On the Singular Limits of the Nonlinear Klein-Gordon Equation

Chi-Kun Lin[1] **and Kung-Chien Wu**[2]

Department of Applied Mathematics
National Chiao Tung University
Hsinchu 30010, TAIWAN

Abstract

We present several singular limits of the modulated nonlinear Klein-Gordon equation. First we derive the hydrodynamical structure of the modulated Klein-Gordon equation then by the hydrodynamical structure we perform the mathematical derivation of the compressible and incompressible Euler equations. Finally, we consider the singular limits directly from the modulated Klein-Gordon equation.

Keywords and phrases: Klein-Gordon equation, modulated energy, Schrödinger equation, wave map, semiclassical limit, nonrelativistic limit, nonrelativistic-semiclassical limit, compressible and incompressible Euler equations.

2000 Mathematical Subject Classification: 35L05, 35Q60, 76Y05.

1 Introduction

The relativistic quantum mechanic equation for a free particle can be derived by writing

$$E^2 = c^2p^2 + m^2c^4,$$

where E is energy, p is momentum, m is mass, and c is the speed of light. The quantum mechanical description of a relativistic free particle results from applying the *corresponding principle*, which allows one to replace classical observable by quantum mechanical operators acting on wave functions [25,

[1]Email: cklin@math.nctu.edu.tw
[2]Email: kjwu.am94g@nctu.edu.tw

27]. Let $\hbar$ denote the Planck constant, then the Schrödinger correspondence principle given by

$$E \to i\hbar\partial_t, \qquad p \to -i\hbar\nabla,$$

will result in the Klein-Gordon equation

$$-\hbar^2\partial_t^2\Psi = -c^2\hbar^2\Delta\Psi + m^2c^4\Psi$$

for wave function Ψ. The Klein-Gordon equation for the complex scalar field is the relativistic version of the Schrödinger equation, which is used to describe spinless particles. It was first considered as a quantum wave equation by Schrödinger in his search for an equation describing de Broglie waves. However, this equation was named after the physicists Oskar Klein and Walter Gordon, who in 1927 proposed that it describes relativistic electrons. Although it turned out that the Dirac equation describes the spinning electron, the Klein-Gordon equation correctly describes the spinless pion [25]. The nonlinear Klein-Gordon equation is easily obtained by adding the term $V'(|\Psi|^2)\Psi$, where V is the potential energy density of the fields;

$$\frac{\hbar^2}{2mc^2}\partial_t^2\Psi - \frac{\hbar^2}{2m}\Delta\Psi + \frac{mc^2}{2}\Psi + V'(|\Psi|^2)\Psi = 0\,. \tag{1.1}$$

Since mc^2t and $\hbar$ have the same dimension of action, $[mc^2t] = [\hbar] = [\text{action}]$, and we may consider the modulated wave function [20]

$$\psi(x,t) = \Psi(x,t)\exp(imc^2t/\hbar)\,, \tag{1.2}$$

where the factor $\exp(imc^2t/\hbar)$ describes the oscillations of the wave function, then ψ satisfies the modulated nonlinear Klein-Gordon equation

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2m}\Delta\psi - V'(|\psi|^2)\psi = \frac{\hbar^2}{2mc^2}\partial_t^2\psi\,. \tag{1.3}$$

The relations between different terms in (1.3) are best seen when the equation is written in terms of dimensionless variables, which will be adorned with carets. The dimensionless independent variables are given by

$$x = L\hat{x}\,, \qquad t = T\hat{t}\,,$$

where L and T denote the reference length and time respectively. We also define the reference velocity by $U = L/T$ and rescale the potential energy as

$$V' = mU^2\hat{V}'\,.$$

Substituting all of these rescaled quantities into the original equation (1.3), and dropping all carets, yields

$$i\varepsilon\partial_t\psi + \frac{1}{2}\varepsilon^2\Delta\psi - V'(|\psi|^2)\psi = \frac{1}{2}\varepsilon^2\nu^2\partial_t^2\psi\,. \tag{1.4}$$

Note that the first important dimensionless parameter ν is given by the ratio of reference velocity and speed of light, $\nu = U/c$, and the scaled Planck constant $\varepsilon = \frac{\hbar}{mU^2T}$ is the second important dimensionless parameter. The two dimensionless parameters ν and ε show the relativistic and quantum effects respectively.

The Klein-Gordon equation is a relativistic version of the Schrödinger equation. Indeed, the right hand side of Eq.(1.4) shows the relativistic effect which is small when considering the light speed c to be large while the Planck constant $\hbar$ is kept fixed. Thus, in the limit as $c \to \infty$ or $\nu \to 0$, Eq.(1.4) goes over into the defocusing nonlinear Schrödinger equation

$$i\varepsilon\partial_t\psi + \frac{1}{2}\varepsilon^2\Delta\psi - V'(|\psi|^2)\psi = 0\,. \tag{1.5}$$

The question of the nonrelativistic limit of the nonlinear Klein-Gordon equation has received considerable attention. In particular, Machihara, Nakanishi and Ozawa [20] gave a very complete answer of this problem, they proved that any finite energy solution converges to the corresponding solution of the nonlinear Schrödinger equation in the energy space, after infinite oscillation in time is removed. The Strichartz estimate plays the most important role to obtain the uniform bound in space and time (see also [24, 26] and references therein). However, to the best of our knowledge, there are very few results concerning the semiclassical or nonrelativistic-semiclassical limit of the nonlinear Klein-Gordon equation. On the other hand, the semiclassical limit of the defocusing nonlinear Schrödinger equation (1.5) is very well studied theoretically and numerically. In this limit the Euler equation for an isentropic compressible flow is formally recovered [4, 5, 6] through the WKB analysis or Madelung transform. We will employ the similar idea to investigate the semiclassical and nonrelativistic-semiclassical limits of the nonlinear Klein-Gordon equation (1.4).

In addition to the introduction this paper is decomposed into three parts. In section 2, according to Noether's theorem we derive the conservation laws associated with the modulated nonlinear Klein-Gordon equation. These conservation laws and the hydrodynamical structure play the most important roles of the singular limits.

In section 3, we discuss the hydrodynamic limits of the Klein-Gordon equation. First, we consider the nonrelativistic-semiclassical limit, i.e. the two parameters $\varepsilon, \nu \to 0$ simultaneously, of the modulated nonlinear Klein-Gordon equation (1.4). To avoid carrying out a double limit, we restrict the case when ν and ε are related. In this situation, the compressible Euler equations are recovered as the nonrelativistic-semiclassical limit. Furthermore, if we rescale the time variable, then the extra degree of the parameter ε enables us to discuss the semiclassical limit no matter when ν is of order $O(1)$ or tends to zero as $\varepsilon \to 0$ and the limit of the current is shown to satisfy the incompressible Euler equations.

In section 4, we investigate the singular limits of the Klein-Gordon equation. We show that the semiclassical limit of the modulated cubic nonlinear Klein-Gordon equation is a relativistic wave map and the associated phase function satisfies a linear relativistic wave equation. The nonrelativistic limit of the modulated defocusing nonlinear Klein-Gordon equation is the defocusing nonlinear Schrödinger equations. The nonrelativistic-semiclassical limit of the modulated cubic nonlinear Klein-Gordon equation is the classical wave map for the limit wave function and the typical linear wave equation for the associated phase function.

