Approximate sampling theorem and the order of smoothness of the Besov space

Stéphane Jaffard [1], Masami Okada [2] and Toshihide Ueno [3]

[1] Université Paris XII-Val de Marne, 61, avenue du Général de Gaulle 94010 Créteil cedex, France
[2] Dept of Math. Sci., Tokyo Metropolitan University, Hachioji, 192-0397, Japan
[3] Division of Functional Genomics, Jichi Medical University, Tochigi, 329-0498, Japan
e-mail: [1]`jaffard@univ-paris12.fr`, [2]`moka@tmu.ac.jp`, [3]`tosshy@jichi.ac.jp`

Abstract

Let φ be a suitable bump function such as the sinc function or the Cardinal B-splines. The sampling approximation of a given function f is defined by

$$S_N(f,\varphi)(x) := \sum_{k\in\mathbb{Z}} f(k/N)\,\varphi(Nx-k).$$

Then under suitable conditions for φ, we have the following asymptotic estimate:

$$\|S_N(f,\varphi)-f\|_{L^p(\mathbb{R})} \le CN^{-s}\|f\|_{B^s_{p,\infty}(\mathbb{R})},$$

where $B^s_{p,\infty}(\mathbb{R})$ is the Besov space with $1/p < s,\ 1 < p < \infty$ and C is a constant depending only on φ, p and s.

Moreover, the asymptotic order of approximation can be shown to be sharp in some cases. In fact, if we choose φ to be the the Cardinal cubic B-spline, then we can show a reversed inequality, which gives the characterization of the Besov smoothness by sampled values of f on dyadic points:

$$\|f\|_{B^s_{p,\infty}(\mathbb{R})} \simeq \sup_{j=1,2,\cdots} 2^{js}\|S_{2^j}(f,\mathbf{N}_4)-f\|_{L^p(\mathbb{R})} + \|f\|_{L^p(\mathbb{R})}$$

$$\simeq \sup_{j=1,2,\cdots} 2^{js}\Big(\sum_{k\in\mathbb{Z}} 2^{-j}\Big|f(\frac{k}{2^j}) - \frac{1}{2}\left\{f(\frac{k-1}{2^j}) + f(\frac{k+1}{2^j})\right\}\Big|^p\Big)^{1/p} + \|f\|_{L^p(\mathbb{R})},$$

for p, s satisfying $1/p < s < 2,\ 1 < p < \infty$.

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1 Introduction

The famous Shannon sampling theorem (also called Whittaker- Kotel'nikov- Someya-Shannon's sampling theorem) gives an exact reproducing formula for band limited analytic functions using the sinc function $\varphi = \sin \pi x/\pi x$. This sinc function has also been known to be useful to compute a numerical solution of partial differential equations, which doesn't necessarily satisfy the band limited condition.

One of the purposes of this paper is to study its generalization to not sufficiently regular functions. We shall give a sharp asymptotic order of the regular sampling approximation of functions in Besov spaces.

Another purpose is to enlarge the class of sampling functions besides the sinc function which may enable various applications of approximation theory of functions. In fact, the cardinal B-spline functions turn out to be another good examples of sampling function which is useful in the characterization of the order of smoothness of Besov functions by means of the asymptotic order of the sampling approximation.

Other purposes are applications of sampling approximation. One obvious application is in numerical analysis taking advantage of the practical way of computation. Another unexpected application is the analysis of data sampled at dyadic points for a Hölder continuous curve.

2 Sampling functions φ

One type of sampling functions is defined by slight modification of the sinc function (let us call it of type I). Namely, a function of type I is a product of the sinc function $\sin \pi x/\pi x$ and a smooth even function of Schwartz class $g(x)$ whose Fourier transform $\widehat{g}(\xi) = \int e^{-ix\xi}\, g(x)\, dx$ has a small support contained in $\{\xi \mid |\xi| < \delta\pi\}$, $\delta < 1/2$, and satisfies $\int \widehat{g}(\xi)\, d\xi = 1$.

Another type of sampling functions, called functions of type II, is defined as an enlarged family of the cardinal B-spline functions. Namely, a function of type II is a compactly supported even continuous function φ vanishing at integer points $k\,(\in \mathbb{Z} \setminus \{0\})$, which also satisfies somewhat stronger conditions than the so-called Strang-Fix condition, i.e., its Fourier transform $\widehat{\varphi}(\xi)$ multiplied by $(1+\xi^2)^\gamma$ with a fixed constant $1/2 < \gamma$ has uniformly bounded second derivative and vanishes at $2\pi k$ $(k \in \mathbb{Z} \setminus \{0\})$ up to at least the second order. The typical example is the cardinal B-spline of order 2 (=degree 1) which also satisfies the two scaling relation. This sampling function can be used in the case where the order of smoothness s of Besov space is not so large (i.e., $0 < s < 2$), which we shall suppose in the sequel for simplicity of presentation. There are many specially designed sampling functions of this type. See [9], [19] and [27] for example.

