Existence of solution for Navier-Stokes equations in modulation spaces

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1 Introduction

We consider the Cauchy problems for Navier-Stokes equations

$$
(\mathrm{P})\begin{cases}\partial_t u-\Delta u+(u\cdot\nabla)u+\nabla\pi=0 & \text{for}\quad (t,x)\in(0,\infty)\times\mathbb{R}^n,\\ \operatorname{div}\, u=0 & \text{for}\quad (t,x)\in(0,\infty)\times\mathbb{R}^n,\\ u(0,x)=u_0(x) & \text{for}\quad x\in\mathbb{R}^n,\end{cases}
$$

in modulation spaces which is defined as follows.

Definition 1.1 *(Modulation spaces) Let $\{\varphi_k\}_{k\in\mathbb{Z}^n}\subset C_0^\infty(\mathbb{R}^n)$ be a partition of unity satisfying the following. Let $\varphi\in C_0^\infty(\mathbb{R}^n)$ satisfy*

$$
\operatorname{supp}\varphi\subset\{\ \xi\in\mathbb{R}^n\ \mid\ |\xi|\le\sqrt{n}\ \},\qquad \sum_{k\in\mathbb{Z}^n}\varphi(\xi-k)=1\quad \textit{for any } \xi\in\mathbb{R}^n.
$$

Let φ_k be defined by

$$
\varphi_k(\xi):=\varphi(\xi-k), \tag{1.1}
$$

$$
\Box_k:=\mathcal{F}^{-1}\varphi_k\mathcal{F}. \tag{1.2}
$$

For $1\le q,\sigma\le\infty, -\infty<s<\infty$, we define $M^s_{q,\sigma}(\mathbb{R}^n)$ by

$$
M^s_{q,\sigma}(\mathbb{R}^n):=\left\{f\in\mathcal{S}'(\mathbb{R}^n)\ \middle|\ ||f||_{M^s_{q,\sigma}(\mathbb{R}^n)}<\infty\right\},
$$

$$
||f||_{M^s_{q,\sigma}(\mathbb{R}^n)}:=\begin{cases}\Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{s\sigma}||\Box_k f||^\sigma_{L^q(\mathbb{R}^n)}\Big)^{\frac{1}{\sigma}} & \textit{for}\quad 1\le\sigma<\infty,\\ \sup_{k\in\mathbb{Z}^n}\langle k\rangle^s||\Box_k f||_{L^q(\mathbb{R}^n)} & \textit{for}\quad \sigma=\infty,\end{cases}
$$

where $\langle k\rangle:=(1+|k|^2)^{\frac{1}{2}}$.

On the study of applications to partial differential equations in modulation spaces, Wang, Zhao and Guo [21] have considered local solutions for nonlinear Schrödinger equations and Navier-Stokes equations for initial data u_0 in $M^0_{2,1}(\mathbb{R}^n)$. Wang and Hudzik [20] have considered global solutions for nonlinear Schrödinger and Klein-Gordon equations with nonlinear term u^p for u_0 in $M^s_{2,\sigma}(\mathbb{R}^n)$ when $\sigma=1$ and $s=0$ for Schrödinger equations, and when $1\le\sigma<np/(np+2s-1)$ and $s=(n+2)/n(p+1)$ for Klein-Gordon

equations. The local and global solutions for nonlinear heat equations and Navier-Stokes equations in $M^0_{q,\sigma}(\mathbb{R}^n)$ are obtained by our previous work [8].

In this paper, we extend our results in [8] for Navier-Stokes equations and nonlinear heat equations. In particular, we treat the derivative index $s \in \mathbb{R}$. It is important to consider the nonlinear partial differential equations in function spaces which have scaling invariance, such as Lebesgue spaces, Sobolev spaces, Besov spaces, Lizorkin-Triebel spaces. In fact, if initial data of the problem (P) is in the Lebesgue space $L^n(\mathbb{R}^n)$ and sufficiently small, there exists a global solution, which was proved by Kato [10], and the index n of Lebesgue space $L^n(\mathbb{R}^n)$ is the critical index which is obtained by the scaling argument for Navier-Stokes equations (P).

On the property of modulation spaces, the scaling argument does not work in general since we define the partition of unity $\{\varphi_k\}_{k\in\mathbb{Z}^n}$ in modulation spaces using the translations of the smooth function with compact support. The detail is studied in the result by Sugimoto and Tomita [17], who have studied the dilation property of modulation spaces. In the result [8] in modulation spaces, the global behavior is controlled by the index q and this point is just same in this paper. Roughly speaking about our theorem, the local behavior of solutions is controlled by the indices s and σ, and the global behavior is controlled by the index q. Here, local behavior is the singularity at $t = 0$ related to the smoothing effect of linear heat equations, and global behavior is the decay rate as $t \to \infty$.

For Navier-Stokes equations (P), Fujita and Kato [6] considered the Cauchy problems in $H^{1/2}_0(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^3$ in 1964. Kato [10] proved the existence of solutions in Lebesgue spaces $L^n(\mathbb{R}^n)$ and showed that the solution exists globally in time when initial data $u_0 \in L^n(\mathbb{R}^n)$ is sufficiently small. In 1985, Giga and Miyakawa [7] considered the Cauchy problems in $L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$. Cannone [4] and Planchon [14] considered global solutions in the case of $n = 3$ for small initial data u_0 in $B^{-1+3/q}_{q,\infty}(\mathbb{R}^3)$ $(3 < q \le 6)$ and $L^3(\mathbb{R}^3)$. Koch and Tataru [11] studied local solutions for initial data $u_0 \in vmo^{-1}$ and global solutions for small initial data $u_0 \in BMO^{-1}$. Miura [13] studied the local solutions, which have time continuity in gmo^{-1}, for initial data $u_0 \in vmo^{-1} \cap gmo^{-1}$. On the Cauchy problems in modulation spaces, Wang, Zhao and Guo [21] proved the existence of local solutions for initial data $u_0 \in M^0_{2,1}(\mathbb{R}^n)$. In our previous work [8], we proved the existence of local solutions for initial data $u_0 \in M^0_{q,\sigma}(\mathbb{R}^n)$ if $1 \le q \le \infty$, $1 \le \sigma \le n/(n-1)$. Addition to the condition of the existence of local solutions, we obtain the global solutions if $q \le n$ and initial data is sufficiently small.