2 Hydrodynamical Structure

A fluid mechanical interpretation for the linear Schrödinger equation was put forth by Madelung in 1927 and applies to nonlinear Schrödinger equations. Indeed, as shown in [5], the same idea also applied to the modulated nonlinear Klein-Gordon equation (1.4). We introduce the complex wave function, the so-called Madelung transformation,

$$\psi = A \exp(iS/\varepsilon), \tag{2.1}$$

in which both A, the amplitude, and S, the action function, are real-valued function. Plugging (2.1) into modulated nonlinear Klein-Gordon equation (1.4) and separating the real and imagine parts, we obtain

$$\partial_t A + \frac{A}{2}\big(\Delta S - \nu^2 \partial_t^2 S\big) + \nabla A \cdot \nabla S - \nu^2 \partial_t A \partial_t S = 0. \tag{2.2}$$

$$\partial_t S + \frac{1}{2}|\nabla S|^2 - \frac{1}{2}\nu^2(\partial_t S)^2 + V'(A^2) = \frac{\varepsilon^2}{2}\frac{\Box_\nu A}{A}, \tag{2.3}$$

where the d'Alerbertian $\Box_\nu$ is defined by $\Box_\nu \equiv \Delta - \nu^2\partial_t^2$. Equation (2.2) turns out to be the continuity equation for the *relativistic quantum fluid* and equation (2.3) is the relativistic quantum Hamilton-Jacobi equation. Introducing the new functions

$$\rho = A^2 = |\psi|^2\,, \tag{2.4}$$

$$u = \nabla S = \frac{i\varepsilon}{2}\frac{1}{|\psi|^2}(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi)\,, \tag{2.5}$$

$$\tau = \partial_t S = \frac{i\varepsilon}{2}\frac{1}{|\psi|^2}(\psi\partial_t\overline{\psi} - \overline{\psi}\partial_t\psi)\,, \tag{2.6}$$

we can rewrite (2.2)–(2.3) as the dispersive perturbation of the compressible Euler type equations

$$\partial_t\big(\rho(1-\nu^2\tau)\big) + \nabla\cdot\big(\rho u\big) = 0\,, \qquad \partial_t u = \nabla\tau\,, \tag{2.7}$$

$$\begin{aligned}&\partial_t\big(\rho u(1-\nu^2\tau)\big) + \nabla\cdot\big(\rho u\otimes u\big) + \nabla P(\rho)\\ &\quad= \frac{\varepsilon^2}{4}\nabla\cdot\Big(\rho\nabla^2\log\rho\Big) - \frac{\varepsilon^2\nu^2}{4}\partial_t\Big(\rho\nabla\partial_t\log\rho\Big),\end{aligned} \tag{2.8}$$

where $P(\rho) = \rho V'(\rho) - V(\rho)$ is the pressure and ∇^2 denotes the Hessian. Eqs. (2.7)–(2.8) is constituted by the Euler, relativistic and quantum parts. If the "Euler part" of these equations is to be hyperbolic, then the pressure $P(\rho)$ must be a strictly increasing function of ρ; in that case, $P'(\rho) = \rho V''(\rho) > 0$. This means that V must be a strictly convex function of ρ and corresponds to a *defocusing* nonlinear Klein-Gordon equation. The compatibility condition $\partial_t u = \nabla\tau$ also implies that

$$u(x,t) = \nabla\bigg(\int_0^t \tau(x,\xi)d\xi + S(x,0)\bigg)\,. \tag{2.9}$$

Defining the Schrödinger part energy density E_S and relativistic part energy density E_K respectively by

$$E_S \;=\; \frac{1}{2}\rho|u|^2 + \frac{\varepsilon^2}{8}\frac{|\nabla\rho|^2}{\rho} + V(\rho) = \frac{\varepsilon^2}{2}|\nabla\psi|^2 + V(|\psi|^2)\,, \tag{2.10}$$

$$E_K \;=\; \frac{1}{2}\nu^2\rho|\tau|^2 + \frac{\varepsilon^2\nu^2}{8}\frac{|\partial_t\rho|^2}{\rho} = \frac{\varepsilon^2\nu^2}{2}|\partial_t\psi|^2\,, \tag{2.11}$$

we obtain from (2.7)–(2.8) the conservation of energy

$$\partial_t(E_S + E_K) + \nabla \cdot \Big((E_S + P(\rho))u \Big) = \frac{\varepsilon^2}{4} \nabla \cdot \left[u\Delta\rho - \nabla \cdot (\rho u) \frac{\nabla \rho}{\rho} \right]. \quad (2.12)$$

In the formal nonrelativistic limit $\nu \to 0$, one neglects the $O(\nu^2)$ terms appearing in (2.7)–(2.8) and the limit densities ρ, u and P satisfy the quantum hydrodynamical equations

$$\partial_t \rho + \nabla \cdot \big(\rho u\big) = 0\,, \quad (2.13)$$

$$\partial_t(\rho u) + \nabla \cdot \big(\rho u \otimes u\big) + \nabla P(\rho) = \frac{\varepsilon^2}{4} \nabla \cdot \Big[\rho \nabla^2 \log \rho \Big], \quad (2.14)$$

which are exactly the fluid formulation of the defocusing nonlinear Schrödinger equation. In this case the relativistic part energy density E_K vanishes and the limit energy density E will be given by

$$E = \frac{1}{2}\rho|u|^2 + \frac{\varepsilon^2}{8}\frac{|\nabla \rho|^2}{\rho} + V(\rho) \quad (2.15)$$

and will satisfy

$$\partial_t E + \nabla \cdot \Big((E + P(\rho))u \Big) = \frac{\varepsilon^2}{4} \nabla \cdot \left[(\rho u)\frac{\Delta \rho}{\rho} - \nabla \cdot (\rho u) \frac{\nabla \rho}{\rho} \right]. \quad (2.16)$$

Next letting $\nu \to 0$ and $\varepsilon \to 0$ simultaneously, both the relativistic and quantum correction terms in (2.7)–(2.8) vanish and the limit densities ρ, u and P will satisfy the compressible Euler equations

$$\partial_t \rho + \nabla \cdot (\rho u) = 0\,, \quad (2.17)$$

$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) = 0\,, \quad (2.18)$$

and the limit energy density E will be given by

$$E = \frac{1}{2}\rho|u|^2 + V(\rho) \quad (2.19)$$

and will satisfy

$$\partial_t E + \nabla \cdot \Big((E + P(\rho))u \Big) = 0 \quad (2.20)$$

hence playing the role of a Lax entropy for the Euler system.

In order to investigate the incompressible limit, we introduce the scaling

$$\widetilde{t} = \varepsilon^{\alpha} t, \qquad \widetilde{x} = x\,, \qquad \alpha > 0\,.$$

After dropping the tilde, the modulated nonlinear Klein-Gordon equation (1.4) becomes

$$i\varepsilon^{1+\alpha}\partial_t\psi - \frac{\varepsilon^{2+2\alpha}\nu^2}{2}\partial_t^2\psi + \frac{\varepsilon^2}{2}\Delta\psi - V'(|\psi|^2)\psi = 0\,. \tag{2.21}$$

For this model the corresponding fluid dynamics equations (2.7)–(2.8) turn out to be

$$\partial_t\big(\rho(1-\nu^2\varepsilon^{2\alpha}\tau)\big) + \nabla\cdot(\rho u) = 0\,, \tag{2.22}$$

$$\begin{aligned} \partial_t\Big(\rho u(1-\nu^2\varepsilon^{2\alpha}\tau) + \frac{\varepsilon^2\nu^2}{4}\rho\nabla\partial_t\log\rho\Big) + \nabla\cdot\big(\rho u\otimes u\big) + \frac{1}{\varepsilon^{2\alpha}}\nabla P(\rho) \\ = \frac{\varepsilon^{2-2\alpha}}{4}\nabla\cdot\Big(\rho\nabla^2\log\rho\Big), \end{aligned} \tag{2.23}$$

and the associated energy equation becomes

$$\partial_t(E_S+E_K) + \nabla\cdot\left(\Big(E_S + \frac{P(\rho)}{\varepsilon^{2\alpha}}\Big)u\right) = \frac{\varepsilon^{2-2\alpha}}{4}\nabla\cdot\left[(\rho u)\frac{\Delta\rho}{\rho} - \nabla\cdot(\rho u)\frac{\nabla\rho}{\rho}\right]. \tag{2.24}$$

where the Schrödinger part energy density E_S and relativistic part energy density E_K are given respectively by

$$E_S = \frac{1}{2}\rho|u|^2 + \frac{\varepsilon^{2-2\alpha}}{8}\frac{|\nabla\rho|^2}{\rho} + \frac{1}{\varepsilon^{2\alpha}}V(\rho)\,, \tag{2.25}$$

$$E_K = \frac{\varepsilon^{2\alpha}\nu^2}{2}\rho|\tau|^2 + \frac{\varepsilon^2\nu^2}{8}\frac{|\partial_t\rho|^2}{\rho}\,. \tag{2.26}$$