Note finally that sampling functions φ of both types vanish at all integer points except at 0, and thus they enable the exact interpolation, i.e., $S_N(f,\varphi)(k/N) = f(k/N)$, $k \in \mathbb{Z}$.

3 Asymptotic error estimate in $L^p(\mathbb{R})$ norm

Let φ be a sampling function described above. Then we have the following asymptotic error estimate for f in the Besov space $B^{1/p}_{p,1}(\mathbb{R})$.

Theorem 3.1 $\|S_N(f,\varphi) - f\|_{L^p(\mathbb{R})} \leq C\, N^{-1/p}\|f\|_{B^{1/p}_{p,1}(\mathbb{R})}$, $1 < p < \infty$, *where C is independent of N and f.*

We shall show later that the order of approximation $O(N^{-1/p})$ as N tends to ∞ is sharp at least in the case of the cardinal cubic B-spline.

4 Outline of the proof of Theorem 3.1

Let us show the outline of the proof. First of all, we have the following formula which shall be crucial in the sequel.

4.1 Poisson summation formula

Let $\widehat{F}$ denote the Fourier transform of F. Then we have

Lemma 4.1 $\widehat{S_N(F,\varphi)}(\xi) = \widehat{\varphi}(\xi/N)\sum_{n=-\infty}^{\infty}\widehat{F}(\xi + 2\pi nN)$.

To prove Lemma 4.1, we apply the Poisson summation formula to $F(t/N)\,\varphi(Nx-t)$ with respect to the t variable and take the Fourier transform with respect to x.

The usefulness of the Poisson summation formula for the regular sampling theorem was kindly suggested to one of the authors by Acad. P. Malliavin in 2006 at ICM Madrid.

4.2 Decomposition of f

Now we choose a function χ of the Schwartz class such that $\widehat{\chi}$ is a non negative cut off function with its support included in $\{\xi \mid 2^{-1}\pi < |\xi| < 2\pi\}$ satisfying $\int_0^\infty \widehat{\chi}(\xi)\xi^{-1}d\xi = 1$.

Then f is decomposed into the low frequency part and the high frequency one, namely, $f = g + h$ where $\widehat{g}(\xi) = \widehat{f}(\xi)\int_0^L \widehat{\chi}(\xi/\lambda)\lambda^{-1}d\lambda$ and $\widehat{h}(\xi) = \widehat{f}(\xi)\int_L^\infty \widehat{\chi}(\xi/\lambda)\lambda^{-1}d\lambda$ with $L = \pi N(1-\delta)/2$. Note that $\operatorname{supp}\widehat{g} \subset \{\xi \mid |\xi| < 2L\}$ and $\operatorname{supp}\widehat{h} \subset \{\xi \mid L/2 < |\xi|\}$.

4.3 Estimate of $S_N(g,\varphi) - g$

With φ of type I, it is easy to see that $S_N(g,\varphi) - g = 0$, comparing the compact supports of $\widehat{g}$ and $\widehat{\varphi}$ in the following formula due to Lemma 4.1 applied to g :

$$\widehat{S_N(g,\varphi)}(\xi) - \widehat{g}(\xi) = \{\widehat{\varphi}(\xi/N) - 1\}\,\widehat{g}(\xi) + \widehat{\varphi}(\xi/N)\sum_{n\neq 0}\widehat{g}(\xi + 2\pi nN). \tag{1}$$

The situation is not so simple for φ of type II. In fact, we have to use the L^p boundedness of an operator defined by a Fourier multiplier, i.e., the Marcinkiewicz multiplier theorem (also, Mikhlin, Hörmander, Krée). Let us first define a smooth non negative cut off function $\rho(\xi)$ with its support included in $\{\xi|\pi/2 < |\xi| < 2\pi\}$ such that $\sum_{j=0}^{[\log_2 N]}\rho^2(\xi/2^j) = 1$ on $\{\xi|\pi < |\xi| < N\pi\}$. Then the first term is decomposed into a sum of functions as follows:

$$\{\widehat{\varphi}(\xi/N) - 1\}\,\widehat{g}(\xi) = N^{-s}\sum_{l=0}^{[\log_2 N]}\{\widehat{\varphi}(\xi/N) - 1\}\,|\xi/N|^{-s}\rho^2(2^l\xi/N)\,|\xi|^s\,\widehat{g}(\xi) + R, \tag{2}$$

where R is the remaining part which is not important. Now we estimate the L^p norm of the Fourier inverse transform of each term separately, applying the Marcinkiewicz multiplier theorem to the Fourier multiplier $\{\widehat{\varphi}(\xi/N) - 1\}\ |\xi/N|^{-s}\rho(2^l\xi/N)$ for each l . Note here that the L^p bound of the operator defined by the Fourier multiplier is $O(2^{(s-2)l})$ by simple computation so that the sum in j is finite and independent of N. Also, recall that the L^p norm of the Fourier inverse transform of $\rho(2^l\xi/N)\,|\xi|^s\,\widehat{g}(\xi)$ is bounded by $\|f\|_{B^s_{p,\infty}(\mathbb{R})}$ according to the definition of the Besov norm.