We study the following integral equation

$$(NS) \qquad u(t) = e^{t\Delta}u_0 - \int_0^t \nabla e^{(t-\tau)\Delta} P(u\otimes u)(\tau)d\tau,$$

where $P := 1 + (-\Delta)^{-1}\nabla\mathrm{div}$ and $\otimes$ denotes tensor product. Let $PM^s_{q,\sigma}(\mathbb{R}^n)$ be defined by

$$PM^s_{q,\sigma}(\mathbb{R}^n) := \{\ u \in [M^s_{q,\sigma}(\mathbb{R}^n)]^n \ \mid \ \mathrm{div}\ u = 0 \text{ in } \mathcal{S}'(\mathbb{R}^n)\ \},$$

$$||u||_{M^s_{q,\sigma}(\mathbb{R}^n)} := \sum_{j=1}^n ||u_j||_{M^s_{q,\sigma}(\mathbb{R}^n)},$$

where $u=(u_1,\cdots,u_n)$. To state our theorems, we introduce the following function spaces $l^{s,\sigma}_{\square}(L^r(0,T;L^q(\mathbb{R}^n)))$ which were introduced by Wang, Hudzik [20].

Definition 1.2 *For $s\in\mathbb{R}$, $1\le\sigma,r,q\le\infty$ and $0<T\le\infty$, we define the function spaces $l^{s,\sigma}_{\square}(L^r(0,T;L^q(\mathbb{R}^n)))$ by*

$$l^{s,\sigma}_{\square}(L^r(0,T;L^q(\mathbb{R}^n))) := \{\ f\in\mathcal{S}'(\mathbb{R}\times\mathbb{R}^n)\ |\ \|f\|_{l^{s,\sigma}_{\square}(L^r(0,T;L^q(\mathbb{R}^n)))}<\infty\ \},$$

$$\|f\|_{l^{s,\sigma}_{\square}(L^r(0,T;L^q(\mathbb{R}^n)))} := \begin{cases} \Big(\sum\limits_{k\in\mathbb{Z}^n}\Big(\langle k\rangle^s\|\square_k f\|_{L^r(0,T;L^q(\mathbb{R}^n))}\Big)^\sigma\Big)^{\frac{1}{\sigma}} & \text{if } \sigma<\infty,\\ \sup\limits_{k\in\mathbb{Z}^n}\langle k\rangle^s\|\square_k f\|_{L^r(0,T;L^q(\mathbb{R}^n))} & \text{if } \sigma=\infty,\end{cases}$$

where $\square_k$ is defined by (1.2).

Theorem 1.3 *Let n, s, q, σ satisfy*

$$n\ge 2,\quad 1\le q\le\infty,\quad 1\le\sigma<\infty,\quad \frac{n(\sigma-1)}{\sigma}-1\le s.$$

Then, for any $u_0\in PM^s_{q,\sigma}(\mathbb{R}^n)$ there exists $T>0$ such that (NS) has a unique solution u in X_T, where

$$X_T := \Big\{\ u\in\Big[C([0,T],M^s_{q,\sigma}(\mathbb{R}^n))\Big]^n\ \ |\ ||u||_{X_T}<\infty,\ \mathrm{div}\ u=0\ \Big\},$$
$$||u||_{X_T} := \sup_{t\in(0,T)}||u(t)||_{M^s_{q,\sigma}(\mathbb{R}^n)}+\|u\|_Z,$$

$$\|u\|_Z := \begin{cases} \|u\|_{l^{0,1}_{\square}(L^2(0,T;L^q(\mathbb{R}^n)))} & \text{if } s=-1,\\ \sup\limits_{t\in(0,T)} t^{\frac{|s|}{2}}\|u(t)\|_{M^0_{q,\sigma}(\mathbb{R}^n)} & \text{if } -1<s<0,\\ \sup\limits_{t\in(0,T)} t^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}\|u(t)\|_{M^s_{q,\nu}(\mathbb{R}^n)} & \text{if } s=\dfrac{n(\sigma-1)}{\sigma}-1\ge 0 \text{ or } 0\le s\le\dfrac{n(\sigma-1)-n}{\sigma},\\ 0 & \text{otherwise},\end{cases}$$

where ν is an arbitrary real number satisfying

$$\frac{1}{\sigma}<\frac{1}{\nu}<\frac{1}{\sigma}+\frac{n(\sigma-1)-\sigma s}{2\sigma n}.$$

In addition, let $u,v\in X_T$ be each solution for initial data u_0, v_0, respectively, it holds that

$$\|u-v\|_{X_T}\to 0\qquad \text{as } v_0\to u_0 \text{ in } [M^s_{q,\sigma}(\mathbb{R}^n)]^n.$$

Furthermore, if $q\le n$ and initial data u_0 is sufficiently small, the solution exists globally in time.

Remark 1.4 *If $s>n(\sigma-1)/\sigma-1$, or initial data u_0 is sufficiently small, we can take the existence time $T>0$ depending only on the norm of initial data u_0. If not so, we take $T>0$ small for each initial data u_0. We prove them in Section 3 together with the proof of Theorem 1.3.*

Remark 1.5 *The range of s is in the interval $[-1, \infty)$ and this lower bound -1 is optimal. In fact, we can show the ill-posedness for Navier-Stokes equations* (NS) *in $PM^s_{2,\sigma}(\mathbb{R}^n)$ if $s < -1$ and $1 \le \sigma < \infty$, which is proved in the appendix of this paper in the way of the result by Bejenaru and Tao [1]. In their way, we show that continuous dependence on initial data fails.*

Remark 1.6 *In some cases, we can show that initial data in Theorem 1.3 is included in the results by Koch and Tataru [11], or Miura [13]. If the derivative index $s = -1$, we have $\sigma = 1$ and*

$$M^{-1}_{q,1}(\mathbb{R}^n) \subset vmo^{-1} \cap gmo^{-1} \quad \textit{if} \quad 1 \le q \le \infty. \tag{1.3}$$

Therefore, if $s = -1$, initial data u_0 for local solutions is included in the result by Miura [13] who proved the existence of local solutions for $u_0 \in vmo^{-1} \cap gmo^{-1}$. For global solutions, we have

$$M^s_{q,\sigma}(\mathbb{R}^n) \subset BMO^{-1} \quad \textit{if} \quad -1 \le s \le 0,\ s = \frac{n(\sigma-1)}{\sigma} - 1 \textit{ and } 1 \le q \le n. \tag{1.4}$$

Therefore, if $-1 \le s \le 0$, initial data u_0 for global solutions is included in the result by Koch, Tataru [11] who proved the existence of global solutions for $u_0 \in BMO^{-1}$. The proofs of (1.3) and (1.4) are given in [9]. However, we don't know such relations in the case $s = n(\sigma-1)/\sigma - 1 > -1$ for local solutions, and the case $s = n(\sigma-1)/\sigma - 1 > 0$ for global solutions.