It follows immediately from the energy equation that

$$\int \frac{\varepsilon^{2\alpha}\nu^2}{2}\rho|\tau|^2 + \frac{\varepsilon^2\nu^2}{8}\frac{|\partial_t\rho|^2}{\rho} + \frac{1}{2}\rho|u|^2 + \frac{\varepsilon^{2-2\alpha}}{8}\frac{|\nabla\rho|^2}{\rho} + \frac{V(\rho)}{\varepsilon^{2\alpha}}dx \le C \tag{2.27}$$

for all $0 < t < \infty$ if the initial energy is bounded. Assuming the minimum of the convex function $V(\rho)$ occurs at $\rho = 1$ then the energy bound (2.27) implies $\rho \to 1$ as $\varepsilon \to 0$. Formally the density ρ goes to 1, thus we expect that the equation (2.22) yields the limit: $\nabla \cdot u = 0$. Writing $\nabla P(\rho) = \nabla(P(\rho) - P(1))$, we deduce from (2.23) that

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla \widetilde{P} = 0\,, \tag{2.28}$$

where $\widetilde{P}$ is the limit of $\frac{P(\rho)-P(1)}{\varepsilon^{2\alpha}}$. In other words, we recover the incompressible Euler equations. The reader is referred to [18] for the detail discussion of the incompressible Euler equations.

3 Hydrodynamic Limits

We apply the modulated energy method to study the hydrodynamic limits, i.e., the compressible and incompressible Euler limits of the modulated Klein-Gordon equations. The modulated energy method was introduced by Brenier [1] to prove the convergence of the Vlasov-Poisson system to the incompressible Euler equations. It was immediately extended by Masmoudi in [22] to general initial data allowing the presence of high oscillations in time (see also [10] for the quantum hydrodynamic model of semiconductor). The same idea is also applied to study various singular limits of other equations, for example the Schrödinger-Poisson equation [29], the Gross-Pitaevskii equation [16] and the coupled nonlinear Schrödinger equation [7, 17].

3.1 Compressible Euler Limit

In this subsection, we consider the convergence towards the compressible Euler equations. In fact, we will consider the so called nonrelativistic-semiclassical limit, i.e. $\nu \to 0$ and $\varepsilon \to 0$ simultaneously. In order to avoid carrying out a double limits the parameters ν and ε must be related. For convenience we set $\nu = \varepsilon^\kappa$ for some $\kappa > 0$, $0 < \varepsilon \ll 1$ and assume the potential energy $V'(|\psi^\varepsilon|^2) = |\psi^\varepsilon|^{2(\gamma-1)}$. Indeed we consider the modulated nonlinear Klein-Gordon equation

$$i\varepsilon\partial_t\psi^\varepsilon - \frac{1}{2}\varepsilon^{2+2\kappa}\partial_t^2\psi^\varepsilon + \frac{1}{2}\varepsilon^2\Delta\psi^\varepsilon - |\psi^\varepsilon|^{2(\gamma-1)}\psi^\varepsilon = 0\,, \tag{3.1}$$

supplemented with the initial conditions:

$$\psi^\varepsilon(x,0) = \psi_0^\varepsilon(x)\,, \qquad \partial_t\psi^\varepsilon(x,0) = \psi_1^\varepsilon(x)\,, \qquad x \in \mathbb{T}^n\,, \tag{3.2}$$

satisfying the energy bound

$$\int_{\mathbb{T}^n} \frac{1}{2}\varepsilon^{2+2\kappa}|\psi_1^\varepsilon|^2 + \frac{1}{2}\varepsilon^2|\nabla\psi_0^\varepsilon|^2 + \frac{1}{\gamma}|\psi_0^\varepsilon|^{2\gamma} dx \leq C\,. \tag{3.3}$$

Here C denotes various positive constants independent of ε.

Associated with (3.1) are the local conservation laws corresponding to charge, momentum (current) and energy conservation. In fact, we have the hydrodynamical variables: Schrödinger part charge ρ_S^ε, relativistic part charge ρ_K^ε, Schrödinger part momentum (current) J_S^ε, relativistic part momentum (current) J_K^ε and energy e^ε given as follows:

$$\begin{aligned}
&\rho_S^\varepsilon = |\psi^\varepsilon|^2, \qquad \rho_K^\varepsilon = \frac{i}{2}\varepsilon^{1+2\kappa}\Big(\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\partial_t\psi^\varepsilon\Big),\\
&J_S^\varepsilon = (J_{S,1}^\varepsilon, J_{S,2}^\varepsilon, ..., J_{S,n}^\varepsilon) = \frac{i}{2}\varepsilon\Big(\psi^\varepsilon\nabla\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\nabla\psi^\varepsilon\Big),\\
&J_K^\varepsilon = (J_{K,1}^\varepsilon, J_{K,2}^\varepsilon, ..., J_{K,n}^\varepsilon) = \frac{1}{2}\varepsilon^{2+2\kappa}\Big(\partial_t\psi^\varepsilon\nabla\overline{\psi^\varepsilon} + \partial_t\overline{\psi^\varepsilon}\nabla\psi^\varepsilon\Big),\\
&e^\varepsilon = \frac{1}{2}\varepsilon^{2+2\kappa}|\partial_t\psi^\varepsilon|^2 + \frac{1}{2}\varepsilon^2|\nabla\psi^\varepsilon|^2 + \frac{1}{\gamma}|\psi^\varepsilon|^{2\gamma}.
\end{aligned} \tag{3.4}$$

The local conservation laws of the modulated Klein-Gordon equation (3.1) are the charge, momentum (current) and energy given below:

(A) Conservation of charge

$$\frac{\partial}{\partial t}\big(\rho_S^\varepsilon - \rho_K^\varepsilon\big) + \nabla\cdot J_S^\varepsilon = 0\,, \tag{3.5}$$

(B) Conservation of momentum (current)

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(J_S^\varepsilon - J_K^\varepsilon\big) + \frac{1}{4}\varepsilon^2\nabla\cdot\Big[2(\nabla\psi^\varepsilon\otimes\nabla\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon}\otimes\nabla\psi^\varepsilon) - \nabla^2(|\psi^\varepsilon|^2)\Big]\\
&+\frac{1}{4}\varepsilon^{2+2\kappa}\nabla\partial_t\Big(\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} + \overline{\psi^\varepsilon}\partial_t\psi^\varepsilon\Big) + \frac{\gamma-1}{\gamma}\nabla|\psi^\varepsilon|^{2\gamma} = 0\,,
\end{aligned} \tag{3.6}$$

(C) Conservation of energy

$$\frac{\partial}{\partial t}e^\varepsilon - \nabla\cdot\Big[\frac{1}{2}\varepsilon^2(\nabla\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon}\partial_t\psi^\varepsilon)\Big] = 0\,. \tag{3.7}$$