The second term can be estimated similarly at each point $\xi/N = 2\pi n$ in view of the fact that $\widehat{\varphi}(\xi/N)\sum_{n\neq 0}\widehat{g}(\xi + 2\pi nN) = \sum_{n\neq 0}\{\widehat{\varphi}(\xi/N) - \widehat{\varphi}(2\pi n)\}\,\widehat{g}(\xi + 2\pi nN)$, and that $\widehat{\varphi}(\xi)$ decays at ∞ with an order faster than $O(|\xi|^{-\mu}), 1 < \mu$.

Consequently, we have

Proposition 4.1 *For p, s satisfying $0 < s < 2,\ 1 < p < \infty$,*
$\|S_N(g,\varphi) - g\|_{L^p(\mathbb{R})} \le C\,N^{-s}\|f\|_{B^s_{p,\infty}(\mathbb{R})}$.

4.4 Estimate of $S_N(h,\varphi)$

To handle the high frequency part $\|S_N(h,\varphi) - h\|_{L^p(\mathbb{R})}$, it suffices to estimate $\|S_N(h,\varphi)\|_{L^p(\mathbb{R})}$ and $\|h\|_{L^p(\mathbb{R})}$ separately.

Since we have the estimate: $\|h\|_{L^p(\mathbb{R})} \le C\,N^{-s}\|f\|_{B^s_{p,\infty}(\mathbb{R})}$, according to a standard property of the Besov space, it suffices to show how to estimate $\|S_N(h,\varphi)\|_{L^p(\mathbb{R})}$.

Let us first see the case $p = 2$. Then we have

Lemma 4.2 $\|S_N(h,\varphi)\|_{L^2(\mathbb{R})} \le C\,N^{-1/2}\|f\|_{B^{1/2}_{2,1}(\mathbb{R})}$.

The proof is rather technical and let us describe the outline of the proof.

1st step: We first choose a function η of the Schwartz class such that $\widehat{\eta} = 1$ on *supp* $\widehat{\chi}$ with its support included in $\{\xi | 2^{-1}\pi < |\xi| < 2\pi\}$, so that if we define $\chi_\lambda(x) = \chi(\lambda x)$ and $\eta_\lambda(x) = \eta(\lambda x)$, then $h(x) = \int_L^\infty \big(\chi_\lambda * \eta_\lambda * f\,\big)\,(x)\lambda\,d\lambda$. Let us denote by $\Delta_y\,f(x)$ the finite difference $\{f(x+y) + f(x-y)\}/2 - f(x)$ of f and recall that $\chi(-x) = \chi(x)$ and $\int_{-\infty}^\infty \chi(x)dx = 0$. Then, we have the following equality.

$$h(x) = 2\int_L^\infty \Big(\int_0^\infty \chi(\lambda y)\Big(\int_{-\infty}^\infty \eta(\lambda(x-r))\,\Delta_y f(r)\,dr\Big)dy\Big)\lambda d\lambda.$$

and thus we obtain

Proposition 4.2

$$S_N(h,\varphi)(x) = 2\int_L^\infty \Big(\int_0^\infty \chi(\lambda y)\Big(\int_{-\infty}^\infty K(x,r)\,\Delta_y f(r)\,dr\Big)dy\Big)\lambda d\lambda, \tag{3}$$

where K is defined by $K(x,r) = \sum_{k\in\mathbb{Z}} \eta(\lambda(k/N - r))\,\varphi(Nx - k)$, the sampling approximation of the translated η_λ.

2nd step: Now, the following inequality is valid for a function $F \in L^2(\mathbb{R})$ in general.

Proposition 4.3

$$\left\| \int_{-\infty}^{\infty} K(\cdot, r) F(r) dr \right\|_{L^2(\mathbb{R})} \leq C\,(\lambda N)^{-1/2}\,\|F\|_{L^2(\mathbb{R})}. \tag{4}$$

To prove this inequality, let us use the Parseval's identity. After computing the Fourier transform with respect to x of $K(x,r)$, we find that the integral with respect to r turns out to be the Fourier transform of $F(r)$. And thus, we have

$$\left\| \int_{-\infty}^{\infty} K(\cdot, r) F(r) dr \right\|_{L^2(\mathbb{R})} = (2\pi)^{-1/2} \left\| \widehat{\varphi}(\xi/N) \sum_{n\in\mathbb{Z}} \widehat{\eta}((\xi+2\pi nN)/\lambda)\lambda^{-1}\widehat{F}(\xi+2\pi nN) \right\|_{L^2(\mathbb{R}_\xi)}.$$

Here, in order to estimate the right hand side, we apply the Cauchy-Schwarz inequality to the sum in n and we note that if φ is of the type I, the integral with respect to ξ can be restricted to the interval $[-2N\pi, 2N\pi]$ and the effective number of non zero terms in the sum with respect to n is of order $O(\lambda/N)$ in view of the support of $\widehat{\eta}$.