Remark 1.7 *The global solutions in Theorem 1.3 are obtained by the smoothing effect of the propagator $e^{t\Delta}$. In fact, the solution $u(t)$ in Theorem 1.3 is small and in the Lebesgue space $L^n(\mathbb{R}^n)$ if $0 < t \le 1$ and initial data u_0 is sufficiently small. Therefore, the solution $u(t)$ exists globally in time by the result of Kato [10], who proved the existence of global solutions for small initial data $u_0 \in L^n(\mathbb{R}^n)$.*

In this paper, we show some properties of modulation spaces and the propagator $e^{t\Delta}$ in the modulation spaces in Section 2. We prove the theorems for Navier-Stokes equations in Section 3 and the claim of Remark 1.5 in Section 4.

2 Preliminaries

In this section, we consider the properties of modulation spaces and the function spaces $l^{s,\sigma}_{\square}(L^r(0,T; L^q(\mathbb{R}^n)))$, and the estimates for the propagator of $e^{t\Delta}$ in those function spaces. For simplicity, we put $l^{s,\sigma}L^r_TL^q := l^{s,\sigma}_{\square}(L^r(0,T; L^q(\mathbb{R}^n)))$ and $\| \cdot \|_{M^s_{q,\sigma}} := \| \cdot \|_{M^s_{q,\sigma}(\mathbb{R}^n)}$ in the following sections. The constant C denotes an absolute positive constant which can change in each line.

2.1 Modulation spaces and $l^{s,\sigma}L^r_TL^q$

In this section, we introduce some properties of modulation spaces and $l^{s,\sigma}L^r_TL^q$.

Proposition 2.1 *[5,18,20] Let $1 \le q, q_1, q_2, \sigma, \sigma_1, \sigma_2 \le \infty$. Then we have the following continuous embeddings.*

(i) *If* $1/q+1/q'=1$, *then*

$$M^0_{q,\min\{q,q'\}}(\mathbb{R}^n)\hookrightarrow L^q(\mathbb{R}^n)\hookrightarrow M^0_{q,\max\{q,q'\}}(\mathbb{R}^n).$$

(ii) *If* $q_1\le q_2,\ \sigma_1\le\sigma_2,\ s_1\ge s_2$, *then*

$$M^{s_1}_{q_1,\sigma_1}(\mathbb{R}^n)\hookrightarrow M^{s_2}_{q_2,\sigma_2}(\mathbb{R}^n).$$

(iii) *If* $\sigma_1\ge\sigma_2,\ \ s_1>s_2,\ \ s_1-s_2>n(1/\sigma_2-1/\sigma_1)$, *then*

$$M^{s_1}_{q,\sigma_1}(\mathbb{R}^n)\hookrightarrow M^{s_2}_{q,\sigma_2}(\mathbb{R}^n).$$

Remark 2.2 *Similar embeddings to (ii) and (iii) in Proposition 2.1 also hold. That is, let* $1\le r\le\infty$, $T>0$ *and* s_j,q_j,σ_j $(j=1,2)$ *be same as Proposition 2.1, we have*

$$l^{s_1,\sigma_1}L^r_TL^{q_1}\hookrightarrow l^{s_2,\sigma_2}L^r_TL^{q_2},$$

if $q_1\le q_2,\sigma_1\le\sigma_2,s_1\ge s_2$ *or* $\sigma_1>\sigma_2,s_1>s_2,s_1-s_2>n(1/\sigma_2-1/\sigma_1)$. *In addition, the embedding in (iii) of Proposition 2.1 does not hold if* $s_1-s_2=n(1/\sigma_2-1/\sigma_1)$ *(see [20]).*

Proposition 2.3 *Let* $s\ge0, 1\le q,r,\sigma,q_j,r_j,\sigma_j\le\infty\,(j=1,2,3,4), 0<T\le\infty$ *satisfy*

$$\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q_3}+\frac{1}{q_4},\quad \frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r_3}+\frac{1}{r_4},\quad \frac{1}{\sigma}=\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-1=\frac{1}{\sigma_3}+\frac{1}{\sigma_4}-1.$$

There exists $C>0$ *such that for any* $f\in l^{s,\sigma_1}L^{r_1}_TL^{q_1}\cap l^{0,\sigma_3}L^{r_3}_TL^{q_3}, g\in l^{0,\sigma_2}L^{r_2}_TL^{q_2}\cap l^{s,\sigma_4}L^{r_4}_TL^{q_4}$, *we have*

$$||fg||_{l^{s,\sigma}L^r_TL^q}\le C||f||_{l^{s,\sigma_1}L^{r_1}_TL^{q_1}}||g||_{l^{0,\sigma_2}L^{r_2}_TL^{q_2}}+C||f||_{l^{0,\sigma_3}L^{r_3}_TL^{q_3}}||g||_{l^{s,\sigma_4}L^{r_4}_TL^{q_4}}. \tag{2.1}$$

Remark 2.4 *In modulation spaces, it is known that*

$$||fg||_{M^s_{q,\sigma}}\le C||f||_{M^s_{q_1,\sigma_1}}||g||_{M^0_{q_2,\sigma_2}}+C||f||_{M^0_{q_3,\sigma_3}}||g||_{M^s_{q_4,\sigma_4}}. \tag{2.2}$$

These $q,q_j,\sigma,\sigma_j\,(j=1,2,3,4)$ *correspond to those of Proposition 2.3, which is proved in the similar way to the proof of (2.2) in [8].*

Lemma 2.5 *Let* $1<\sigma,\sigma_1,\sigma_2<\infty$, $0\le\alpha<n/\sigma, 0\le\beta<n(1+1/\sigma-1/\sigma_2)-\alpha$ *satisfy* $1/\sigma-(\alpha+\beta)/n=1/\sigma_1+1/\sigma_2-1$, *and* $\sigma\ge\sigma_1$, *there exists* $C>0$ *such that for any* $f\in L^{\sigma_1}(\mathbb{R}^n)$ *and* $g\in L^{\sigma_2}(\mathbb{R}^n)$, *we have*

$$\left\||x|^{-\alpha}\left((|\cdot|^{-\beta}f)*g\right)\right\|_{L^\sigma(\mathbb{R}^n)}\le C\|f\|_{L^{\sigma_1}(\mathbb{R}^n)}\|g\|_{L^{\sigma_2}(\mathbb{R}^n)}.$$

Remark 2.6 *Similar lemma to Lemma 2.5 can be verified in sequence spaces instead of Lebesgue spaces. That is, for any* $\{a_k\}_{k\in\mathbb{Z}^n}\in l^{\sigma_1}(\mathbb{Z}^n)$ *and* $\{b_k\}_{k\in\mathbb{Z}^n}\in l^{\sigma_2}(\mathbb{Z}^n)$, *we have*

$$\left\|\left\{\langle k\rangle^{-\alpha}\sum_{m\in\mathbb{Z}^n}\frac{a_{k-m}}{\langle k-m\rangle^\beta}b_m\right\}_{k\in\mathbb{Z}^n}\right\|_{l^\sigma(\mathbb{Z}^n)}\le C\left\|\{a\}_{k\in\mathbb{Z}^n}\right\|_{l^{\sigma_1}(\mathbb{Z}^n)}\left\|\{b\}_{k\in\mathbb{Z}^n}\right\|_{l^{\sigma_2}(\mathbb{Z}^n)}, \tag{2.3}$$

where $\sigma, \sigma_1, \sigma_2, \alpha, \beta$ are same as Lemma 2.5. We can also obtain the same estimate on the condition of $\sigma \geq \sigma_2$ instead of $\sigma \geq \sigma_1$.