They play the crucial roles of the proof of the hydrodynamics limits. Motivated by Brenier's pioneer work [1], we can define the modulated energy of (3.1) as

$$
\begin{aligned}
H^\varepsilon(t) = & \frac{\varepsilon^2}{2}\int_{\mathbb{T}^n} |(\nabla - i\varepsilon^{-1}u)\psi^\varepsilon|^2 dx \\
& + \frac{1}{2}\int_{\mathbb{T}^n} |\varepsilon^{1+\kappa}\partial_t\psi^\varepsilon|^2 dx + \int_{\mathbb{T}^n} \Theta(\rho_S^\varepsilon, \rho) dx
\end{aligned} \tag{3.8}
$$

where

$$
\Theta(\rho_S^\varepsilon, \rho) = \frac{1}{\gamma}\Big((\rho_S^\varepsilon)^\gamma - \rho^\gamma\Big) - \rho^{\gamma-1}\big(\rho_S^\varepsilon - \rho\big) \tag{3.9}
$$

is a convex function, minimum occurs at $\rho_S^\varepsilon = \rho$ and satisfies $\Theta(\rho_S^\varepsilon) \geq 0$. We also assume

$$
\begin{aligned}
H^\varepsilon(0) = & \frac{\varepsilon^2}{2}\int_{\mathbb{T}^n} |(\nabla - i\varepsilon^{-1}u_0)\psi_0^\varepsilon|^2 dx + \frac{1}{2}\int_{\mathbb{T}^n} |\varepsilon^{1+\kappa}\psi_1^\varepsilon|^2 dx \\
& + \int_{\mathbb{T}^n} \Theta(|\psi_0^\varepsilon|^2, \rho_0) dx \to 0 \quad \text{as} \quad \varepsilon \to 0\,,
\end{aligned} \tag{3.10}
$$

i.e., we consider the well-prepared initial data. Therefore to obtain the hydrodynamic limit, we only need to show that the modulated energy $H^\varepsilon(t)$ tends to zero as $\varepsilon \to 0$. Indeed, we have the following theorem [15].

Theorem 3.1 *Let $\gamma > 1$ for even spatial dimension n and $\gamma \geqslant \frac{n}{n-1}$ for odd n. Let ψ^ε be the solution of the modulated nonlinear Klein-Gordon equation* (3.1)–(3.2) *and the initial condition $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^{s+1}(\mathbb{T}^n) \oplus H^s(\mathbb{T}^n)$, $s > \frac{n}{2}+1$, satisfying* (3.3) *and* (3.10). *Then there exist $T_* > 0$ such that*

$$
\|(\rho_S^\varepsilon - \rho)(\cdot,t)\|_{L^\gamma(\mathbb{T}^n)} \to 0\,, \qquad \|\rho_K^\varepsilon(\cdot,t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \to 0\,, \tag{3.11}
$$

$$
\|(J_S^\varepsilon - \rho u)(\cdot,t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \to 0\,, \qquad \|J_K^\varepsilon(\cdot,t)\|_{L^1(\mathbb{T}^n)} \to 0\,, \tag{3.12}
$$

for $t \in [0, T_*)$ as $\varepsilon \downarrow 0$, where $(\rho, u) \in C\big([0,T_*); H^s(\mathbb{T}^n)\big)$ is the unique local smooth solution of the γ-law compressible Euler equations

$$
\begin{cases}
\partial_t\rho + \nabla\cdot(\rho u) = 0\,, & x \in \mathbb{T}^n, \quad t \in [0, T_*)\,, \\
\partial_t(\rho u) + \nabla\cdot(\rho u \otimes u) + \nabla P(\rho) = 0\,, & \\
\rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x)\,, & x \in \mathbb{T}^n\,.
\end{cases} \tag{3.13}
$$

where $0 < \rho_0 \in H^s(\mathbb{T}^n)$, $u_0 \in H^s(\mathbb{T}^n)$ *and the equation of states is given by* $P(\rho) = \frac{\gamma-1}{\gamma}\rho^\gamma$.

3.2 Incompressible Euler Limit

In this subsection, we study the convergence from the modulated nonlinear Klein-Gordon equation to the incompressible Euler equations. To obtain the incompressible limit, the time variable has to be rescaled, $t \to \varepsilon^\alpha t, \alpha > 0$ and potential energy is given by $V'(|\psi^\varepsilon|^2) = (|\psi^\varepsilon|^\gamma - 1)|\psi^\varepsilon|^{\gamma-2}, \gamma \geq 2$. More precisely, we will investigate the time-scaled modulated nonlinear Klein-Gordon equation

$$i\varepsilon^{1+\alpha}\partial_t\psi^\varepsilon - \frac{\varepsilon^{2+2\alpha}\nu^2}{2}\partial_t^2\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon - (|\psi^\varepsilon|^\gamma - 1)|\psi^\varepsilon|^{\gamma-2}\psi^\varepsilon = 0\,, \tag{3.14}$$

supplemented with initial conditions

$$\psi^\varepsilon(x,0) = \psi_0^\varepsilon(x)\,, \qquad \partial_t\psi^\varepsilon(x,0) = \psi_1^\varepsilon(x)\,, \qquad x \in \mathbb{T}^n\,, \tag{3.15}$$

satisfying the uniform bound

$$\int_{\mathbb{T}^n} \frac{1}{2}\nu^2\varepsilon^2|\psi_1^\varepsilon|^2 + \frac{1}{2}\varepsilon^{2-2\alpha}|\nabla\psi_0^\varepsilon|^2 + \frac{1}{\gamma\varepsilon^{2\alpha}}\big(|\psi_0^\varepsilon|^\gamma - 1\big)^2 dx < C\,. \tag{3.16}$$

We will consider the limit as the scaled Planck constant $\varepsilon \to 0$ and the parameter ν is kept fixed. To prove the incompressible limit of (3.14) we have to define the hydrodynamical variables; Schrödinger part charge ρ_S^ε, relativistic part charge ρ_K^ε, Schrödinger part momentum (current) J_S^ε, relativistic part momentum J_K^ε and energy e^ε as follows:

$$\begin{aligned}
&\rho_S^\varepsilon = |\psi^\varepsilon|^2, \qquad \rho_K^\varepsilon = \frac{i}{2}\nu^2\varepsilon^{1+\alpha}\Big(\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\partial_t\psi^\varepsilon\Big)\,,\\
&J_S^\varepsilon = (J_{S,1}^\varepsilon, J_{S,2}^\varepsilon, ..., J_{S,n}^\varepsilon) = \frac{i}{2}\varepsilon^{1-\alpha}\Big(\psi^\varepsilon\nabla\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\nabla\psi^\varepsilon\Big)\,,\\
&J_K^\varepsilon = (J_{K,1}^\varepsilon, J_{K,2}^\varepsilon, ..., J_{K,n}^\varepsilon) = \frac{\nu^2\varepsilon^2}{2}\Big(\partial_t\psi^\varepsilon\nabla\overline{\psi^\varepsilon} + \partial_t\overline{\psi^\varepsilon}\nabla\psi^\varepsilon\Big)\,,\\
&e^\varepsilon = \frac{1}{2}\nu^2\varepsilon^2|\partial_t\psi^\varepsilon|^2 + \frac{1}{2}\varepsilon^{2-2\alpha}|\nabla\psi^\varepsilon|^2 + \frac{1}{\gamma\varepsilon^{2\alpha}}\Big(|\psi^\varepsilon|^\gamma - 1\Big)^2.
\end{aligned} \tag{3.17}$$

The local conservation laws associated with the rescaled modulated nonlinear Klein-Gordon equation (3.14) are the charge, momentum and energy given respectively by:
(A) Conservation of charge

$$\frac{\partial}{\partial t}\left(\rho_S^\varepsilon - \rho_K^\varepsilon\right) + \nabla \cdot J_S^\varepsilon = 0\,, \tag{3.18}$$

