega(t) = \sup_{|\theta_1 - \theta_2| \leq t} |f(e^{i\theta_1}) - f(e^{i\theta_2})|.$$

In 1932, Hardy and Littlewood [2] shows the following theorem called a Hardy-Littlewood theorem.

Theorem 2.1 (cf. [1] p. 74)**.** *Let f be a holomorphic function on the unit disk Δ and continuous on $\overline{\Delta} = \Delta \cup \partial\Delta$. Suppose that there exists $\alpha \in (0, 1]$ such that*

$$|f(e^{i\theta_1}) - f(e^{i\theta_2})| = O(|\theta_1 - \theta_2|^{\alpha}).$$

Then

$$|f'(z)| = O((1 - |z|)^{\alpha - 1})$$

holds as $|z| \to 1$.

In this section, we shall show the following theorem of Hardy-Littlewood type for holomorphic functions whose modulus of continuity is $|\log|\theta||^{-\alpha}$.

Theorem 2.2. *Let f be a holomorphic function on the unit disk Δ and continuous on $\overline{\Delta} = \Delta \cup \partial\Delta$. Suppose that there exists $\alpha > 0$ such that*

$$\bigl|f(e^{i\theta_1}) - f(e^{i\theta_2})\bigr| = O(|\log|\theta_1 - \theta_2||^{-\alpha}), \tag{2.1}$$

if $|\theta_1-\theta_2|<\delta$ *for some* $\delta\in(0,1)$. *Then,*

$$|f'(z)|=O((1-|z|)^{-1}|\log(1-|z|)|^{-\alpha}) \tag{2.2}$$

holds as $|z|\to 1$.

Proof. By Cauchy's integral formula,

$$f'(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(f(e^{it})-f(e^{i\varphi}))e^{it}}{(e^{it}-z)^2}dt. \qquad (z=re^{i\varphi})$$

Thus, we have

$$|f'(z)|\le\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|f(e^{i(t+\varphi)})-f(e^{i\varphi})|}{1-2r\cos t+r^2}dt.$$

Since

$$1-2r\cos t+r^2\ge(1-r)^2+\frac{4rt^2}{\pi^2},$$

it follows from (2.1) that

$$|f'(z)|\le\frac{A}{2\pi}\int_{-\delta}^{\delta}\frac{|\log|t||^{-\alpha}}{(1-r)^2+4r(t/\pi)^2}dt+B. \tag{2.3}$$

Setting $C_r=\pi^2(1-r)^2/4r$ and $t=\sqrt{C_r}\tan\theta$, we have

$$\begin{aligned}I(z):=\frac{A}{2\pi}\int_{-\delta}^{\delta}\frac{|\log|t||^{-\alpha}}{(1-r)^2+4r(t/\pi)^2}dt&=\frac{A\pi}{8r}\int_{-\delta}^{\delta}\frac{|\log|t||^{-\alpha}}{t^2+C_r}dt\\&=\frac{A\pi}{8r\sqrt{C_r}}\int_{-\beta_r}^{\beta_r}\frac{d\theta}{\left|\log\sqrt{C_r}+\log|\tan\theta|\right|^{\alpha}}\\&=\frac{A\pi}{4r\sqrt{C_r}}\int_0^{\beta_r}\frac{d\theta}{\left|\log\sqrt{C_r}+\log|\tan\theta|\right|^{\alpha}},\end{aligned}$$

where $\beta_r=\arctan\frac{\delta}{\sqrt{C_r}}\in(0,\frac{\pi}{2})$. As $r=|z|\to1$, $C_r\to0$ and $\beta_r\to\frac{\pi}{2}$. We take $r>0$ sufficiently close to 1 so that $C_r<1$.

When $\theta\in(0,\frac{\pi}{4}]$, $\tan\theta\in(0,1]$. Hence

$$\log\sqrt{C_r}+\log|\tan\theta|\le\log\sqrt{C_r}<0,$$

and

$$\left|\log\sqrt{C_r}+\log|\tan\theta|\right|^{-\alpha}\le\left|\log\sqrt{C_r}\right|^{-\alpha}.$$

Thus, we have

$$\int_0^{\pi/4}\frac{d\theta}{\left|\log\sqrt{C_r}+\log|\tan\theta|\right|^{\alpha}}=O\left(|\log(1-|z|)|^{-\alpha}\right), \tag{2.4}$$

because $C_r = O((1-|z|)^2)$.

Next, we take a constant $\lambda \in (\frac{1}{2}, 1)$ and put $\gamma_r := \arctan\left(\frac{1}{\sqrt{C_r}}\right)^{\lambda}$. We may assume that $\gamma_r < \beta_r$. When $\theta \in (\frac{\pi}{4}, \gamma_r]$, $\tan\theta \in (1, C_r^{-\lambda/2})$ and we have

$$\log\sqrt{C_r} + \log|\tan\theta| \leq (1-\lambda)\log\sqrt{C_r} < 0.$$

This implies

$$\left|\log\sqrt{C_r} + \log|\tan\theta|\right|^{-\alpha} \leq (1-\lambda)^{-\alpha}\left|\log\sqrt{C_r}\right|^{-\alpha}$$

and we have

$$\int_{\pi/4}^{\gamma_r} \frac{d\theta}{\left|\log\sqrt{C_r} + \log|\tan\theta|\right|^{\alpha}} = O\left(|\log(1-|z|)|^{-\alpha}\right). \tag{2.5}$$