On the other hand, if φ is of the type II, we have to take into account the decay order of $\widehat{\varphi}(\xi/N)$, although the rest is essentially the same.

Consequently, putting $F = \Delta_y f$ in the proposition, we have obtained

$$\left\| \int_{-\infty}^{\infty} K(\cdot, r)\, \Delta_y f(r)\, dr \right\|_{L^2(\mathbb{R})} \leq C(\lambda N)^{-1/2}\|\Delta_y f\|_{L^2(\mathbb{R})}\,. \tag{5}$$

3rd step: By the Minkowski's inequality applied to the right hand side of Prop. 4.2,

$$\begin{aligned} \|S_N(h,\varphi)\|_{L^2(\mathbb{R})} &\leq 2\int_L^{\infty} \Big(\int_0^{\infty} |\chi|(\lambda y)\, C(\lambda N)^{-1/2}\|\Delta_y f\|_{L^2(\mathbb{R})}\, dy \Big) \lambda d\lambda \\ &\leq 2\int_0^{\infty} \|\Delta_y f\|_{L^2(\mathbb{R})} \Big(\int_{yL}^{\infty} t^{1/2}|\chi(t)|\, dt \Big) y^{-3/2}\, dy\, N^{-1/2}, \end{aligned}$$

where we have made the change of variables $\lambda y = t$.
Now, recall that $\int_0^1 \|\Delta_y f\|_{L^2(\mathbb{R})}\, y^{-1/2} dy/y \leq \|f\|_{B^{1/2}_{2,1}(\mathbb{R})}$ by another definition of the Besov norm (see e.g., [26] p.189). Since the integral $\int_1^{\infty} \|\Delta_y f\|_{L^2(\mathbb{R})}\, y^{-1/2} dy/y$ is easier to treat, the outline of the proof of Lemma 4.2 is finally finished.

4.5 Estimates in $L^\infty(\mathbb{R})$ and $L^1(\mathbb{R})$, and interpolation

In this subsection, let C denotes constants which may not always be the same. Note that the inequality $\|S_N(h,\varphi)\|_{L^\infty(\mathbb{R})} \leq C\|h\|_{L^\infty(\mathbb{R})}$ is straightforward owing to the decay property of φ. Also, by a basic property of the convolution, $\|h\|_{L^\infty(\mathbb{R})} \leq C\|f\|_{L^\infty(\mathbb{R})}$, which gives

$$\|S_N(h,\varphi)\|_{L^\infty(\mathbb{R})} \leq C\|f\|_{L^\infty(\mathbb{R})}. \tag{6}$$

Let us proceed to estimate the $L^1(\mathbb{R})$ norm of $S_N(h,\varphi)$.

Lemma 4.3

$$\|S_N(h,\varphi)\|_{L^1(\mathbb{R})} \leq CN^{-1}\|f\|_{B^1_{1,1}(\mathbb{R})}\,. \tag{7}$$

Note first that, by the Minkowski's inequality, Lemma 4.3 is derived from the following estimate as in the preceding case of $L^2(\mathbb{R})$ estimate up to some modification.

Proposition 4.4

$$\|K(\cdot,r)\|_{L^1(\mathbb{R})} \leq C(1/N + 1/\lambda). \tag{8}$$

To show this estimate, recall that the Fourier transform of $K(\cdot,r)$ is equal to

$$\widehat{\varphi}(\xi/N)\sum_{n\in\mathbb{Z}}\widehat{\eta}\left((\xi+2\pi nN)/\lambda\right)e^{-ir(\xi+2\pi nN)}/\lambda, \tag{9}$$

which means that $K(\cdot,r)$ is equal to

$$\sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty}\eta(\lambda(u-r))N\varphi(N(x-u))e^{-i2\pi nNu}du. \tag{10}$$

Here, again, it suffices to notice that the effective number of non zero terms in the sum with respect to n is of order $O(\lambda/N)$ in view of the support of $\widehat{\eta}$, in the case where φ is of the type I. Also, the same result holds for φ of type II with the same argument thanks to the decay order of φ, as we have seen before. Therefore, the proof is done.

Finally, we apply the interpolation theorem of the Besov spaces (see e.g., [2] p.153), which yields

$$\|S_N(h,\varphi)\|_{L^p(\mathbb{R})} \leq CN^{-1/p}\|f\|_{B^{1/p}_{p,1}(\mathbb{R})}\,,\ 1<p<\infty \tag{11}$$

Therefore, summing up preceding estimates for g and h, the proof of Theorem 3.1 is now immediate.

