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# Cauchy problem and Kato smoothing for water waves with surface tension

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## § 1. Introduction

This note is a short introduction to our paper [2]. We are interested in this work in the study of the Cauchy problem for the water waves system with surface tension in arbitrary dimension. Water waves are waves on the free surface of a fluid (think of the interface between air and water for the oceans, lakes, canals...). This system is of hyperbolic-dispersive type. Its solutions correspond to solutions of the incompressible Euler equations for a potential flow in a domain with a free boundary. A popular form is the following system

$$\begin{cases} \partial_t \eta - G(\eta)\psi = 0, \\ \partial_t \psi + g\eta - \kappa H(\eta) + \dfrac{1}{2}\left|\nabla\psi\right|^2 - \dfrac{1}{2}\dfrac{\left(\nabla\eta\cdot\nabla\psi + G(\eta)\psi\right)^2}{1+|\nabla\eta|^2} = 0, \end{cases} \tag{1}$$

where $\eta, \psi \colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ are the unknowns, $g, \kappa$ are positive constants,

$$H(\eta) = \operatorname{div}\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right),$$

and $G(\eta)$ is the Dirichlet–Neumann operator whose definition is given below.

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Concerning this problem, there are many results starting from the pionneering work of K. Beyer and M. Günther [8]. See D. M. Ambrose and N. Masmoudi [7], B. Schweiser [23], T. Iguchi [16], J. Shatah and C. Zeng [22], M. Ming and Z. Zhang [20], F. Rousset and N. Tzvetkov [21].

In this paper we present a sharp existence and uniqueness result for the water waves system with bottom boundary, for data of low regularity and without any condition of regularity on the bottom (see Theorem 2.1). Our assumptions ensure that the initial velocity has Lipschitz regularity, which is the natural assumption to make (as long as dispersive effects are not taken into account). Moreover, in the one dimensional case, we prove a smoothing effect of Kato type (gain of $1/4$ derivative) with the natural weights in the estimates (see Theorem 2.2).

To prove these results, inspired by the work by Lannes [18], we begin by a careful paralinearization of the equations (in the sense of Bony's theory) using the results of Alazard and Metivier [1]. We then show that the system can be arranged into a symmetric system of Schrödinger type (see equation (10)) to which we can apply the usual energy estimate method (for Theorem 2.1) and Doi's method [13, 14] (for Theorem 2.2).

## § 2. Main results

### § 2.1. The problem

In a domain $\Omega_t \subset \mathbb{R}^{d+1}$ (which depends on time $t$) which is located between a free hypersurface $\Sigma_t$ and a fixed known bottom $\Gamma$ we consider a potential flow whose velocity $v = \nabla_{x,y}\phi$ is such that

$$\Delta_{x,y}\phi = 0 \quad \text{in } \Omega_t, \quad \partial_n \phi = 0 \quad \text{on } \Gamma.$$

The problem is then given by two equations: a kinematic condition (which states that the free surface moves with the fluid), and a dynamic condition (that expresses a balance of forces across the free surface). The system reads

$$\begin{cases} \partial_t \eta = \partial_y \phi - \nabla \eta \cdot \nabla \phi & \text{on } \Sigma_t = \{y = \eta(t,x)\}, \\ \partial_t \phi + \dfrac{1}{2} \left| \nabla_{x,y}\phi \right|^2 + g\eta = \kappa H(\eta) & \text{on } \Sigma_t, \end{cases}$$

Since $\phi$ is an harmonic function satisfying $\partial_n \phi = 0$ on the bottom $\Gamma$, it suffices to determine its trace on the free surface $\phi|_\Sigma$. Following Zakharov, set

$$\psi(t,x) = \phi(t,x,\eta(t,x)).$$

Then $(\eta,\phi)$ is solution if and only if $(\eta,\psi)$ solves the system (1) where, by notation,

$$G(\eta)\psi = \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi\Big|_{y=\eta} = \partial_y\phi - \nabla\eta\cdot\nabla\phi\Big|_{y=\eta}.$$