The proof of Lemma 2.5 is given in [9]. The following proposition is necessary for the critical case of Theorem 1.3, which is the case of $s = n(\sigma - 1)/\sigma - 1$.

Proposition 2.7 *Let $1 \leq q, q_1, q_2, \leq \infty$, $1 < \sigma, \sigma_1, \sigma_2 < \infty$, $0 < T \leq \infty$ and $0 < s < n/\sigma$. If*

$$\begin{aligned} \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \quad & \sigma \geq \sigma_1, \sigma_2, \\ \frac{1}{\sigma} - \frac{s}{n} = \frac{1}{\sigma_1} + \frac{1}{\sigma_2} - 1, & \end{aligned} \tag{2.4}$$

Then, there exists $C > 0$ such that, for any $u \in M^s_{q_1,\sigma_1}(\mathbb{R}^n)$ and $v \in M^s_{q_2,\sigma_2}(\mathbb{R}^n)$, we have

$$\|uv\|_{M^s_{q,\sigma}} \leq C\|u\|_{M^s_{q_1,\sigma_1}} \|v\|_{M^s_{q_2,\sigma_2}}. \tag{2.5}$$

Proof. To prove (2.5), we have

$$\begin{aligned} \|uv\|^\sigma_{M^s_{q,\sigma}(\mathbb{R}^n)} &\leq \sum_{k\in\mathbb{Z}^n} \Big(\langle k\rangle^s \sum_{k_1\in\mathbb{Z}^n} \|\Box_{k_1} u\|_{L^{q_1}(\mathbb{R}^n)} \sum_{|k_2|\leq 3\sqrt{n}} \|\Box_{k-k_1-k_2} v\|_{L^{q_2}(\mathbb{R}^n)} \Big)^\sigma \\ &\leq C \sum_{k\in\mathbb{Z}^n} \Big(\sum_{k_1\in\mathbb{Z}^n} \langle k_1\rangle^s \|\Box_{k_1} u\|_{L^{q_1}(\mathbb{R}^n)} \sum_{|k_2|\leq\sqrt{n}} \|\Box_{k-k_1-k_2} v\|_{L^{q_2}(\mathbb{R}^n)} \Big)^\sigma \\ &\quad + C \sum_{k\in\mathbb{Z}^n} \Big(\sum_{k_1\in\mathbb{Z}^n} \|\Box_{k_1} u\|_{L^{q_1}(\mathbb{R}^n)} \sum_{|k_2|\leq\sqrt{n}} \langle k-k_1-k_2\rangle^s \|\Box_{k-k_1-k_2} v\|_{L^{q_2}(\mathbb{R}^n)} \Big)^\sigma. \end{aligned}$$

Here, we used $\langle k\rangle^s \leq C(\langle k_1\rangle^s + \langle k-k_1-k_2\rangle^s)$ if $|k_2| \leq 3\sqrt{n}$. Therefore, we obtain the desired estimate applying (2.3). In fact, the first term in the right hand side is estimated by (2.3) with

$$\alpha := 0, \quad \beta := s, \quad a_k := \langle k\rangle^s \sum_{|k_2|\leq\sqrt{n}} \|\Box_{k-k_2} v\|_{L^{q_2}(\mathbb{R}^n)}, \quad b_k := \langle k\rangle^s \|\Box_k u\|_{L^{q_1}(\mathbb{R}^n)}.$$

The second term follows similarly. ❑

2.2 The properties for $e^{t\Delta}$

In this section, we consider the properties of $e^{t\Delta}$ in modulation spaces.

Proposition 2.8 *Let $1 \leq q, r, \sigma, \nu \leq \infty$, $s, \widetilde{s} \in \mathbb{R}$.*

(i) *If $q \geq r$, there exists a constant $C > 0$ such that*

$$||e^{t\Delta} f||_{M^s_{q,\sigma}} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} ||f||_{M^s_{r,\sigma}}. \tag{2.6}$$

(ii) *If $\sigma \le \nu$, there exists a constant $C > 0$ such that*

$$||e^{t\Delta}f||_{M^0_{q,\sigma}} \le C\left(1 + t^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}\right)||f||_{M^0_{q,\nu}}. \tag{2.7}$$

(iii) *If $s \le \tilde{s}$, there exists constant $C > 0$ such that*

$$||e^{t\Delta}f||_{M^{\tilde{s}}_{q,\sigma}} \le C(1 + t^{-\frac{\tilde{s}-s}{2}})||f||_{M^s_{q,\sigma}}. \tag{2.8}$$

Proof. (i) and (ii) was proved in [8]. Therefore, we prove (iii) and only treat the case $\tilde{s} = 0$ and $s \le 0$ for simplicity. On the low frequency part of the norm, that is, let k satisfy $|k| \le 3\sqrt{n}$, we have from $L^q(\mathbb{R}^n)$ boundedness of $e^{t\Delta}$

$$\sum_{|k|\le 3\sqrt{n}} \|\Box_k e^{t\Delta} f\|^\sigma_{L^q} \le C \sum_{|k|\le 3\sqrt{n}} \langle k\rangle^{s\sigma} \|\Box_k f\|^\sigma_{L^q} \le C\|f\|^\sigma_{M^s_{q,\sigma}}.$$

On the high frequency part of the norm, we use Lemma 2.5 in [8];

$$\|\Box_k e^{t\Delta} f\|_{L^q} \le Ce^{-ct|k|^2}\|\Box_k f\|_{L^q} \quad \text{if } |k| \ge 3\sqrt{n}. \tag{2.9}$$

So that, we have

$$\|\Box_k e^{t\Delta} f\|_{L^q} \le C|k|^{|s|}e^{-ct|k|^2}\langle k\rangle^s\|\Box_k f\|_{L^q} \le Ct^{-\frac{|s|}{2}}\langle k\rangle^s\|\Box_k f\|_{L^q}.$$

Taking the sequence norm $l^\sigma(\mathbb{Z}^n)$, we obtain the desired estimate. ❑