Let us state a slightly different version of Theorem 3.1 as its corollary:

Corollary 4.1

$$\|S_N(f,\varphi)-f\|_{L^p(\mathbb{R})} \leq CN^{-s}\|f\|_{B^s_{p,\infty}(\mathbb{R})}\,,\ 1/p<s<2,\ 1<p<\infty, \tag{12}$$

where the constant C is independent of N and f.

Let us show how to derive Corollary 4.1 for the sake of completeness, since it is somewhat involved. From the the proof of Theorem 3.1, it follows that we only need to prove $\|S_N(h,\varphi)\|_{L^p(\mathbb{R})} \leq CN^{-s}\|f\|_{B^s_{p,\infty}(\mathbb{R})}$, for $1/p<s<2$, $1<p<\infty$. First, we define a smooth non-negative function θ by $\theta(\xi)=\sum_{l=1}^{\infty}\rho^2(\xi/2^l)$ so that $\theta(\xi)=0$, for $|\xi|\leq\pi$, and $\theta(\xi)=1$, for $2\pi<|\xi|$. Then, the above inequality (11) implies $\|S_N(h,\varphi)\|_{L^p(\mathbb{R})} \leq CN^{-1/p}\|f_0\|_{B^{1/p}_{p,1}(\mathbb{R})}$, where f_0 is defined by its Fourier transform such as $\widehat{f_0}(\xi)=\theta(4\pi\xi/L)\widehat{f}(\xi)$. Finally, we know that $\|f_0\|_{B^{1/p}_{p,1}(\mathbb{R})}$ is dominated by $CN^{1/p-s}\|f\|_{B^s_{p,\infty}(\mathbb{R})}$, according to the property of the Besov norm, from which follows the proof.

Remark 1 *The case where φ is the sinc function is excluded from our consideration. Incidentally, however, we can obtain similar results in this case by analogous arguments. In particular, we have the same estimate as above for $1<p\leqq 2$. This may explain why the sinc function is still so useful in the approximate sampling of not band limited functions.*

5 Inverse inequality

Note first that from Corollary 4.1 with $N = 2^j$, we have by the Minkowski's inequality

$$\|S_{2^{j+1}}(f,\varphi) - S_{2^j}(f,\varphi)\|_{L^p(\mathbb{R})} \le C_0\, 2^{-js}\, \|f\|_{B^s_{p,\infty}(\mathbb{R})},\ j = 1,2,\cdots,\ 1/p < s < 2,\ 1 < p < \infty.$$

And we also observe that, under the same assumption on p and s,

$$\sup_{j=1,2,\cdots} 2^{js}\, \|S_{2^j}(f,\varphi) - f\|_{L^p(\mathbb{R})} \le C_1\, \|f\|_{B^s_{p,\infty}(\mathbb{R})}\,. \tag{13}$$

In this section, we are concerned with the inequality inverse to (13).
Let us choose $\varphi(x) = \mathbf{N}_2(x+1)$, $\mathbf{N}_4(x+2)$, the translated cardinal B-spline of order 2, 4 respectively, which are defined by the convolution of the characteristic function of the unit interval $[0,1]$:

$$\mathbf{N}_2(x) = \mathbb{I}_{[0,1]} * \mathbb{I}_{[0,1]}\,(x),\ \ \mathbf{N}_4(x) = \mathbb{I}_{[0,1]} * \mathbb{I}_{[0,1]} * \mathbb{I}_{[0,1]} * \mathbb{I}_{[0,1]}\,(x). \tag{14}$$

Recall that both $\varphi = \mathbf{N}_2$ and $\varphi = \mathbf{N}_4$ satisfy the so called two scaling relation which is ubiquitous in the wavelet theory:

$$\begin{aligned}
\mathbf{N}_2(x) &= 2^{-1}\,\mathbf{N}_2(2x) + \mathbf{N}_2(2x-1) + 2^{-1}\,\mathbf{N}_2(2x-2),\\
\mathbf{N}_4(x) &= 8^{-1}\,\mathbf{N}_4(2x) + 2^{-1}\,\mathbf{N}_4(2x-1) + (3/4)\,\mathbf{N}_4(2x-2) + 2^{-1}\,\mathbf{N}_4(2x-3) + 8^{-1}\,\mathbf{N}_4(2x-4).
\end{aligned}$$