## § 2.2. Assumptions

We assume that, for each time $t$, one has

$$\Omega_t = \Omega_{1,t}\cap\Omega_2$$

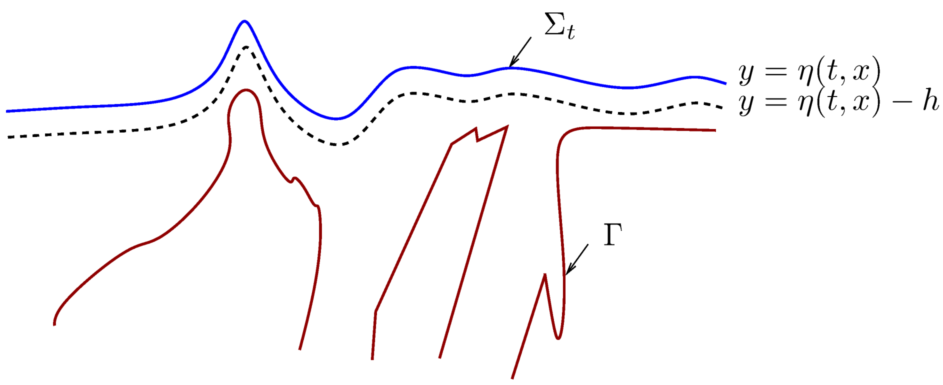
where $\Omega_{1,t}$ is the half space located below the free surface $\Sigma_t$,

$$\Omega_{1,t} = \{\,(x,y)\in\mathbb{R}^d\times\mathbb{R}\,:\,y<\eta(t,x)\,\}\qquad (d\geq 1)$$

for some unknown function $\eta$ and $\Omega_2$ contains a fixed strip around $\Sigma_t$, that means that there exists $h>0$ s.t.

$$\{(x,y)\in\mathbb{R}^d\times\mathbb{R}\,:\,\eta(t,x)-h\leq y\leq\eta(t,x)\}\subset\Omega_2,$$

for all $t\in[0,T]$. We shall also assume that the domain $\Omega_2$ (and hence the domain $\Omega_t=\Omega_{1,t}\cap\Omega_2$) is connected.




The domain

We work in a fluid domain such that there is uniformly a minimum depth of water. We emphasize that no regularity assumption is made on the bottom $\Gamma=\partial\Omega_t\setminus\Sigma_t$. We consider both cases of infinite depth and bounded depth bottoms (and all cases in-between). Finally, we could consider the case where the free surface is a graph over a given smooth hypersurface and the bottom is time dependent.

## § 2.3. Remark

Many interesting features are revealed by analyzing the linearized of system (1) at the origin. Since $G(0)\simeq|D_x|$, we find

$$\begin{cases}\partial_t\eta-|D_x|\,\psi=0,\\ \partial_t\psi-\Delta\eta=0,\end{cases}$$

and hence

$$\partial_t \Phi + i\,|D_x|^{\frac{3}{2}}\,\Phi = 0 \quad \text{with} \quad \Phi = |D_x|^{\frac{1}{2}}\,\eta + i\psi.$$

We make two observations. Firstly, note that $\psi$ and $|D_x|^{1/2}\,\eta$ have the same regularity (which can also be understood by looking at the Hamiltonian). Secondly, we expect dispersive estimates.

## § 2.4. Main results

The main results in [2] are the following theorems.

**Theorem 2.1.** *Let $d \geq 1$, $s > 2 + \frac{d}{2}$ and $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$ s.t.*

$$\operatorname{dist}(\Sigma_0, \Gamma) > 0.$$

*Then there exists $T^\star > 0$ such that the Cauchy problem for (1) with initial data $(\eta_0, \psi_0)$ has a unique maximal solution $(\eta, \psi) \in C^0\big([0, T^\star[; H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)\big)$.*

**Theorem 2.2.** *Consider the case $d = 1$. Let $s > 5/2$ and $T > 0$. If*

$$(\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R})),$$

*is a solution of (1), such that $\operatorname{dist}(\Sigma_t, \Gamma) > 0$, then*

$$\langle x \rangle^{-\frac{1}{2}-\delta}(\eta, \psi) \in L^2(0, T; H^{s+\frac{3}{4}}(\mathbb{R}) \times H^{s+\frac{1}{4}}(\mathbb{R})),$$

*for all $\delta > 0$.*

*Remark.* $(i)$ Here, $s > 2 + \frac{d}{2}$ appears to be the natural threshold of regularity (as it controls the Lipschitz norm of the non-linearities). This gives rise to many technical difficulties, which would be avoided if we assumed $s > 3 + \frac{d}{2}$. In addition, we allow general bottoms.

$(ii)$ Other dispersive estimates (such as Strichartz estimates) will be considered in a forthcoming paper.