Proposition 2.9 *Let $1 \le q \le \infty$, $1 \le \nu \le \sigma < \infty$, $s \in \mathbb{R}$ and $\alpha \ge 0$. If $\alpha > 0$ or $\nu < \sigma$, then we have*

$$\lim_{T\searrow 0} \sup_{t\in(0,T)} t^{\frac{\alpha}{2}+\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}||e^{t\Delta}f||_{M^{s+\alpha}_{q,\nu}} = 0 \quad \textit{for any } f \in M^s_{q,\sigma}(\mathbb{R}^n).$$

Proof. This proposition is proved in the similar way to Proposition 2.8 in [8]. ❑

Proposition 2.10 *Let $1 \le q \le \infty$, $1 \le \sigma < \infty$ and $0 < \alpha \le 1$, then we have*

(i) $\|e^{t\Delta}f\|_{l^{0,\sigma}L^{\frac{2}{\alpha}}_T L^q} \le C(1 + T^{\frac{\alpha}{2}})\|f\|_{M^{-\alpha}_{q,\sigma}}$ *for any $f \in M^{-\alpha}_{q,\sigma}(\mathbb{R}^n)$.*

(ii) $\lim_{T\searrow 0} \|e^{t\Delta}f\|_{l^{0,\sigma}L^{\frac{2}{\alpha}}_T L^q} = 0$ *for any $f \in M^{-\alpha}_{q,\sigma}(\mathbb{R}^n)$.*

Remark 2.11 *The way of the proof of Proposition 2.10 is using (2.9) and the argument of Proposition 2.8 in [8]. The details are given in [9].*

Proposition 2.12 *Let $s \in \mathbb{R}$, $1 \le r, q, \sigma \le \infty$ and $0 < T \le \infty$.*

(i) *There exists $C > 0$ such that for any $f \in l^{s,\sigma}L^1_T L^q$, we have*

$$\sup_{t\in(0,T)} \left\|\int_0^t e^{(t-\tau)\Delta}f(\tau)d\tau\right\|_{M^s_{q,\sigma}} \le C\|f\|_{l^{s,\sigma}L^1_T L^q}.$$

(ii) *There exists $C > 0$ such that for any $f \in l^{s-\frac{2}{r},\sigma} L^1_T L^q$, we have*

$$\left\| \int_0^t e^{(t-\tau)\Delta} f(\tau) d\tau \right\|_{l^{s,\sigma} L^r_T L^q} \leq C(1 + T^{\frac{1}{r}}) \|f\|_{l^{s-\frac{2}{r},\sigma} L^1_T L^q}.$$

Proof. We show (i). Since $\Box_k$ and $e^{t\Delta}$ are commutative and $e^{t\Delta}$ is a bounded operator in Lebesgue spaces, we have

$$\left\| \Box_k \int_0^t e^{(t-\tau)\Delta} f(\tau) d\tau \right\|_{L^q(\mathbb{R}^n)} \leq \int_0^T \|\Box_k f(\tau)\|_{L^q(\mathbb{R}^n)} d\tau.$$

Summing over $k \in \mathbb{Z}^n$ after multiplying $\langle k \rangle^s$ in both sides, we obtain the desired estimate.

We show (ii) in the case $s = 0$ for simplicity. We treat the low and high frequency separately. First, we consider the estimate of low frequency. In this case, we assume $|k| \leq 3\sqrt{n}$, and applying the boundedness of the propagator $e^{t\Delta}$, we have

$$\begin{aligned} \left\| \Box_k \int_0^t e^{(t-\tau)\Delta} f(\tau) d\tau \right\|_{L^r_T L^q} &\leq C T^{\frac{1}{r}} \int_0^T \|\Box_k f(\tau)\|_{L^q(\mathbb{R}^n)} d\tau \\ &\leq C T^{\frac{1}{r}} \langle k \rangle^{-\frac{2}{r}} \|\Box_k f\|_{L^1_T L^q}. \end{aligned} \tag{2.10}$$

We treat the high frequency next. In this case, we assume $|k| \geq 3\sqrt{n}$ and have from (2.9)

$$\left\| \Box_k \int_0^t e^{(t-\tau)\Delta} f(\tau) d\tau \right\|_{L^r_T L^q} \leq C \left\| \int_0^t e^{-c(t-\tau)|k|^2} \|\Box_k f(\tau)\|_{L^q(\mathbb{R}^n)} d\tau \right\|_{L^r_T}.$$

Applying Hausdorff-Young's inequality, we have

$$\left\| \Box_k \int_0^t e^{(t-\tau)\Delta} f(\tau) d\tau \right\|_{L^r_T L^q} \leq C \langle k \rangle^{-\frac{2}{r}} \|\Box_k f\|_{L^1_T L^q}. \tag{2.11}$$

Taking the sequence norm $l^\sigma(\mathbb{Z}^n)$, we obtain the estimate (ii) by (2.10) and (2.11). ❑

2.3 The boundedness of the operator P

We show a kind of boundedness of $P = 1 + (-\Delta)^{-1}\nabla$ div in modulation spaces.

Lemma 2.13 *Let $1 \leq q \leq \infty$, $f \in \mathcal{S}'(\mathbb{R}^n)$.*

(i) *If $|k| \leq 3\sqrt{n}$, we have*

$$||\Box_k \nabla P f||_{L^q(\mathbb{R}^n)} \leq C \|\Box_k f\|_{L^q(\mathbb{R}^n)} \quad \textit{if} \quad \|\Box_k f\|_{L^q(\mathbb{R}^n)} < \infty.$$

(ii) *If $|k| \geq 3\sqrt{n}$, we have*

$$||\Box_k P f||_{L^q(\mathbb{R}^n)} \leq C \|\Box_k f\|_{L^q(\mathbb{R}^n)} \quad \textit{if} \quad \|\Box_k f\|_{L^q(\mathbb{R}^n)} < \infty.$$

Proof. We prove (i) first. We use the homogeneous Besov space $\dot{B}^0_{q,1}(\mathbb{R}^n)$, let $\{\phi_j\}_{j\in\mathbb{Z}}$ be Littlewood Paley's decomposition. We have from the embedding $\dot{B}^0_{q,1}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ and the boundedness of P in the homogeneous Besov spaces

$$||\Box_k \nabla P f||_{L^q(\mathbb{R}^n)} \le C \sum_{-\infty<j\le n} 2^j \|\phi_j * (\Box_k f)\|_{L^q(\mathbb{R}^n)} \le C\|\Box_k f\|_{L^q(\mathbb{R}^n)}.$$

This is the desired estimate. We can also prove (ii) since the operator P is the bounded operator in the homogeneous Besov spaces and k is away from the origin. ❑

From Lemma 2.13, we obtain the following proposition.