Therefore, putting $c_{j,k} = f(k/2^j)$, we can derive by simple computation,

$$S_{2^{j+1}}(f,\mathbf{N}_2)(x) - S_{2^j}(f,\mathbf{N}_2)(x) = \sum_{k\in\mathbb{Z}} \{c_{j+1,2k+1} - 2^{-1}(c_{j,k} + c_{j,k+1})\}\mathbf{N}_2(2^{j+1}x - 2k - 1),$$

$$\begin{aligned}
S_{2^{j+1}}(f,\mathbf{N}_4)(x) - S_{2^j}(f,\mathbf{N}_4)(x) &= 4^{-1}\sum_{k\in\mathbb{Z}} \{c_{j,k} - 2^{-1}(c_{j,k-1} + c_{j,k+1})\}\mathbf{N}_4(2^{j+1}x - 2k)\\
&\quad + \sum_{k\in\mathbb{Z}} \{c_{j+1,2k+1} - 2^{-1}(c_{j,k} + c_{j,k+1})\}\mathbf{N}_4(2^{j+1}x - 2k - 1).
\end{aligned}$$

Consequently, we get the following equivalence (see [11] p.145).

Proposition 5.1

$$\begin{aligned}
\|S_{2^{j+1}}(f,\mathbf{N}_2) - S_{2^j}(f,\mathbf{N}_2)\|^p_{L^p(\mathbb{R})} &\simeq \sum_{k\in\mathbb{Z}} 2^{-j}\big|c_{j+1,2k+1} - 2^{-1}(c_{j,k} + c_{j,k+1})\big|^p,\\
\|S_{2^{j+1}}(f,\mathbf{N}_4) - S_{2^j}(f,\mathbf{N}_4)\|^p_{L^p(\mathbb{R})} &\simeq \sum_{k\in\mathbb{Z}} 2^{-j}\big|c_{j,k} - 2^{-1}(c_{j,k-1} + c_{j,k+1})\big|^p\\
&\quad + \sum_{k\in\mathbb{Z}} 2^{-j}\big|c_{j+1,2k+1} - 2^{-1}(c_{j,k} + c_{j,k+1})\big|^p.
\end{aligned}$$

As an immediate consequence, we obtain

Proposition 5.2

$$\|S_{2^{j+1}}(f,\mathbf{N}_2) - S_{2^j}(f,\mathbf{N}_2)\|_{L^p(\mathbb{R})} \le C_2\, \|S_{2^{j+1}}(f,\mathbf{N}_4) - S_{2^j}(f,\mathbf{N}_4)\|_{L^p(\mathbb{R})}. \tag{15}$$

Therefore, by the telescope argument again, we have

Corollary 5.1

$$\|S_{2^j}(f,\mathbf{N}_2)-f\|_{L^p(\mathbb{R})}\le C_3\sum_{i=j}^{\infty}\|S_{2^i}(f,\mathbf{N}_4)-f\|_{L^p(\mathbb{R})}. \tag{16}$$

This means that $\|S_{2^j}(f,\mathbf{N}_2)-f\|_{L^p(\mathbb{R})}$ is essentially dominated by $\|S_{2^j}(f,\mathbf{N}_4)-f\|_{L^p(\mathbb{R})}$.

Now, we can state a characterization of the Besov norm by establishing the inverse inequality of the preceding estimate as follows.

Theorem 5.3

$$\|f\|_{B^s_{p,\infty}(\mathbb{R})}\simeq\sup_{j=1,2,\cdots}2^{js}\|S_{2^j}(f,\mathbf{N}_4)-f\|_{L^p(\mathbb{R})}+\|f\|_{L^p(\mathbb{R})}\,,\ 1/p<s<2,\ 1<p<\infty. \tag{17}$$

Note that we have to prove only the inequality "$\le$ " of Theorem 5.3 .

Let us employ the cut off function θ again. Then by the definition of the Besov norm, we have for $0<s$,

$$\|f\|_{B^s_{p,\infty}(\mathbb{R})}\simeq\sup_{j=1,2,\cdots}2^{js}\,\|\mathfrak{F}^{-1}[\theta^2(\xi/2^j)\widehat{f}(\xi)]\|_{L^p(\mathbb{R})}\ +\ \|f\|_{L^p(\mathbb{R})}, \tag{18}$$

where $\mathfrak{F}^{-1}[G]$ means the inverse Fourier transform of G.

Secondly, we use a cheap trick, namely, we observe that $\|\mathfrak{F}^{-1}[\theta^2(\xi/2^j)\widehat{f}(\xi)]\|_{L^p(\mathbb{R})}$ is not greater than

$$\begin{aligned}&\|\,\mathfrak{F}^{-1}[\theta^2(\xi/2^j)\{\widehat{\mathbf{N}_2}(\xi/2^j)\sum_{n\in\mathbb{Z}}\widehat{f}(\xi+2^{j+1}\pi n)-\widehat{f}(\xi)\}]\,\|_{L^p(\mathbb{R})}\\&\quad+\|\,\mathfrak{F}^{-1}[\theta^2(\xi/2^j)\,\widehat{\mathbf{N}_2}(\xi/2^j)\sum_{n\in\mathbb{Z}}\widehat{f}(\xi+2^{j+1}\pi n)]\,\|_{L^p(\mathbb{R})}.\end{aligned}$$