(B) Conservation of momentum

$$\begin{aligned}&\frac{\partial}{\partial t}\left(J_S^\varepsilon - J_K^\varepsilon\right) + \frac{1}{4}\varepsilon^{2-2\alpha}\nabla \cdot \left[2(\nabla\psi^\varepsilon \otimes \nabla\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon} \otimes \nabla\psi^\varepsilon) - \nabla^2(|\psi^\varepsilon|^2)\right]\\ &+\frac{1}{4}\nu^2\varepsilon^2\nabla\partial_t\left(\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} + \overline{\psi^\varepsilon}\partial_t\psi^\varepsilon\right)\\ &+\frac{1}{\gamma\varepsilon^{2\alpha}}\nabla\left((\gamma-1)|\psi^\varepsilon|^{2\gamma} - (\gamma-2)|\psi^\varepsilon|^\gamma\right) = 0\,,\end{aligned} \tag{3.19}$$

(C) Conservation of energy

$$\frac{\partial}{\partial t}e^\varepsilon - \nabla \cdot \left[\frac{1}{2}\varepsilon^{2-2\alpha}(\nabla\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon}\partial_t\psi^\varepsilon)\right] = 0\,. \tag{3.20}$$

Similarly, we define the modulated energy

$$\begin{aligned}H^\varepsilon(t) = &\frac{\varepsilon^{2-2\alpha}}{2}\int_{\mathbb{T}^n}\left|(\nabla - i\varepsilon^{\alpha-1}u)\psi^\varepsilon\right|^2dx + \frac{\nu^2\varepsilon^2}{2}\int_{\mathbb{T}^n}|\partial_t\psi^\varepsilon|^2dx\\ &+\frac{1}{\gamma\varepsilon^{2\alpha}}\int_{\mathbb{T}^n}\left((\rho_S^\varepsilon)^{\frac{\gamma}{2}} - 1\right)^2dx\end{aligned} \tag{3.21}$$

which satisfies the well-prepared initial condition

$$\begin{aligned}H^\varepsilon(0) = &\frac{\varepsilon^{2-2\alpha}}{2}\int_{\mathbb{T}^n}|(\nabla - i\varepsilon^{\alpha-1}u_0)\psi_0^\varepsilon|^2dx + \frac{\nu^2\varepsilon^2}{2}\int_{\mathbb{T}^n}|\psi_1^\varepsilon|^2dx\\ &+\frac{1}{\gamma\varepsilon^{2\alpha}}\int_{\mathbb{T}^n}\left(|\psi_0^\varepsilon|^\gamma - 1\right)^2dx \to 0 \quad \text{as} \quad \varepsilon \to 0\,.\end{aligned} \tag{3.22}$$

To obtain the hydrodynamic limit, we also need to show that the modulated energy $H^\varepsilon(t)$ tends to zero as $\varepsilon \to 0$. We have the following theorem [15].

Theorem 3.2 *Let $\alpha > 0$, $\gamma \geq 2$ and ψ^ε be the solution of the time scale modulated nonlinear Klein-Gordon equation* (3.14) *with initial condition $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^{s+1}(\mathbb{T}^n) \oplus H^s(\mathbb{T}^n)$, $s > \frac{n}{2}+1$, satisfying* (3.16) *and* (3.22). *Then there exist $T_* > 0$ such that*

$$\|(\rho_S^\varepsilon - 1)(\cdot, t)\|_{L^\gamma(\mathbb{T}^n)} \to 0\,, \qquad \|\rho_K^\varepsilon(\cdot, t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \to 0\,, \tag{3.23}$$

$$\|(J_S^\varepsilon - \rho_S^\varepsilon u)(\cdot, t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \to 0\,, \qquad \|J_K^\varepsilon(\cdot, t)\|_{L^1(\mathbb{T}^n)} \to 0\,, \tag{3.24}$$

for $t \in [0, T_*)$ as $\varepsilon \downarrow 0$, where $u \in C\big([0, T_*); H^s(\mathbb{T}^n)\big)$ is the unique local smooth solution of the incompressible Euler equations

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla \pi = 0\,, \qquad \nabla \cdot u = 0\,, \\[2ex] u(x, 0) = u_0(x)\,, \qquad \nabla \cdot u_0 = 0\,. \end{cases} \tag{3.25}$$

4 Singular Limits

In this section we discuss the singular limits of the modulated nonlinear Klein-Gordon equation and the detail of the proof is referred to [13, 14]. The main idea is based on the conservation laws of charge and energy;

$$\frac{\partial}{\partial t}\left[|\psi|^2 + \frac{i}{2}\varepsilon\nu^2(\overline{\psi}\partial_t\psi - \psi\partial_t\overline{\psi})\right] + \nabla \cdot \left[\frac{i}{2}\varepsilon(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi)\right] = 0\,, \tag{4.1}$$

$$\frac{\partial}{\partial t}\left[\Big(\nu^2|\partial_t\psi|^2 + |\nabla\psi|^2\Big) + \frac{2}{\varepsilon^2}V\right] - \nabla \cdot \Big[(\nabla\psi\partial_t\overline{\psi} + \nabla\overline{\psi}\partial_t\psi)\Big] = 0\,. \tag{4.2}$$

Examining the charge equation (4.1) we see that although $|\psi|^2$, Schrödinger part, is positive-definite but Klein-Gordon part $\frac{i}{2}\varepsilon\nu^2(\overline{\psi}\partial_t\psi - \psi\partial_t\overline{\psi})$ is not. Here we face one of the major difficulties with the Klein-Gordon equation. However, the energy density is positive-definite and can be employed to obtain the estimate of the Schrödinger part charge. Thus we introduce the charge-energy inequality to establish the singular limits. This is consistent with Einstein's relativity of mass-energy equivalent.

4.1 Semiclassical Limit

First we discuss the semiclassical limit of the modulated cubic nonlinear Klein-Gordon equation

$$i\partial_t\psi^\varepsilon - \frac{1}{2}\varepsilon\nu^2\partial_t^2\psi^\varepsilon + \frac{\varepsilon}{2}\Delta\psi^\varepsilon - \Big(\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon}\Big)\psi^\varepsilon = 0\,, \tag{4.3}$$

supplemented with initial conditions

$$\psi^\varepsilon(x,0) = \psi_0^\varepsilon(x)\,, \quad \partial_t\psi^\varepsilon(x,0) = \psi_1^\varepsilon(x)\,, \qquad x \in \mathbb{T}^n\,. \tag{4.4}$$

Notice that the 4th term $\frac{|\psi^\varepsilon|^2-1}{\varepsilon}$ of (4.3) can be served as the density fluctuation of the sound wave which is similar to the acoustic wave as discussed in the low Mach number limit of the compressible fluid [2, 11, 19, 21]. For this model (4.3)–(4.4) we have the following existence result.

Theorem 4.1 *Let $\nu, T > 0$ and $0 < \varepsilon \ll 1$. Given initial data $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$ and $\frac{|\psi_0^\varepsilon|^2-1}{\varepsilon} \in L^2(\mathbb{T}^n)$, there exists a function ψ^ε such that*

$$\psi^\varepsilon \in L^\infty([0,T];H^1(\mathbb{T}^n)) \cap C([0,T];L^2(\mathbb{T}^n))\,, \tag{4.5}$$

$$\partial_t\psi^\varepsilon \in L^\infty([0,T];L^2(\mathbb{T}^n)) \cap C([0,T];H^{-1}(\mathbb{T}^n))\,, \tag{4.6}$$

$$\frac{|\psi^\varepsilon|^2-1}{\varepsilon} \in L^\infty([0,T];L^2(\mathbb{T}^n))\,, \tag{4.7}$$

and satisfies the weak formulation of (4.3) *given by*

$$\begin{aligned} 0 \;&=\; i\Big\langle \psi^\varepsilon(\cdot,t_2) - \psi^\varepsilon(\cdot,t_1), \varphi\Big\rangle - \frac{1}{2}\varepsilon\nu^2\Big\langle \partial_t\psi^\varepsilon(\cdot,t_2) - \partial_t\psi^\varepsilon(\cdot,t_1), \varphi\Big\rangle \\ &-\; \frac{\varepsilon}{2}\int_{t_1}^{t_2}\Big\langle \nabla\psi^\varepsilon(\cdot,\tau), \nabla\varphi\Big\rangle d\tau - \int_{t_1}^{t_2}\Big\langle\Big(\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\Big)\psi^\varepsilon(\cdot,\tau), \varphi\Big\rangle\, d\tau\,, \end{aligned} \tag{4.8}$$