# § 3. Paralinearization of the Dirichlet to Neumann operator

When $\eta \in C^\infty(\mathbb{R}^d)$ and $\Omega$ has no bottom, the analysis of the Dirichlet-Neumann operator $G(\eta)$ is well known. In this section we give an expression of $G(\eta)$ for $\eta$ of low regularity. This is achieved by using the paradifferential calculus of J.-M. Bony. This analysis allows to simplify many nonlinear arguments (such as Nash-Moser, commutators estimates, computations of changes of variables...).

### § 3.1. Definition of the Dirichlet to Neumann operator

We first show that one can define a **variationnal** solution to

$$\Delta_{x,y}\phi = 0 \quad \text{in } \Omega, \quad \phi\big|_{\Sigma} = \psi, \quad \partial_n\phi\big|_{\Gamma} = 0.$$

Notice that, for this definition, no smoothness on the bottom is required. However, as soon as it is smooth enough, say $C^{1,1}$, the boundary condition $\partial_n\phi(x,y) = 0$ on $\Gamma$ is satisfied in a classical sense.

We set

$$G(\eta)\psi = \partial_y\phi - \nabla\eta\cdot\nabla\phi\Big|_{y=\eta}.$$

Then we have,

**Proposition 3.1.** *If $\psi \in H^{\sigma}(\mathbb{R}^d)$ and $\eta \in H^{s+\frac{1}{2}}(\mathbb{R}^d)$ where*

$$1 \le \sigma \le s, \quad s > 2 + \frac{d}{2},$$

*then*

$$G(\eta)\psi \in H^{\sigma-1}(\mathbb{R}^d).$$

To do so we localize the analysis near $\Sigma$ ($\phi$ is smooth inside) and then use standard elliptic regularity in a strip (from Alvarez-Samaniego & Lannes [6]).

### § 3.2. Paralinearization of the DN operator

When $\eta \in C^{\infty}(\mathbb{R}^d)$ and $\Omega$ has no bottom, it is well known (see [24]) that $G(\eta)$ is a classical elliptic pseudo-differential operator of order 1, whose symbol has an asymptotic expansion of the form

$$\lambda^{(1)}(x,\xi) + \lambda^{(0)}(x,\xi) + \lambda^{(-1)}(x,\xi) + \cdots$$

where $\lambda^{(k)}$ are homogeneous of degree $k$ in $\xi$, and the principal symbol $\lambda^{(1)}$ and the sub-principal symbol $\lambda^{(0)}$ are given by (cf [17])

$$\begin{aligned} \lambda^{(1)} &= \sqrt{\left(1+|\nabla\eta|^2\right)|\xi|^2 - (\nabla\eta\cdot\xi)^2}, \\ \lambda^{(0)} &= \frac{1+|\nabla\eta|^2}{2\lambda^{(1)}}\left\{\operatorname{div}\left(\alpha^{(1)}\nabla\eta\right) + i\partial_\xi\lambda^{(1)}\cdot\nabla\alpha^{(1)}\right\}, \end{aligned} \tag{2}$$

with

$$\alpha^{(1)} = \frac{1}{1+|\nabla\eta|^2}\left(\lambda^{(1)} + i\nabla\eta\cdot\xi\right).$$

The symbols $\lambda^{(-1)}, \ldots$ are defined by induction and we can prove that $\lambda^{(k)}$ involves only derivatives of $\eta$ of order $|k|+2$.

There are also various results when $\eta \notin C^\infty$. Expressing $G(\eta)$ as a singular integral operator, it was proved by Craig, Schanz and C. Sulem [12] that

$$\eta \in C^{k+1},\ \psi \in H^{k+1} \text{ with } k \in \mathbb{N} \Rightarrow G(\eta)\psi \in H^k. \tag{3}$$

Moreover, when $\eta$ is a given function with limited smoothness, it is known that $G(\eta)$ is a pseudo-differential operator with symbol of limited regularity[1]. In this direction, for $\sigma \in H^{s+1}(\mathbb{R}^2)$ with $s$ large enough, it follows from the analysis by Lannes ([18]) and a small additional work that

$$G(\eta)\psi = \mathrm{Op}(\lambda^{(1)})\psi + r(\eta,\psi), \tag{4}$$

where the remainder $r(\eta,\psi)$ is such that

$$\psi \in H^s(\mathbb{R}^d) \Rightarrow r(\eta,\psi) \in H^s(\mathbb{R}^d).$$

This implies that, if $\eta \in H^{s+1}(\mathbb{R}^d)$ and $\psi \in H^{s+1}(\mathbb{R}^d)$ for some $s$ large enough, then $G(\eta)\psi \in H^s(\mathbb{R}^d)$. This result was first established by Craig and Nicholls in [11] and Wu in [26, 27] by different methods. We refer to [18] for comments on the estimates associated to these regularity results as well as to [6] for the rather different case where one considers various dimensional parameters.