The first term is easily estimated above by

$$C_4\|\,\mathfrak{F}^{-1}[\widehat{\mathbf{N}_2}(\xi/2^j)\sum_{n\in\mathbb{Z}}\widehat{f}(\xi+2^{j+1}\pi n)-\widehat{f}(\xi)]\,\|_{L^p(\mathbb{R})}=C_4\|S_{2^j}(f,\mathbf{N}_2)-f\|_{L^p(\mathbb{R})}\,. \tag{19}$$

The second term needs a technical observation, namely, the function $m=\theta\,\widehat{\mathbf{N}_2}/(\widehat{\mathbf{N}_2}-\widehat{\mathbf{N}_4})$ is a Fourier multiplier, since on the support of θ, $\widehat{\mathbf{N}_2}/(\widehat{\mathbf{N}_2}-\widehat{\mathbf{N}_4})$ is bounded. Therefore, by the Marcinkiewicz multiplier theorem it is estimated above by

$$C_5\|\,\mathfrak{F}^{-1}[\theta(\xi/2^j)\{\widehat{\mathbf{N}_2}(\xi/2^j)-\widehat{\mathbf{N}_4}(\xi/2^j)\}\sum_{n\in\mathbb{Z}}\widehat{f}(\xi+2^{j+1}\pi n)]\,\|_{L^p(\mathbb{R})}, \tag{20}$$

which is also dominated by

$$\begin{aligned}C_6\|S_{2^j}(f,\mathbf{N}_2)-S_{2^j}(f,\mathbf{N}_4)\|_{L^p(\mathbb{R})}&\le C_6\{\|S_{2^j}(f,\mathbf{N}_2)-f\|_{L^p(\mathbb{R})}+\|S_{2^j}(f,\mathbf{N}_4)-f\|_{L^p(\mathbb{R})}\}\\&\le C_7\sum_{i=j}^{\infty}\|S_{2^i}(f,\mathbf{N}_4)-f\|_{L^p(\mathbb{R})},\end{aligned}$$

due to Corollary 5.1. Consequently, we have

$$2^{js}\,\|\mathfrak{F}^{-1}[\theta^2(\xi/2^j)\widehat{f}(\xi)]\|_{L^p(\mathbb{R})} \le C_8\,2^{js}\sum_{i=j}^{\infty}\|S_{2^i}(f,\mathbf{N}_4)-f\|_{L^p(\mathbb{R})}$$
$$= C_8\sum_{i=j}^{\infty}2^{(j-i)s}\,2^{is}\|S_{2^i}(f,\mathbf{N}_4)-f\|_{L^p(\mathbb{R})}.$$

Since $\sum_{i=j}^{\infty}2^{(j-i)s}$ is finite for $0<s$, the rest of the proof is now immediate.

6 Applications

6.1 Numerical solutions of PDE

Let us describe a historical connection of our results to the numerical computation of solutions to time dependent PDEs. For this purpose, we mainly follow the the book of H. L. Resnikoff - R. O. Wells, Jr.[18]. See, for example, [1], [7] and [29] among others, for other references of wavelet methods. Following the book, let us denote by $\varphi=\varphi_{coif}$ the Coifman's scaling function of a suitable degree. According to the book, Resnikoff - Wells and Tian - Wells proved the following asymptotic error estimate for r-times differentiable functions f in mid-1990's:

$$\|f-S_{2^j}(f,\varphi_{coif})\|_{L^2(\mathbb{R})}\le C_r\,2^{-jr}\quad (j=1,2,\ldots),$$

with C_r depending on f but not on j.

In the second part of their book, they compute numerical solutions of the Dirichlet boundary value problems for the two dimensional Laplacian.

Inspired from their method, the asymptotic error estimate was investigated further and its strengthened version was stated in [28] as follows, (see [4] for another closely related interesting results). Let us put $\varphi=\varphi_{coif}$ as above. Then, if f belongs to the Sobolev space $W_p^r(\mathbb{R})$, $r\le m$, $1\le p\le\infty$, we have

$$\|f-S_{2^j}(f,\varphi_{coif})\|_{L^p(\mathbb{R})}\le C_{\varphi,p,r}\,2^{-jr}\|f^{(r)}\|_{L^p(\mathbb{R})}\qquad (j=1,2,\ldots), \tag{21}$$

where, $C_{\varphi,p,r}$ is a constant depending only on φ, p and r. It has been one of our motivations to generalize this asymptotic error estimate for sampling approximation further to functions in Besov spaces with respect to more general sampling functions.

Then, it is natural to use this sampling approximation to compute numerical solutions of time dependent partial differential equations as follows.