for every $[t_1,t_2] \subset [0,T]$ and for all $\varphi \in C_0^\infty(\mathbb{T}^n)$. Moreover, for all $t \in [0,T]$, it satisfies the charge-energy inequality

$$\int_{\mathbb{T}^n} |\psi^\varepsilon|^2 + \nu^2|\partial_t\psi^\varepsilon|^2 + |\nabla\psi^\varepsilon|^2 + \frac{1}{2}\Big(\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\Big)^2 dx \le 2C_1 + \Big(1 + 2\varepsilon^2\nu^2\Big)C_2\,, \tag{4.9}$$

where

$$\begin{aligned} C_1 &= \int_{\mathbb{T}^n} |\psi_0^\varepsilon|^2 + \frac{i}{2}\varepsilon\nu^2(\psi_1^\varepsilon\overline{\psi_0^\varepsilon} - \overline{\psi_1^\varepsilon}\psi_0^\varepsilon)dx\,, \\ C_2 &= \int_{\mathbb{T}^n} \nu^2|\psi_1^\varepsilon|^2 + |\nabla\psi_0^\varepsilon|^2 + \frac{1}{2}\Big(\frac{|\psi_0^\varepsilon|^2-1}{\varepsilon}\Big)^2 dx\,, \end{aligned} \tag{4.10}$$

are the initial charge and energy respectively.

The charge is constituted by the Schrödinger and the Klein-Gordon parts. Although the Schrödinger part is positive definite but the Klein-Gordon part is not necessary positive definite. However, the total charge can be bounded by the energy of the modulated Klein-Gordon equation.

We assume the initial datum satisfy $|\psi_0^\varepsilon| = 1$ almost everywhere and $(\psi_0^\varepsilon, \psi_1^\varepsilon) \to (\psi_0, 0)$ in $H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$ as $\varepsilon \to 0$, thus $|\psi_0| = 1$ almost everywhere. First we deduce from the charge-energy inequality (4.9) that

$$\{\psi^\varepsilon\}_\varepsilon \quad \text{is bounded in} \quad L^\infty([0,T]; H^1(\mathbb{T}^n))\,, \tag{4.11}$$

$$\{\partial_t\psi^\varepsilon\}_\varepsilon \quad \text{is bounded in} \quad L^\infty([0,T]; L^2(\mathbb{T}^n))\,, \tag{4.12}$$

$$\left\{\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon}\right\}_\varepsilon \quad \text{is bounded in} \quad L^\infty([0,T]; L^2(\mathbb{T}^n))\,, \tag{4.13}$$

then the classical compactness argument shows that there exists a subsequence still denoted by $\{\psi^\varepsilon\}_\varepsilon$ and a function ψ satisfying

$$\psi \in L^\infty([0,T]; H^1(\mathbb{T}^n)), \qquad \partial_t\psi \in L^\infty([0,T]; L^2(\mathbb{T}^n))$$

such that

$$\psi^\varepsilon \rightharpoonup \psi \quad \text{weakly} * \text{in} \quad L^\infty([0,T]; H^1(\mathbb{T}^n))\,, \tag{4.14}$$

$$\partial_t\psi^\varepsilon \rightharpoonup \partial_t\psi \quad \text{weakly} * \text{in} \quad L^\infty([0,T]; L^2(\mathbb{T}^n))\,. \tag{4.15}$$

Also, from (4.13), we have

$$|\psi^\varepsilon|^2 \to 1 \quad \text{a.e. and strongly in } L^2(\mathbb{T}^n). \tag{4.16}$$

Note that (4.13) only shows that $\left\{\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\right\}_\varepsilon$ is a weakly relative compact set in $L^\infty([0,T]; L^2(\mathbb{T}^n))$. In order to overcome the difficulty caused by nonlinearity, the fourth term on the right hand side of (4.8), we use the compactness argument to prove $\psi^\varepsilon \to \psi$ strongly in $C([0,T]; L^2(\mathbb{T}^n))$. Moreover, the quantity $\frac{|\psi^\varepsilon(x,t)|^2-1}{\varepsilon}$ is bounded in $L^\infty([0,T]; L^2(\mathbb{T}^n))$, and hence it converges weakly $*$ to some function $w \in L^\infty([0,T]; L^2(\mathbb{T}^n))$. To find the explicit form of w, we define two functions $W(\psi^\varepsilon)$ and $Z(\psi^\varepsilon)$ respectively by

$$W(\psi^\varepsilon) = \frac{i}{2}\Big(\psi^\varepsilon\nabla\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\nabla\psi^\varepsilon\Big)\,, \quad Z(\psi^\varepsilon) = \frac{i}{2}\nu^2\Big(\overline{\psi^\varepsilon}\partial_t\psi^\varepsilon - \psi^\varepsilon\partial_t\overline{\psi^\varepsilon}\Big)\,. \tag{4.17}$$

We rewrite the conservation of charge (4.1) as

$$\frac{|\psi^\varepsilon(x,t)|^2-1}{\varepsilon} = -Z(\psi^\varepsilon) + Z(\psi^\varepsilon(x,0)) - \int_0^t \text{div } W(\psi^\varepsilon) d\tau \,. \tag{4.18}$$

To obtain the compactness of the sequence $\left\{\frac{|\psi^\varepsilon(x,t)|^2-1}{\varepsilon}\right\}_\varepsilon$, we have to treat the compactness of $\{Z(\psi^\varepsilon)\}_\varepsilon$ and $\{W(\psi^\varepsilon)\}_\varepsilon$ separately. In fact, we have $Z(\psi^\varepsilon) \rightharpoonup Z(\psi)$, $Z(\psi^\varepsilon(x,0)) \rightharpoonup 0$ and

$$\int_0^t \text{div } W(\psi^\varepsilon) d\tau \rightharpoonup \int_0^t \text{div } W(\psi) d\tau$$

in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$, thus

$$\frac{|\psi^\varepsilon(x,t)|^2-1}{\varepsilon} \rightharpoonup -Z(\psi) - \int_0^t \text{div } W(\psi) d\tau \tag{4.19}$$

in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$, and the limit function w is given explicitly by

$$w = -Z(\psi) - \int_0^t \text{div } W(\psi) d\tau \,.$$

Letting $\varepsilon \to 0$ we can show that the limit wave function ψ satisfies

$$i\partial_t \psi + \left[Z(\psi) + \int_0^t \text{div } W(\psi) d\tau \right] \psi = 0 \tag{4.20}$$

in the sense of distribution. Using $|\psi|^2 = 1$ and differentiating (4.20) with respect to time variable t, one can show ψ satisfies the relativistic wave map equation

$$\big(1+\nu^2\big)\partial_t^2\psi - \Delta\psi = \Big[|\nabla\psi|^2 - \big(1+\nu^2\big)|\partial_t\psi|^2\Big]\psi\,, \quad |\psi| = 1 \quad \text{a.e.} \tag{4.21}$$

supplemented with the initial conditions

$$\psi(x,0) = \psi_0(x), \quad \partial_t\psi(x,0) = 0, \qquad x \in \mathbb{T}^n, \qquad |\psi_0| = 1 \quad \text{a.e.} \tag{4.22}$$

Using the fact $|\psi| = 1$ and writing $\psi = e^{i\theta}$ shows

$$\big(1+\nu^2\big)\partial_t^2\theta = \Delta\theta\,, \qquad \theta(x,0) = \arg\,\psi_0\,, \quad \partial_t\theta(x,0) = 0\,, \tag{4.23}$$

i.e., θ is a distribution solution of the linear relativistic wave equation, the ν^2 terms in (4.21) and (4.23) show the relativistic effect.