A fundamental difference with these results is that we shall determine the full structure of $G(\eta)$ by performing a full paralinearizaton of $G(\eta)\psi$ with respect to $\psi$ and $\eta$. In our case the function $\eta$ will not be $C^\infty$ but only at least $C^2$, so we shall set

$$\lambda = \lambda^{(1)} + \lambda^{(0)}, \tag{5}$$

which will be well-defined in the $C^2$ case.

*Remark.* If $d = 1$ or $\eta = 0$ then $\lambda$ simplifies to $\lambda(x,\xi) = |\xi|$ (this is one of the key dichotomy between 2D waves and 3D waves). Also, directly from (2), one can check the following formula (which holds for all $d \geq 1$)

$$\mathrm{Im}\,\lambda^{(0)} = -\frac{1}{2}(\partial_\xi \cdot \partial_x)\lambda^{(1)}, \tag{6}$$

which reflects the fact that the Dirichlet-Neumann operator is a symmetric operator.

This symbol belongs to the following symbol classes.

[1] We do not explain here the way we define pseudo-differential operators with symbols of limited smoothness since this problem will be fixed by using paradifferential operators, and since all that matters in (4) is the regularity of the remainder term $r(\sigma,\psi)$.

**Definition 3.2.** Given $\rho \geq 0$ and $m \in \mathbb{R}$, $\Gamma^m_\rho(\mathbb{R}^d)$ denotes the space of locally bounded functions $a(x,\xi)$ on $\mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$, which are $C^\infty$ with respect to $\xi$ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbb{N}^d$ and all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ belongs to $W^{\rho,\infty}(\mathbb{R}^d)$ and there exists a constant $C_\alpha$ such that,

$$\forall\, |\xi| \geq \frac{1}{2}, \quad \left\| \partial_\xi^\alpha a(\cdot,\xi) \right\|_{W^{\rho,\infty}} \leq C_\alpha (1+|\xi|)^{m-|\alpha|}.$$

Here are some examples: 1) If $a = a(x) \in W^{\rho,\infty}(\mathbb{R}^d)$ then $a \in \Gamma^0_\rho(\mathbb{R}^d)$. 2) If $a = a(x,\xi)$ homogeneous of order $m$ in $\xi$, with regularity $W^{\rho,\infty}$ in $x$, then $a \in \Gamma^m_\rho(\mathbb{R}^d)$. 2bis) If $\eta \in H^r(\mathbb{R}^d)$ then

$$\lambda^{(1)} \in \Gamma^1_{r-1-d/2}(\mathbb{R}^d), \quad \lambda^{(0)} \in \Gamma^0_{r-2-d/2}(\mathbb{R}^d),$$

where $\lambda^{(1)}, \lambda^{(0)}$ are given by (2).

Following Bony [9] the paradifferential operator $T_a$ is defined by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int \chi(\xi-\eta,\eta)\widehat{a}(\xi-\eta,\eta)\psi(\eta)\widehat{u}(\eta)\, d\eta,$$

where $\widehat{a}(\theta,\xi) = \int e^{-ix\cdot\theta} a(x,\xi)\, dx$,

$$\psi(\eta) = 0 \quad \text{for } |\eta| \leq 1, \qquad \psi(\eta) = 1 \quad \text{for } |\eta| \geq 2.$$

$$\chi(\theta,\eta) = 1 \quad \text{for } |\theta| \leq \varepsilon_1 |\eta|, \quad \chi(\theta,\eta) = 0 \quad \text{for } |\theta| \geq \varepsilon_2 |\eta|.$$

for $\varepsilon_1, \varepsilon_2$ small enough.

We have the same symbolic calculus as for ΨDOs, except that the symbolic calculus is finite instead of being asymptotic.