Let $u(x,t)$ be the exact solution of a PDE and suppose that it is smooth enough with respect to the x variable. Then we denote by u_d a numerical solution define as follows which is expected to be close to u if we choose a sufficiently large fixed number J:

$$u(x,t)\approx u_d(x,t):=\sum_{k\in\mathbb{Z}}c_k(t)\,\varphi(2^Jx-k).$$

Here $c_k(t)$ are functions in t to be computed numerically, since the PDE for $u(x,t)$ is reduced to a system of ODE for $c_k(t)$ after replacing u_t,u_x, etc., of the PDE by their counterparts $(u_d)_t\,,(u_d)_x$, etc., in view of the good approximation property: $u_t\approx(u_d)_t=\sum_{k\in\mathbb{Z}}\dot{c}_k(t)\,\varphi(2^Jx-k)$, and $u_x\approx(u_d)_x=2^J\sum_{k\in\mathbb{Z}}c_k(t)\,\varphi'(2^Jx-k),$ etc.

6.2 The order of smoothness of functions

Let $f(x)$ be a kind of α-Hölder continuous function ($0 < \alpha < 1$) like the stock price curve defined on $\mathbb{R}$ and let us consider the case where the function $f(x)$ belongs to $B^s_{p,\infty}(\mathbb{R}), s = \alpha + 1/p$. Then our problem is to find the information on α from the observed value of $f(x)$ at dyadic points $k/2^j, k \in \mathbb{Z},\ j = 1, 2, \cdots$, i.e., from the sampling approximation $S_{2^j}(f, \varphi)$ of f.
Let us call α the critical order of smoothness of f if either

(i) $f(x)$ belongs to $B^s_{p,\infty}(\mathbb{R})$, $s = \alpha + 1/p$ and not to $B^t_{p,\infty}(\mathbb{R})$, $t > \alpha + 1/p$, or

(ii) $f(x)$ belongs to $B^s_{p,\infty}(\mathbb{R})$, $s < \alpha + 1/p$ and not to $B^t_{p,\infty}(\mathbb{R})$, $t = \alpha + 1/p$.

Let us show that this critical order can be determined using only the value of f on the dyadic points $\{k/2^j\}$.

Let us define β by

$$\beta = -\limsup_{j\to\infty} \frac{1}{jp} \log_2 \Big(\sum_{k\in\mathbb{Z}} \big| c_{j,k} - \frac{1}{2}(c_{j,k-1} + c_{j,k+1}) \big|^p \Big) . \tag{22}$$

Then we have

Theorem 6.1 $\alpha = \beta$.

Let us sketch the proof for the sake of completeness. We first note an elementary equivalence: For any sequence $\{F_j\}$ which is convergent to F,

$$\sup_{j=1,2,\cdots} 2^{js} \|F_j - F\| < \infty \ \Leftrightarrow\ \limsup_{j\to\infty} 2^{js} \|F_j - F_{j+1}\| < \infty.$$

By means of this equivalence, Theorem 5.3 implies that

$$f \in B^s_{p,\infty}(\mathbb{R}) \Leftrightarrow \sup_{j=1,2,\cdots} 2^{js} \|S_{2^j}(f, \mathbf{N}_4) - f\|_{L^p(\mathbb{R})} < \infty$$
$$\Leftrightarrow \limsup_{j\to\infty} 2^{js} \|S_{2^{j+1}}(f, \mathbf{N}_4) - S_{2^j}(f, \mathbf{N}_4)\|_{L^p(\mathbb{R})} < \infty.$$

Then, we use the second equivalence of Proposition 5.1 and compute the $\log_2$ of the both hand sides. The rest of the proof is elementary.

Remark 2 *It follows from preceding arguments, with an additinal computation, that we actually have the equivalence for* p, q, s *satisfying* $1 < p < \infty$, $1 \le q \le \infty$, $1/p < s < 2$*:*

$$\|f\|_{B^s_{p,q}(\mathbb{R})} \simeq \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big(\|S_{2^j}(f, \mathbf{N}_4) - f\|_{L^p(\mathbb{R})} \Big)^q \Big)^{1/q} + \|f\|_{L^p(\mathbb{R})}. \tag{23}$$

Therefore, under the condition that $1 < p < \infty$, $1 \le q \le \infty$, $0 < \alpha$ *and* $\alpha + 1/p < 2$, *we have also the equivalence:*

$$\|f\|_{B^{\alpha+1/p}_{p,q}(\mathbb{R})} \simeq \Big(\sum_{j=0}^{\infty} 2^{j\alpha q} \Big(\sum_{k\in\mathbb{Z}} \big| c_{j,k} - \frac{1}{2}(c_{j,k-1} + c_{j,k+1}) \big|^p \Big)^{q/p} \Big)^{1/q} + \|f\|_{L^p(\mathbb{R})}. \tag{24}$$

See [7] and [11] p.362 for closeley related equivalences where the coefficients in the expansion by the cardinal B-spline are different, i.e., they are given by the inner product.

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