Theorem 4.2 *Let $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$, $|\psi_0^\varepsilon| = 1$ a.e. and $(\psi_0^\varepsilon, \psi_1^\varepsilon) \to (\psi_0, 0)$ in $H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$, $|\psi_0| = 1$ a.e., and let ψ^ε be the corresponding weak solution of the modulated cubic nonlinear Klein-Gordon equation* (4.3)–(4.4). *Then the weak limit ψ, satisfying $|\psi| = 1$ a.e., solves the relativistic wave map* (4.21)–(4.22). *Moreover, let $\psi = e^{i\theta}$ then the phase function θ satisfies the relativistic wave equation* (4.23).

If we further assume the asymptotic expansion

$$\psi^\varepsilon = \psi + \varepsilon\omega + O(\varepsilon^2)$$

and let $R = \omega\overline{\psi} + \overline{\omega}\psi$ and $I = -i(\omega\overline{\psi} - \overline{\omega}\psi)$ be the real and imaginary parts of $2\omega\overline{\psi}$ respectively then R and I formally satisfy

$$\partial_t R = -\partial_t^2\theta, \qquad R(x,0) = \omega_0\overline{\psi_0} + \overline{\omega_0}\psi_0. \tag{4.24}$$

and

$$(1+\nu^2)\partial_t^2 I - \Delta I = \partial_t^2\theta(2R + 4\nu^2\partial_t\theta) + 2\nabla R \cdot \nabla\theta - \partial_t|\nabla\theta|^2, \tag{4.25}$$

$$I(x,0) = -i(\omega_0\overline{\psi_0} - \overline{\omega_0}\psi_0), \qquad \partial_t I(x,0) = -|\nabla\psi_0|^2 + i\nu^2\Big(\omega_1\overline{\psi_0} - \overline{\omega_1}\psi_0\Big)$$

respectively. The detail is referred to [14].

4.2 Nonrelativistic Limit

We also discuss the non-relativistic limit of the modulated nonlinear Klein-Gordon equation

$$i\varepsilon\partial_t\psi^\nu - \frac{1}{2}\varepsilon^2\nu^2\partial_t^2\psi^\nu + \frac{\varepsilon^2}{2}\Delta\psi^\nu - |\psi^\nu|^p\psi^\nu = 0\,, \quad p > 0 \tag{4.26}$$

$$\psi^\nu(x,0) = \psi_0^\nu(x)\,, \quad \partial_t\psi^\nu(x,0) = \psi_1^\nu(x)\,, \qquad x \in \mathbb{T}^n\,, \tag{4.27}$$

by the compactness method as the previous subsection. For the complete study of the non-relativistic limit of the Cauchy problem for the nonlinear Klein-Gordon equation we will refer to Machihara-Nakanishi-Ozawa [20] (see also [24, 26] and references therein).

First, we state the existence theorem of the modulated Klein-Gordon equation (4.26)–(4.27).

Theorem 4.3 *Let $p, \varepsilon, T > 0$ and $0 < \nu \ll 1$. Given initial data $(\psi_0^\nu, \psi_1^\nu) \in H^1 \cap L^{p+2}(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$, there exists a function ψ^ν such that*

$$\psi^\nu \in L^\infty([0,T]; H^1(\mathbb{T}^n)) \cap C([0,T]; L^2(\mathbb{T}^n)), \tag{4.28}$$

$$\partial_t \psi^\nu \in L^\infty([0,T]; L^2(\mathbb{T}^n)) \cap C([0,T]; H^{-1}(\mathbb{T}^n)), \tag{4.29}$$

$$\psi^\nu \in L^\infty([0,T]; L^{p+2}(\mathbb{T}^n)), \tag{4.30}$$

and satisfies the weak formulation of (4.26) *given by*

$$\begin{aligned} 0 &= -\frac{1}{2}\varepsilon^2\nu^2\Big\langle \partial_t\psi^\nu(\cdot,t_2) - \partial_t\psi^\nu(\cdot,t_1), \varphi\Big\rangle + i\varepsilon\Big\langle \psi^\nu(\cdot,t_2) - \psi^\nu(\cdot,t_1), \varphi\Big\rangle \\ &- \frac{\varepsilon^2}{2}\int_{t_1}^{t_2}\Big\langle \nabla\psi^\nu(\cdot,\tau), \nabla\varphi\Big\rangle d\tau - \int_{t_1}^{t_2}\Big\langle |\psi^\nu|^p\psi^\nu(\cdot,\tau), \varphi\Big\rangle d\tau, \end{aligned} \tag{4.31}$$

for every $[t_1, t_2] \subset [0,T]$ and for all $\varphi \in C_0^\infty(\mathbb{T}^n)$. Moreover, ψ^ν satisfies the charge-energy inequality

$$\int_{\mathbb{T}^n} |\psi^\nu|^2 + \frac{1}{2}\varepsilon^2\nu^2|\partial_t\psi^\nu|^2 + \frac{\varepsilon^2}{2}|\nabla\psi^\nu|^2 + \frac{|\psi^\nu|^{p+2}}{p+2}dx \le 2C_1 + (1+2\nu^2)C_2, \tag{4.32}$$

where C_1 and C_2 are the initial charge and energy given respectively by

$$\begin{aligned} C_1 &= \int_{\mathbb{T}^n} |\psi_0^\nu|^2 + \frac{i}{2}\varepsilon\nu^2(\psi_1^\nu\overline{\psi_0^\nu} - \overline{\psi_1^\nu}\psi_0^\nu)dx, \\ C_2 &= \int_{\mathbb{T}^n} \frac{1}{2}\varepsilon^2\nu^2|\psi_1^\nu|^2 + \frac{\varepsilon^2}{2}|\nabla\psi_0^\nu|^2 + \frac{1}{p+2}|\psi_0^\nu|^{p+2}dx. \end{aligned} \tag{4.33}$$

The proof of the theorem is the standard energy method. To study the nonrelativistic limit we will assume the initial condition (ψ_0^ν, ψ_1^ν) converges strongly in $H^1(\mathbb{T}^n) \cap L^{p+2}(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$ to $(\psi_0, 0)$ as ν tends to 0. We deduce from the charge-energy inequality (4.32) that

$$\{\psi^\nu\}_\nu \quad \text{is bounded in} \quad L^\infty([0,T]; H^1(\mathbb{T}^n)), \tag{4.34}$$

$$\{\nu\partial_t\psi^\nu\}_\nu \quad \text{is bounded in} \quad L^\infty([0,T]; L^2(\mathbb{T}^n)), \tag{4.35}$$

$$\{\psi^\nu\}_\nu \quad \text{is bounded in} \quad L^\infty([0,T]; L^{p+2}(\mathbb{T}^n)). \tag{4.36}$$

In the case of semiclassical limit, we have $L_t^\infty L_x^2$ bound for $\partial_t\psi$, but for non-relativistic limit, we only have $L_t^\infty L_x^2$ bound for $\nu\partial_t\psi$, hence to show $\psi^\nu \to \psi$

in $C([0,T];L^2(\mathbb{T}^n))$, we need further argument than semiclassical limit. But $\psi^\nu \to \psi$ in $C([0,T];L^2(\mathbb{T}^n))$ is not enough to overcome the difficulty caused by nonlinearity, we need more integrability, in fact, we have $\psi^\nu \to \psi$ in $L^\infty([0,T];L^{p+1}(\mathbb{T}^n))$. Thus, passage to the limit $\nu \to 0$ we deduce that the limit wave function ψ is a distribution solution of the defocusing nonlinear Schrödinger equation;

$$i\varepsilon\partial_t\psi + \frac{\varepsilon^2}{2}\Delta\psi - |\psi|^p\psi = 0\,, \qquad (x,t) \in \mathbb{T}^n \times (0,T)\,, \tag{4.37}$$

$$\psi(x,0) = \psi_0(x)\,, \qquad x \in \mathbb{T}^n\,. \tag{4.38}$$

Theorem 4.4 *Let* $(\psi_0^\nu, \psi_1^\nu) \in H^1(\mathbb{T}^n) \cap L^{p+2}(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$*,* $(\psi_0^\nu, \psi_1^\nu) \to (\psi_0, 0)$ *in* $H^1 \cap L^{p+2}(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$*, and* ψ^ν *be the corresponding weak solution of the modulated nonlinear Klein-Gordon equation* (4.26)–(4.27)*. Then the weak limit* ψ *of* $\{\psi^\nu\}_\nu$ *solves the defocusing nonlinear Schrödinger equation* (4.37)–(4.38)*.*