Finally, we consider the case where $\theta \in (\gamma_r, \beta_r]$. Since $\arctan x = \int_0^x \frac{1}{t^2+1}dt$, we have

$$\begin{aligned}\beta_r - \gamma_r &= \int_{(\sqrt{C_r})^{-\lambda}}^{\delta/\sqrt{C_r}} \frac{dx}{x^2+1} \\ &\leq \left(\frac{\delta}{\sqrt{C_r}} - \frac{1}{\sqrt{C_r^{\lambda}}}\right)\frac{\sqrt{C_r^{2\lambda}}}{\sqrt{C_r^{2\lambda}}+1} = O\left(C_r^{\lambda-1/2}\right).\end{aligned}$$

On the other hand,

$$\log\sqrt{C_r} + \log|\tan\theta| \leq \log\delta < 0,$$

because $\tan\theta \leq \frac{\delta}{\sqrt{C_r}}$. Therefore, we conclude

$$\begin{aligned}\int_{\gamma_r}^{\beta_r} \frac{d\theta}{\left|\log\sqrt{C_r} + \log|\tan\theta|\right|^{\alpha}} &\leq (\beta_r - \gamma_r)|\log\delta|^{-\alpha} \\ &= O((1-|z|)^{2\lambda-1}).\end{aligned} \tag{2.6}$$

Combining (2.4), (2.5) and (2.6), we have

$$I(z) = O\left((1-|z|)^{-1}|\log(1-|z|)|^{-\alpha}\right).$$

Thus, we complete the proof of the theorem. □

§ 3. Conformal mappings on invariant components of Kleinian groups

Let G be a finitely generated non-elementary Kleinian group. The group G is said to have *bounded geometry* if there exists a constant $\varepsilon > 0$ such that the injectivity radius with respect to the hyperbolic metric at any point in $\mathbb{H}^3/G$ is greater than ε.

We also assume that G has a simply connected invariant component D and denote by φ a Riemann mapping from the unit disk Δ onto D as before. Many things are known for Kleinian groups with bounded geometry (cf. [5]). For example, the limit set of G is locally connected whenever it is connected. Particularly, H. Miyachi ([6]) shows the following;

Proposition 3.1. *Let G be a Kleinian group with bounded geometry having a simply connected invariant component D and $\varphi : \Delta \to D$ a Riemann mapping. Then, φ has a continuous extension to $\partial\Delta$ and*

$$|\varphi(e^{i\theta_1}) - \varphi(e^{i\theta_2})| = O(|\log|\theta_1 - \theta_2||^{-\alpha})$$

holds as $|\theta_1 - \theta_2| \to 0$.

From this proposition and Theorem 2.1, we immediately obtain a theorem which is shown in [9] by a different method;

Theorem 3.2. *Let G, D and φ be the same ones as in Proposition 3.1. Then,*

$$|\varphi'(z)| = O\left((1 - |z|)^{-1} |\log(1 - |z|)|^{-\alpha}\right) \tag{3.1}$$

holds as $|z| \to 1$.

Remark. In [9], we have also shown that if G is a regular b-group, then

$$|\varphi'(z)| = O\left((1 - |z|)^{-1} |\log(1 - |z|)|^{-2}\right) \tag{3.2}$$

and we obtain the modulus of continuity on $\partial\Delta$,

$$|\varphi(e^{i\theta_1}) - \varphi(e^{i\theta_2})| = O\left(|\log|\theta_1 - \theta_2||^{-1}\right). \tag{3.3}$$

by using (3.2). From Theorem 2.1 it seems to be difficult to show (3.2) from (3.3). Actually, Nolder and Oberlin [7] show the following;

Proposition 3.3. *Let $\omega(t)$ be a differentiable non-negative increasing function on $[0, \infty)$ having the decreasing derivative $\omega'(t)$. The following are equivalent:*

1. *If f is a holomorphic function with the modulus of continuity θ, then*

$$|f'(z)| = O(\omega'(1 - |z|)).$$

2.

$$\limsup_{t \to 0+} \frac{\omega(t)}{t\omega'(t)} < \infty.$$

In our case, (3.3) implies $\omega(t) = (-\log t)^{-1}$ for small $t > 0$ and $\omega'(t) = t^{-1}(\log t)^{-2}$. However,

$$\limsup_{t \to 0+} \frac{\omega(t)}{t\omega'(t)} = \lim_{t \to 0+} (-\log t) = +\infty.$$

Hence, the second condition is not satisfied and we can not apply the above proposition to get (3.3) from (3.2).

We have also characterized quasi-Fuchsian groups in terms of the growth of derivatives of Riemann mappings of invariant components.

Proposition 3.4 ([9]). *Let G be a Kleinian group having a simply connected invariant component D with $\partial D \subset \mathbb{C}$ and φ a conformal mapping of the unit disk Δ onto D. Suppose that D/G has no punctures. Then, the following conditions are equivalent.*

1. *There exist constants $\alpha > 0$, $A > 0$ and a point $\zeta_0 \in D$ such that for any $z \in \varphi^{-1}(G\zeta_0) \setminus \varphi^{-1}(\infty)$,*

$$|\varphi'(z)| \leq \frac{A}{(1 - |z|)|\log(1 - |z|)|^{2+\alpha}} \tag{3.4}$$

   holds.

2. *G is a quasi-Fuchsian group.*

From Theorem 2.1 and the above one, immediately we have;

Theorem 3.5. *Let G, D and φ be the same ones as above. Suppose that $\varphi : \Delta \to D$ has the continuous extension to $\partial\Delta$. If the extension φ on $\partial\Delta$ satisfies*

$$|\varphi(e^{i\theta_1}) - \varphi(e^{i\theta_2})| = O(|\log|\theta_1 - \theta_2||^{-2-\alpha})$$

for some $\alpha > 0$. Then G is a quasi-Fuchsian group.

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