1. If $a \in \Gamma^m_0(\mathbb{R}^d)$ then $T_a$ is of order $m$ (bounded from $H^{\mu+m}$ to $H^\mu$ for all $\mu \in \mathbb{R}$).
2. If $a \in \Gamma^m_\rho(\mathbb{R}^d)$ and $b \in \Gamma^{m'}_\rho(\mathbb{T}^d)$ with $\rho > 0$, then $T_aT_b - T_{a\#b}$ is of order $m + m' - \rho$, with
$$a\#b = \sum_{|\alpha|<\rho} \frac{1}{i^\alpha \alpha!} \partial_\xi^\alpha a \partial_x^\alpha b.$$
3. if $a \in \Gamma^m_\rho(\mathbb{R}^d)$ then $(T_a)^* - T_{a^*}$ is of order $m - \rho$ where
$$a^* = \sum_{|\alpha|<\rho} \frac{1}{i^\alpha \alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{a}.$$
4. We shall also use operators norms estimates (from [19]), parametrices for elliptic operators, Gårding inequality...

If $a = a(x)$, $T_a$ is called a paraproduct.

1. If $u \in H^s(\mathbb{R}^d)$, $s > d/2$ and $F$ is $C^\infty$ with $F(0) = 0$, then

$$F(u) - T_{F'(u)}u \in H^{2s-\frac{d}{2}}(\mathbb{R}^d).$$

2. If $a \in H^\alpha(\mathbb{R}^d)$ and $b \in H^\beta(\mathbb{R}^d)$, $\alpha, \beta > d/2$, then

$$ab - T_a b - T_b a \in H^{\alpha+\beta-\frac{d}{2}}(\mathbb{R}^d).$$

**Proposition 3.3** (from [1, 2]). *Let $d \geq 1$, $s > 2 + d/2$.*

- *If $\eta \in H^{s+\frac{1}{2}}(\mathbb{R}^d)$ and $\psi \in H^\sigma(\mathbb{R}^d)$ with $1 \leq \sigma \leq s-1$, then*

$$G(\eta)\psi = T_\lambda \psi + F(\eta, \psi),$$

  *where $F(\eta, \psi) \in H^\sigma(\mathbb{R}^d)$.*

- *If $\eta \in H^{s+\frac{1}{2}}(\mathbb{R}^d)$ and $\psi \in H^s(\mathbb{R}^d)$, then*

$$\underbrace{G(\eta)\psi}_{\in H^{s-1}} = \underbrace{T_\lambda\big(\psi - T_{\mathfrak{B}}\eta\big)}_{\in H^{s-1}} - \underbrace{T_V \cdot \nabla\eta}_{H^{s-\frac{1}{2}}} + f(\eta, \psi). \tag{7}$$

  *where*

$$f(\eta, \psi) \in H^{s+\frac{1}{2}}(\mathbb{R}^d),$$

  *and*

$$\mathfrak{B} = \frac{\nabla\eta \cdot \nabla\psi + G(\eta)\psi}{1 + |\nabla\eta|^2}, \quad V = \nabla\psi - \mathfrak{B}\nabla\eta.$$

*Remark.* $(i)$ It is well known that $\mathfrak{B}$ and $V$ play a key role in the study of the water waves. These are simply the projection of the velocity field on the vertical and horizontal directions.

$(ii)$ If $d = 1$, (7) simplifies to

$$G(\eta)\psi - \big(|D_x|\left(\psi - T_{\mathfrak{b}}\eta\right) - T_V\partial_x\eta\big) \in H^{s+\frac{1}{2}}(\mathbb{R}).$$

$(iii)$ The good unknown $\psi - T_{\mathfrak{B}}\eta$ contains all the geometry. This corresponds to the so called good unknown of Alinhac (as introduced in [5]).

$(iv)$ Up to considering lower order terms in the symbol of the DN, one can prove an identity at any order (with $f \in H^{2s-\frac{3+d}{2}}(\mathbb{R}^d)$). This tool is useful for the study of 3D progressive water waves (Iooss & Plotnikov [17], Alazard & Métivier [1]).

*Proof.* One basic approach toward the analysis of solutions of a boundary value problem is to flatten the boundary. We map a neighborhood of the free surface to a strip via the correspondance

$$v(x, z) = \phi\big(x, \rho(x, z)\big) \quad \text{with} \quad \rho(x, z) := hz + \eta(x),$$

where $(x,z)\mapsto(x,\rho(x,z))$ is a diffeomorphism from $\mathbb{R}^d\times[-1,0]$ to a strip $S\subset\Omega\cup\Sigma$. As defined, $v$ satisfies

$$\begin{cases}\alpha\partial_z^2 v+\Delta v+\beta\cdot\nabla\partial_z v-\gamma\partial_z v=0,\\ v\big|_{z=0}=\psi,\end{cases}$$

where

$$\alpha=\frac{1+|\nabla\eta|^2}{h},\quad \beta=-\frac{2}{h}\nabla\eta,\quad \gamma=\frac{1}{h}\Delta\eta.$$

We want to compute the normal derivatives of $u$ at the boundary in terms of tangential derivatives.