Proposition 2.14 *Let $s \in \mathbb{R}$, $1 \le r, q, \sigma \le \infty$ and $T > 0$. Then, there exists $C > 0$ such that we have*

$$\|\nabla P f\|_{M^s_{q,\sigma}} \le C\|f\|_{M^{s+1}_{q,\sigma}} \quad \textit{if} \ \ f \in M^{s+1}_{q,\sigma}(\mathbb{R}^n).$$
$$\|\nabla P f\|_{l^{s,\sigma}L^r_T L^q} \le C\|f\|_{l^{s+1,\sigma}L^r_T L^q} \quad \textit{if} \ \ f \in l^{s+1,\sigma}L^r_T L^q.$$

3 Proof of Theorem 1.3

We consider the following integral equation;

$$\Psi(u)(t) = e^{t\Delta}u_0 - \int_0^t \nabla e^{(t-\tau)\Delta} P(u \otimes u) d\tau,$$

and for simplicity, we define $\|\cdot\|_Y$ by

$$\|u\|_Y := \sup_{t\in(0,T)} \|u(t)\|_{M^s_{q,\sigma}}.$$

We treat the case $s = -1$ in Step 1, the case $-1 < s < 0$ in Step 2, the case $s = n(\sigma-1)/\sigma - 1 \ge 0$ or $0 \le s \le \{n(\sigma-1) - n\}/\sigma$ in Step 3, and the last case in Step 4. The claim of Remark 1.4 in the case of initial data being sufficiently small is shown in Step 1, and the case of $s > -1$ and $s > n(\sigma-1)/\sigma - 1$ can be proved by the following estimate in each case. There exists $T > 0$ $\alpha, \beta > 0$ such that

$$\|\Psi(u)\|_{X_T} \le C\|u\|_{M^s_{q,\sigma}} + C(T^\alpha + T^\beta)\|u\|^2_{X_T},$$

which we show in the following steps.

Step 1. By the condition of Theorem 1.3, we have $\sigma = 1$ if $s = -1$. We show the following estimates.

$$\|\Psi(u)\|_Y \le C\|u_0\|_{M^{-1}_{q,1}} + C\|u\|^2_Z, \tag{3.1}$$
$$\|\Psi(u)\|_Z \le \|e^{t\Delta}u_0\|_Z + C(1 + T^{\frac{1}{2}})\|u\|^2_Z. \tag{3.2}$$

On the norm $\|\cdot\|_Y$, the first term in the right hand side is obtained by (2.7). We have from Proposition 2.14, Proposition 2.12, Proposition 2.3 and Remark 2.2

$$\begin{aligned}\left\|\int_0^t \nabla e^{(t-\tau)\Delta}P(u\otimes u)d\tau\right\|_{M^{-1}_{q,1}} &\le C\left\|\int_0^t e^{(t-\tau)\Delta}(u\otimes u)d\tau\right\|_{M^0_{q,1}}\\ &\le C\|u\otimes u\|_{l^{0,1}L^1_TL^q}\\ &\le C\|u\|^2_{l^{0,1}L^2_TL^{2q}}\\ &\le C\|u\|^2_Z.\end{aligned}$$

On the norm $\|\cdot\|_Z$, we apply Proposition 2.14, Proposition 2.12 and Remark 2.2

$$\begin{aligned}\left\|\int_0^t \nabla e^{(t-\tau)\Delta}P(u\otimes u)d\tau\right\|_{Z} &\le C\left\|\int_0^t e^{(t-\tau)\Delta}(u\otimes u)d\tau\right\|_{l^{1,1}L^2_TL^q}\\ &\le C(1+T^{\frac{1}{2}})\|u\otimes u\|_{l^{0,1}L^1_TL^q}\\ &\le C(1+T^{\frac{1}{2}})\|u\|^2_Z.\end{aligned}$$

Therefore, we obtain the desired estimate. ¿From Proposition 2.10, the norm $\|e^{t\Delta}u_0\|_Z$ is small for sufficiently small $T>0$, which depend on each u_0 and isn't taken depending only on the norm of u_0. And we can apply Banach's fixed point theorem in a certain complete metric space to obtain the desired solution. In fact, the way of the proof is similar to that of [8]. That is, we define the complete metric space for small $\varepsilon>0$;

$$\begin{gathered}\left\{\, u\in\left[C([0,T],M^{-1}_{q,1}(\mathbb{R}^n))\right]^n \;\middle|\; \|u\|_Y\le C\|u_0\|_{M^{-1}_{q,1}}+\varepsilon,\ \|u\|_Z\le 2\varepsilon \,\right\},\\ d(u,v):=\|u-v\|_Y+\|u-v\|_Z,\end{gathered}$$

and can apply Banach's fixed point theorem.

Step 2. By the condition of Theorem 1.3, we have $\sigma<2$ if $s<0$, $n\ge 2$ and $s\ge n(\sigma-1)/\sigma-1$. We show the following estimate. There exists $\alpha_1,\alpha_2>0$ and $\beta_1,\beta_2\ge 0$ such that we have

$$\|\Psi(u)\|_Y\le C\|u_0\|_{M^s_{q,\sigma}}+C(T^{\alpha_1}+T^{\beta_1})\|u\|^2_Z, \tag{3.3}$$
$$\|\Psi(u)\|_Z\le \|e^{t\Delta}u_0\|_Z+C(T^{\alpha_2}+T^{\beta_2})\|u\|^2_Z. \tag{3.4}$$

We define a real number ν satisfying $1/\nu=2/\sigma-1$. On the norm $\|\cdot\|_Y$, we apply Proposition 2.14, (2.7), (2.8), (2.2) and Proposition 2.1 and have

$$\begin{aligned}\left\|\int_0^t \nabla e^{(t-\tau)\Delta}P(u\otimes u)d\tau\right\|_{M^s_{q,\sigma}} &\le C\int_0^t \|e^{(t-\tau)\Delta}(u\otimes u)\|_{M^{s+1}_{q,\sigma}}d\tau\\ &\le C\int_0^t\left(1+(t-\tau)^{-\frac{1+s}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}\right)\|u\otimes u\|_{M^0_{q,\nu}}d\tau\\ &\le C\int_0^t\left(1+(t-\tau)^{-\frac{1+s}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}\right)\|u\|^2_{M^0_{2q,\sigma}}d\tau\\ &\le C\|u\|^2_Z\int_0^t\left(1+(t-\tau)^{-\frac{1+s}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}\right)\tau^s d\tau\\ &\le C\|u\|^2_Z\left(T^{1+s}+T^{1-\frac{1+s}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})+s}\right).\end{aligned}$$

Here, it is necessary that the exponent of T being nonnegative, and it is satisfied by $s \geq n(\sigma-1)/\sigma-1$. On the norm $\|\cdot\|_Z$, we apply Proposition 2.14, (2.7), (2.8), (2.2) and Proposition 2.1, and have

$$
\begin{aligned}
t^{\frac{|s|}{2}} \left\| \int_0^t \nabla e^{(t-\tau)\Delta} P(u\otimes u) d\tau \right\|_{M^0_{q,\sigma}} &\leq t^{\frac{|s|}{2}} \left\| \int_0^t e^{(t-\tau)\Delta} (u\otimes u) d\tau \right\|_{M^1_{q,\sigma}} \\
&\leq C t^{\frac{|s|}{2}} \int_0^t \left(1+(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}\right) \|u\otimes u\|_{M^0_{q,\nu}} d\tau \\
&\leq C\|u\|_Z^2 t^{\frac{|s|}{2}} \left(t^{1+s} + t^{\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})+s}\right).
\end{aligned}
$$

It is necessary that the exponent of t being nonnegative, and it is satisfied by $s \geq n(\sigma-1)/\sigma-1$. Therefore, we can apply Banach's fixed point theorem and obtain the desired solution.