1) First of all, by means of a elliptic regularity result from [6] we show that $v\in H^{s+\frac12}\big([-1,0]\times\mathbb{R}^d\big)$.

2) Key step: paralinearization. Introduce

$$u=v-T_{\frac{\partial_z v}{\partial_z\rho}}\rho.$$

Then using the paradifferential calculus we find after some computations the following,

$$T_\alpha\partial_z^2 u+\Delta u+T_\beta\cdot\nabla\partial_z u-T_\gamma\partial_z u\in C_z^0\big([-1,0];H_x^{2s-\frac{5+d}{2}}(\mathbb{R}^d)\big). \tag{8}$$

(Notice that, in terms of the paracomposition operators introduced by Alinhac in [4], we have $u-\chi^*\phi\in H^{2s-1/2-d/2}(\mathbb{R}^d)$ where $\chi\colon(x,z)\mapsto(x,\rho(x,z))$ is the diffeomorphism used to map the domain $\Omega$ to the strip $S$. Therefore, the key identity (8) could also be seen as a direct consequence of the results in [4].)

3) Elliptic factorization. By using symbolic calculus for paradifferential operators, we next show that there exist two symbols such that

$$(\partial_z-T_a)(\partial_z-T_A)u=F\in C_z^0\big([-1,0];H_x^{s-\frac12+0}(\mathbb{R}^d)\big).$$

4) Elliptic regularity. Now introduce $w:=(\partial_z-T_A)u$, which satisfies

$$\partial_z w-T_a w=F\in C_z^0(H_x^{s-\frac12+0}).$$

This yields

$$(\partial_z u-T_A u)|_{z=0}=w(0)\in H^{s+\frac12}(\mathbb{R}^d).$$

This yields $\partial_z u$ on the boundary $\{z=0\}$ in terms of tangential derivatives, modulo an admissible remainder. □

### § 3.3. Paralinearization of the full system

Consider a given solution $(\eta,\psi)$ of (1) such that

$$\eta\in C^0\big([0,T];H^{s+\frac12}(\mathbb{R}^d)\big),\quad \psi\in C^0\big([0,T];H^s(\mathbb{R}^d)\big),$$

for some $s>2+d/2$, with $d\geq 1$.

**Proposition 3.4.** *Introduce $U = \psi - T_{\mathfrak{B}}\eta$. Then*

$$\begin{cases} \partial_t\eta + T_V \cdot \nabla\eta - T_\lambda U = f_1, \\ \partial_t U + T_V \cdot \nabla U + T_h\eta = f_2, \end{cases} \tag{9}$$

*where $\lambda$ is the symbol of $G(\eta)$, $h$ is the symbol of $-H(\eta)$ and*

$$\|(f_1,f_2)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s)} \le C\left(\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s)}\right).$$

## § 4. Symmetrization of the equations

### § 4.1. Construction of a paradifferential symmetrizer

**Proposition 4.1.** *There exists a **symmetrizer** $S$ of the form*

$$S = \begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix},$$

*which conjugates $\left(\begin{smallmatrix} 0 & -T_\lambda \\ T_h & 0 \end{smallmatrix}\right)$ to a skew-symmetric operator so that*

$$S\begin{pmatrix} 0 & -T_\lambda \\ T_h & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & -T_\gamma \\ (T_\gamma)^* & 0 \end{pmatrix} S,$$

*for some symbol $\gamma$ of order $3/2$. Here the notation $A \sim B$ means that $A - B$ is of order* order $(A)$ + order $(B) - 3/2$.

*Proof.* We seek $p, q, \gamma$ such that

$$\begin{cases} T_pT_\lambda \sim T_\gamma T_q, \\ T_qT_h \sim T_\gamma T_p, \\ T_\gamma \sim (T_\gamma)^*. \end{cases}$$

Example: if $\eta = 0$, then $\lambda = |\xi|$, $h = \xi^2$. We obtain the desired symmetrization with

$$p = |\xi|^{\frac{1}{2}}, \quad q = 1, \quad \gamma = |\xi|^{\frac{3}{2}}.$$

Therefore we seek $p, q, \gamma$ under the form

$$p = p^{(1/2)} + p^{(-1/2)}, \quad q = q^{(0)} + q^{(-1)}, \quad \gamma = \gamma^{(3/2)} + \gamma^{(1/2)},$$

where $a^{(m)}$ is a symbol homogeneous in $\xi$ of order $m \in \mathbb{R}$.

Using

$$h^{(1)} = -\frac{i}{2}(\partial_x \cdot \partial_\xi)h^{(2)}, \quad \operatorname{Im}\lambda^{(0)} = -\frac{1}{2}(\partial_x \cdot \partial_\xi)\lambda^{(1)},$$

and

$$h^{(2)} = \left(c\lambda^{(1)}\right)^2 \quad \text{with} \quad c = \left(1 + |\nabla\eta|^2\right)^{-\frac{3}{4}}.$$

we find (in a systematic way) the following explicit solution

$$\begin{aligned}
q &= \left(1 + |\nabla\eta|^2\right)^{-\frac{1}{2}}, \\
p &= \left(1 + |\nabla\eta|^2\right)^{-\frac{5}{4}} \sqrt{\lambda^{(1)}} + p^{(-1/2)}, \\
\gamma &= \sqrt{h^{(2)}\lambda^{(1)}} + \sqrt{\frac{h^{(2)}}{\lambda^{(1)}}} \frac{\operatorname{Re}\lambda^{(0)}}{2} - \frac{i}{2}(\partial_\xi \cdot \partial_x)\sqrt{h^{(2)}\lambda^{(1)}},
\end{aligned}$$

where

$$p^{(-1/2)} = \frac{1}{\gamma^{(3/2)}} \left\{ q^{(0)} h^{(1)} - \gamma^{(1/2)} p^{(1/2)} + i\partial_\xi \gamma^{(3/2)} \cdot \partial_x p^{(1/2)} \right\}.$$

□

## § 4.2. Reduction

Introduce the new unknown

$$\Phi = T_p \eta + i T_q U$$

where recall that $U = \psi - T_{\mathfrak{B}}\eta$. Then $\Phi \in C^0([0,T]; H^s(\mathbb{R}^d))$ and

$$\partial_t \Phi + T_V \cdot \nabla\Phi + iT_\gamma \Phi = F, \tag{10}$$

where $F \in L^\infty(0,T; H^s(\mathbb{R}^d))$. Moreover

$$\|F\|_{L^\infty(0,T;H^s\times H^s)} \leq C\left(\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s)}\right),$$

for some function $C$ depending only on $\operatorname{dist}(\Sigma_0, \Gamma)$.

## § 4.3. A priori estimate

There are many subtleties to prove the existence and the uniqueness of the solutions. Here we shall only mention how we prove *a priori* estimates. Paradifferential calculus is the most simple tool to prove nonlinear estimates. However, we need to explain how to deal with low regularity indexes, what are the good norms on $H^s$ to prove energy estimates and how we define mollifiers.

Energy estimates: To obtain an $H^s$ estimate for $\Phi$, instead of using $(I-\Delta)^{s/2}$, we commute the reduced system with

$$T_{\left(\gamma^{(3/2)}\right)^{\frac{2s}{3}}}.$$

The symbol $\left(\gamma^{(3/2)}\right)^{\frac{2s}{3}}$ is homogeneous of order $s$, elliptic, and such that

$$\left\{\left(\gamma^{(3/2)}\right)^{\frac{2s}{3}},\gamma\right\}=0.$$

Calculus with low regularity:

One can check that the sub-principal symbols $\lambda^{(0)}, h^{(1)}, p^{(-1/2)}, \gamma^{(1/2)}$ depend only linearly on $\nabla^2\eta$. This observation and some technical remarks about paradifferential operators are the key to prove a result valid for $s>2+d/2$ ($s>3+d/2$ is easier).