Step 3. We show the following estimate. There exists $\alpha_1, \alpha_2 > 0$ and $\beta_1, \beta_2 \geq 0$ such that we have

$$
\begin{aligned}
\|\Psi(u)\|_Y &\leq C\|u_0\|_{M^s_{q,\sigma}} + C(T^{\alpha_1}+T^{\beta_1})\|u\|_Z^2, \\
\|\Psi(u)\|_Z &\leq \|e^{t\Delta}u_0\|_Z + C(T^{\alpha_2}+T^{\beta_2})\|u\|_Z^2.
\end{aligned}
$$

We define $\tilde{\nu}$ satisfying

$$
\frac{1}{\tilde{\nu}} - \frac{s}{n} = \frac{2}{\nu} - 1. \tag{3.5}
$$

By the conditions for s, σ, ν in the theorem, we have $\sigma < \tilde{\nu} < \infty$. On the norm $\|\cdot\|_Y$, we apply Proposition 2.14, (2.7) and (2.8) and have

$$
\left\| \int_0^t \nabla e^{(t-\tau)\Delta} P(u\otimes u) d\tau \right\|_{M^s_{q,\sigma}} \leq C \int_0^t \left(1+(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde{\nu}})}\right) \|u\otimes u\|_{M^s_{q,\tilde{\nu}}} d\tau.
$$

From the relationship (3.5), we apply (2.5) and Proposition 2.1, and have

$$
\begin{aligned}
\left\| \int_0^t \nabla e^{(t-\tau)\Delta} P(u\otimes u) d\tau \right\|_{M^s_{q,\sigma}} &\leq C \int_0^t \left(1+(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde{\nu}})}\right) \|u\|^2_{M^s_{2q,\nu}} d\tau \\
&\leq C\|u\|_Z^2 \int_0^t \left(1+(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde{\nu}})}\right) \tau^{-n(\frac{1}{\nu}-\frac{1}{\sigma})} d\tau \\
&\leq C\|u\|_Z^2 \left(T^{1-n(\frac{1}{\nu}-\frac{1}{\sigma})} + T^{\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde{\nu}})-n(\frac{1}{\nu}-\frac{1}{\sigma})}\right).
\end{aligned}
$$

Here, it is necessary that the exponent of T being nonnegative, and it is satisfied by $s \geq n(\sigma-1)/\sigma-1$. On the norm $\|\cdot\|_Z$, we also apply Proposition 2.14, (2.7), (2.8), (2.5) and Proposition 2.1 similarly and have

$$
\begin{aligned}
& t^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})} \left\| \int_0^t \nabla e^{(t-\tau)\Delta} P(u\otimes u) d\tau \right\|_{M^s_{q,\nu}} \\
&\leq C t^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})} \int_0^t \left(1+(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}\right) \|u\otimes u\|_{M^s_{q,\tilde{\nu}}} d\tau \\
&\leq C\|u\|_Z^2 t^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})} \left(t^{1-n(\frac{1}{\nu}-\frac{1}{\sigma})} + t^{\frac{1}{2}-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})-n(\frac{1}{\nu}-\frac{1}{\sigma})}\right).
\end{aligned}
$$

It is necessary that the exponent of t being nonnegative, and it is satisfied by $s \geq n(\sigma-1)/\sigma-1$. Therefore, we can apply Banach's fixed point theorem.

Step 4. The first term of $\Psi(u)$ is estimated by (2.7);

$$\|e^{t\Delta}u_0\|_Y \le C\|u_0\|_{M^s_{q,\sigma}}.$$

Let $\tilde{\nu}$ satisfy

$$\sigma < \tilde{\nu} < \infty, \quad \frac{1}{\tilde{\nu}} - \frac{s}{n} = \frac{2}{\sigma} - 1.$$

Applying Proposition 2.14, (2.7), (2.8), (2.5) and Proposition 2.1, we have

$$\begin{aligned}\left\| \int_0^t \nabla e^{(t-\tau)\Delta} P(u\otimes u) d\tau \right\|_{M^s_{q,\sigma}} &\le C \int_0^t \left(1 + (t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde{\nu}})}\right) \|u\otimes u\|_{M^s_{q,\tilde{\nu}}} d\tau \\ &\le C\|u\|_Y^2 \left(T + T^{\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde{\nu}})}\right).\end{aligned}$$

Here, the integrability is guaranteed by $s > n(\sigma-1)/\sigma - 1$. And the condition $\nu < \infty$ is satisfied by $s > \{n(\sigma-1)-n\}/\sigma$. The existence time $T > 0$ can be taken depending only on the norm of u_0 since the exponent of T is positive.

3.1 Ill-posedness for (NS)

We show the claim of Remark 1.5 that states ill-posedness for (NS) in $PM^s_{2,\sigma}(\mathbb{R}^n)$ if $s < -1$. We show that continuous dependence on initial data fails by the way of Bejenaru and Tao [1]. It is sufficient by their way to show that for $s < -1$, the map from $\left(B_D, \|\cdot\|_{M^s_{2,1}}\right)$ to $\left(B_S, \|\cdot\|_{C([0,1],M^s_{2,\sigma})}\right)$ defined by

$$f \mapsto \int_0^t \nabla e^{(t-\tau)\Delta} P\left((e^{\tau\Delta}f)\otimes(e^{\tau\Delta}f)\right) d\tau$$

is discontinuous, where

$$\begin{aligned}B_D &:= \{\, f \in PM^{-1}_{2,1}(\mathbb{R}^n) \mid \|f\|_{M^{-1}_{2,1}} \le \varepsilon \,\}, \\ B_S &:= \left\{\, u \in \left[C([0,1], M^{-1}_{2,1}(\mathbb{R}^n))\right]^n \,\middle|\, \|u\|_{C([0,1],M^{-1}_{2,1}(\mathbb{R}^n))} + \|u\|_{l^{0,1}_1 L^2 L^2} \le C\varepsilon \,\right\},\end{aligned}$$

and $\varepsilon > 0$ is a small constant. We remark that we can take the existence time $T > 0$ being greater than 1 by Remark 1.4 if the initial data is sufficiently small. To show discontinuous, we show there exists a bounded sequence $\{f_N\}_{N=1}^{\infty}$ in $PM^{-1}_{2,1}(\mathbb{R}^n)$ such that we have

$$\|f_N\|_{M^s_{2,\sigma}} \to 0 \quad \text{as } N\to\infty, \tag{3.6}$$

$$\sup_{t\in(0,1)} \left\| \int_0^t \nabla e^{(t-\tau)\Delta} P(e^{\tau\Delta}f_N)\otimes(e^{\tau\Delta}f_N) d\tau \right\|_{M^s_{2,\sigma}} \ge c \quad \text{for any large } N, \tag{3.7}$$

where c is a positive constant.

For $j = 1, 2$, we define j-th component of $\mathcal{F}[f_N]$ by

$$N(-1)^{j-1}\xi_j^{-1}\left\{\chi(\xi - Ne_1) + \chi(-\xi - Ne_1) + \chi(\xi - Ne_2) + \chi(-\xi - Ne_2)\right\}, \tag{3.8}$$

where $e_1 := (1, 0, \cdots 0)$ and $e_2 := (0, 1, 0, \cdots 0)$ are unit vectors on $\mathbb{R}^n$, $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ and χ is a characteristic function whose support is the following set;

$$\{\ \xi \in \mathbb{R}^n \mid 1 \le \xi_j \le 2 \text{ for } j = 1, 2, \cdots, n\ \}.$$

For $j = 3, \cdots, n$, we define j-th component by 0. We remark that f_N satisfies div $f_N = 0$. Then, we have

$$\|f_N\|_{M^s_{2,\sigma}} \le C_0 N^{1+s} \tag{3.9}$$

and obtain (3.6). Though we must consider $\varepsilon C_0^{-1} f_N \in B_D$, we omit the absolute constant εC_0^{-1} and consider f_N for simplicity.

To prove (3.7), we take $t = 1/N^2$ and will show

$$\left\| \int_0^{\frac{1}{N^2}} e^{(\frac{1}{N^2} - \tau)\Delta} P\nabla \left((e^{\tau\Delta} f_N) \otimes (e^{\tau\Delta} f_N) \right) d\tau \right\|_{M^s_{2,\sigma}} \ge c. \tag{3.10}$$

We have from Plancherel's theorem

$$\begin{aligned} &\left\| \int_0^{\frac{1}{N^2}} e^{(\frac{1}{N^2} - \tau)\Delta} P\nabla \left((e^{\tau\Delta} f_N) \otimes (e^{\tau\Delta} f_N) \right) d\tau \right\|_{M^s_{2,\sigma}} \\ &\ge \left\| \Box_0 \int_0^{\frac{1}{N^2}} e^{(\frac{1}{N^2} - \tau)\Delta} P\nabla \left((e^{\tau\Delta} f_N) \otimes (e^{\tau\Delta} f_N) \right) d\tau \right\|_{L^2(\mathbb{R}^n)} \\ &\ge \left\| \int_0^{\frac{1}{N^2}} \varphi_0 e^{-(\frac{1}{N^2} - \tau)|\xi|^2} \left(1 - \frac{\xi_1}{|\xi|^2} \sum_{l=1}^n \xi_l \right) \sum_{j=1}^2 \xi_j \mathcal{F} \left[(e^{\tau\Delta} f_N^j)(e^{\tau\Delta} f_N^1) \right] d\tau \right\|_{L^2(E)}, \end{aligned} \tag{3.11}$$

where f_N^j denotes the j-th component of f_N, the last inequality is obtained by considering the first component, and E satisfies $E \subset \operatorname{supp} \varphi_0 \cup \{\ \xi \in \mathbb{R}^n \,|\, 1/100 \le \xi_j \le 1 \text{ for } j = 1, 2, \cdots, n\}$ and

$$\left(1 - \frac{\xi_1}{|\xi|^2} \sum_{l=1}^n \xi_l \right) > 0 \quad \text{for } \xi \in E. \tag{3.12}$$

And we also have for $\xi \in E$ and $\tau \in (0, N^{-2})$

$$\begin{aligned} \mathcal{F} \left[(e^{\tau\Delta} f_N^j)(e^{\tau\Delta} f_N^1) \right] &= N^2 (-1)^{j-1} \left\{ \xi_j^{-1} e^{-\tau|\xi|^2} \chi(\xi - Ne_1) \right\} * \left\{ \xi_1^{-1} e^{-\tau|\xi|^2} \chi(-\xi - Ne_1) \right\} \\ &+ N^2 (-1)^{j-1} \left\{ \xi_j^{-1} e^{-\tau|\xi|^2} \chi(-\xi - Ne_1) \right\} * \left\{ \xi_1^{-1} e^{-\tau|\xi|^2} \chi(\xi - Ne_1) \right\} \\ &+ N^2 (-1)^{j-1} \left\{ \xi_j^{-1} e^{-\tau|\xi|^2} \chi(\xi - Ne_2) \right\} * \left\{ \xi_1^{-1} e^{-\tau|\xi|^2} \chi(-\xi - Ne_2) \right\} \\ &+ N^2 (-1)^{j-1} \left\{ \xi_j^{-1} e^{-\tau|\xi|^2} \chi(-\xi - Ne_2) \right\} * \left\{ \xi_1^{-1} e^{-\tau|\xi|^2} \chi(\xi - Ne_2) \right\} \\ &=: I_j + II_j + III_j + IV_j. \end{aligned}$$

For $j = 1$, we have

$$|I_1| + |II_1| \le CN^2 N^{-2} = C, \tag{3.13}$$

$$|III_1 + IV_1| \ge cN^2. \tag{3.14}$$

For $j = 2$, we have

$$|I_2 + II_2 + III_2 + IV_2| \le CN^2 N^{-1} = CN, \tag{3.15}$$

Therefore, we have from (3.11), (3.12), (3.13), (3.14) and (3.15)

$$\begin{aligned}
&\left\| \int_0^{\frac{1}{N^2}} e^{(\frac{1}{N^2}-\tau)\Delta} P\nabla \left((e^{\tau\Delta} f_N) \otimes (e^{\tau\Delta} f_N) \right) d\tau \right\|_{M^s_{2,\sigma}} \\
&\ge c \left\| \varphi_0 \left(1 - \frac{\xi_1}{|\xi|^2} \sum_{l=1}^n \xi_l \right) \left\{ \xi_1 N^2 - C\xi_2 N \right\} \int_0^{\frac{1}{N^2}} d\tau \right\|_{L^2(E)} \\
&\ge c
\end{aligned}$$

for sufficiently large $N \in \mathbb{N}$. Therefore, we obtain (3.7).

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