Mollifiers: Because of lack of commutations, we cannot use usual mollifiers of the form $\chi(\varepsilon D_x)$. Instead we use the following variant. Given $\varepsilon\in[0,1]$, we define $J_\varepsilon$ as the paradifferential operator with symbol $\jmath_\varepsilon=\jmath_\varepsilon(t,x,\xi)$ given by

$$\jmath_\varepsilon=\jmath_\varepsilon^{(0)}+\jmath_\varepsilon^{(-1)}=\exp\big(-\varepsilon\gamma^{(3/2)}\big)-\frac{i}{2}(\partial_x\cdot\partial_\xi)\exp\big(-\varepsilon\gamma^{(3/2)}\big).$$

Then

$$\{\jmath_\varepsilon^{(0)},\gamma^{(3/2)}\}=0,\quad \operatorname{Im}\jmath_\varepsilon^{(-1)}=-\frac{1}{2}(\partial_x\cdot\partial_\xi)\jmath_\varepsilon^{(0)}.$$

Of course, for any $\varepsilon>0$, $\jmath_\varepsilon\in C^0([0,T];\Gamma^m_{3/2}(\mathbb{R}^d))$ for all $m\le 0$. However, the important fact is that $\jmath_\varepsilon$ is uniformly bounded in $C^0([0,T];\Gamma^0_{3/2}(\mathbb{R}^d))$ for all $\varepsilon\in[0,1]$. Therefore, we have the following uniform estimates:

$$\begin{aligned}&\|J_\varepsilon T_\gamma-T_\gamma J_\varepsilon\|_{H^\mu\to H^\mu}\le C(\|\nabla\eta\|_{W^{3/2,\infty}}),\\&\|(J_\varepsilon)^*-J_\varepsilon\|_{H^\mu\to H^{\mu+3/2}}\le C(\|\nabla\eta\|_{W^{3/2,\infty}}).\end{aligned}$$

## § 5. The Kato smoothing effect

We want to prove that if

$$(\eta,\psi)\in C^0([0,T];H^{s+\frac12}(\mathbb{R})\times H^s(\mathbb{R})),$$

then

$$\langle x\rangle^{-\frac12-\delta}(\eta,\psi)\in L^2(0,T;H^{s+\frac34}(\mathbb{R})\times H^{s+\frac14}(\mathbb{R})),$$

for all $\delta>0$.

The proof of this result follows essentially Doi's strategy for proving analog results on Schrödinger equations, and we construct first a symbol having a positive Poisson bracket with the principal symbol of the operator $|D_x|^{\frac34}T_c|D_x|^{\frac34}$. Then we can apply a non classical Gårding inequality to conclude.

**1. Reduction.** Recall that $\Phi$ solves

$$\partial_t\Phi+T_V\partial_x\Phi+i\,|D_x|^{\frac34}T_c\,|D_x|^{\frac34}\Phi=F\in L^\infty(0,T;H^s(\mathbb{R})).$$

**2. Doi's Lemma.**

**Lemma 5.1.** *There exists a symbol $a = a(x,\xi)$ homogeneous of degree 0 in $\xi$ and $C^\infty$ in $x$, such that*

$$\forall \delta > 0, \exists K > 0/ \quad \left\{ c\left|\xi\right|^{\frac{3}{2}}, a \right\}(t,x,\xi) \geq K \frac{\left|\xi\right|^{1/2}}{\left(1+\left|x\right|\right)^{1+\delta}}.$$

To see this, the key ingredient is

$$\begin{aligned}\left\{ c\left|\xi\right|^{\frac{3}{2}}, \frac{x\xi}{\left|\xi\right|} \right\} &= \partial_\xi (c\left|\xi\right|^{\frac{3}{2}})\partial_x \left(\frac{x\xi}{\left|\xi\right|}\right) - \partial_x (c\left|\xi\right|^{\frac{3}{2}})\partial_\xi \left(\frac{x\xi}{\left|\xi\right|}\right) \\ &= c(\partial_\xi \left|\xi\right|^{\frac{3}{2}})\frac{\xi}{\left|\xi\right|} - 0 \\ &= \frac{3}{2} c \left|\xi\right|^{\frac{1}{2}}.\end{aligned}$$

**3. A Gårding inequality.** Let $d \geq 1$ and $\delta > 0$. Assume that $d \in \Gamma^{1/2}_{1/2}(\mathbb{R}^d)$ is such that

$$d(x,\xi) \geq K \langle x \rangle^{-1-2\delta} \left|\xi\right|^{\frac{1}{2}}.$$

Then we have

$$\langle T_d u, u \rangle \geq a \left\| \langle x \rangle^{-\frac{1}{2}-\delta} u \right\|^2_{H^{\frac{1}{4}}} - A \left\| u \right\|^2_{L^2}.$$

**4.** We conclude by means of classical arguments and the nonlinear estimates used to study the Cauchy problem.

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