4.3 Nonrelativistic-Semiclassical Limit

In this section we consider the nonrelativistic-semiclassical limit of the modulated cubic nonlinear Klein-Gordon equation

$$i\partial_t\psi^\varepsilon - \frac{1}{2}\varepsilon^{1+2\alpha}\partial_t^2\psi^\varepsilon + \frac{\varepsilon}{2}\Delta\psi^\varepsilon - \Big(\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon}\Big)\psi^\varepsilon = 0\,, \tag{4.39}$$

$$\psi^\varepsilon(x,0) = \psi_0^\varepsilon(x)\,, \quad \partial_t\psi^\varepsilon(x,0) = \psi_1^\varepsilon(x)\,, \qquad x \in \mathbb{T}^n\,. \tag{4.40}$$

Theorem 4.5 *Given* $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$ *and* $\frac{|\psi_0^\varepsilon|^2-1}{\varepsilon} \in L^2(\mathbb{T}^n)$*, there exists a function* ψ^ε *such that*

$$\psi^\varepsilon \in L^\infty([0,T];H^1(\mathbb{T}^n)) \cap C([0,T];L^2(\mathbb{T}^n))\,, \tag{4.41}$$

$$\partial_t\psi^\varepsilon \in L^\infty([0,T];L^2(\mathbb{T}^n)) \cap C([0,T];H^{-1}(\mathbb{T}^n))\,, \tag{4.42}$$

$$\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon} \in L^\infty([0,T];L^2(\mathbb{T}^n))\,, \tag{4.43}$$

and satisfies the weak formulation of (4.39) *given by*

$$\begin{aligned} 0 \;&=\; -\frac{1}{2}\varepsilon^{1+2\alpha}\Big\langle \partial_t\psi^\varepsilon(\cdot,t_2) - \partial_t\psi^\varepsilon(\cdot,t_1), \varphi\Big\rangle + i\Big\langle \psi^\varepsilon(\cdot,t_2) - \psi^\varepsilon(\cdot,t_1), \varphi\Big\rangle \\ &- \frac{\varepsilon}{2}\int_{t_1}^{t_2}\Big\langle \nabla\psi^\varepsilon(\cdot,\tau), \nabla\varphi\Big\rangle d\tau - \int_{t_1}^{t_2}\Big\langle \Big(\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\Big)\psi^\varepsilon(\cdot,\tau), \varphi\Big\rangle\, d\tau, \end{aligned} \tag{4.44}$$

for every $[t_1, t_2] \subset [0, T]$ *and for all* $\varphi \in C_0^\infty(\mathbb{T}^n)$*. Moreover, it satisfies the charge-energy inequality*

$$\int_{\mathbb{T}^n} |\psi^\varepsilon|^2 + \varepsilon^{2\alpha}|\partial_t \psi^\varepsilon|^2 + |\nabla\psi^\varepsilon|^2 + \frac{1}{2}\left(\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon}\right)^2 dx \le 2C_1 + (1 + 2\varepsilon^{2+2\alpha})C_2 \tag{4.45}$$

where C_1 *and* C_2 *denote the initial charge and energy given respectively by*

$$\begin{aligned} C_1 &= \int_{\mathbb{T}^n} |\psi_0^\varepsilon|^2 + \frac{i}{2}\varepsilon^{1+2\alpha}(\psi_1^\varepsilon\overline{\psi_0^\varepsilon} - \overline{\psi_1^\varepsilon}\psi_0^\varepsilon)dx\,, \\ C_2 &= \int_{\mathbb{T}^n} \varepsilon^{2\alpha}|\psi_1^\varepsilon|^2 + |\nabla\psi_0^\varepsilon|^2 + \frac{1}{2}\left(\frac{|\psi_0^\varepsilon|^2 - 1}{\varepsilon}\right)^2 dx\,. \end{aligned} \tag{4.46}$$

Similar to the semiclassical limit, we also assume $|\psi_0^\varepsilon| = |\psi_0| = 1$ and $(\psi_0^\varepsilon, \psi_1^\varepsilon) \to (\psi_0, 0)$ in $H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$ as $\varepsilon \to 0$. It follows immediately from the charge-energy inequality (4.45) that

$$\{\psi^\varepsilon\}_\varepsilon \quad \text{is bounded in} \quad L^\infty([0,T]; H^1(\mathbb{T}^n))\,, \tag{4.47}$$

$$\{\varepsilon^\alpha\partial_t\psi^\varepsilon\}_\varepsilon \quad \text{is bounded in} \quad L^\infty([0,T]; L^2(\mathbb{T}^n))\,, \tag{4.48}$$

$$\left\{\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon}\right\}_\varepsilon \quad \text{is bounded in} \quad L^\infty([0,T]; L^2(\mathbb{T}^n))\,. \tag{4.49}$$

We deduce from (4.49) that

$$|\psi^\varepsilon|^2 \to 1 \quad \text{a.e. and strongly in } L^2(\mathbb{T}^n)$$

as ε tends to 0. Note that (4.49) only shows that the quantity $\{\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\}_\varepsilon$ is a weakly relative compact set in $L^\infty([0,T]; L^2(\mathbb{T}^n))$, then (up to a subsequence) the sequence $\{\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\}_\varepsilon$ converges weakly $*$ to some function in $L^\infty([0,T]; L^2(\mathbb{T}^n))$. From equation of continuity, we can show

$$\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon} \rightharpoonup -\int_0^t \text{div } W(\psi)d\tau \tag{4.50}$$

weakly $*$ in $L^\infty([0,T]; L^2(\mathbb{T}^n))$, and thus

$$\left(\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon}\right)\psi^\varepsilon \rightharpoonup -\psi\int_0^t \text{div } W(\psi)d\tau \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{T}^n)\,. \tag{4.51}$$

We deduce from the above convergent results that the limit ψ satisfies $|\psi| = 1$ a.e. and

$$i\partial_t\psi + \left(\int_0^t \text{div } W(\psi)d\tau\right)\psi = 0 \tag{4.52}$$

in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$. Similar discussion as the case of semiclassical limit using $|\psi| = |\psi_0| = 1$ a.e., we can prove that ψ satisfies the classical wave map equation

$$\partial_t^2\psi - \Delta\psi = \Big(|\nabla\psi|^2 - |\partial_t\psi|^2\Big)\psi\,, \qquad |\psi| = 1 \quad \text{a.e.} \tag{4.53}$$

$$\psi(x,0) = \psi_0(x)\,, \quad \partial_t\psi(x,0) = 0\,, \qquad x\in\mathbb{T}^n\,. \tag{4.54}$$

Using the fact $|\psi| = |\psi_0| = 1$ again and writing $\psi = e^{i\theta}$ shows

$$\partial_t^2\theta = \Delta\theta, \qquad \theta(x,0) = \arg\,\psi_0\,, \quad \partial_t\theta(x,0) = 0\,. \tag{4.55}$$

Theorem 4.6 *Let* $(\psi_0^\varepsilon,\psi_1^\varepsilon)\in H^1(\mathbb{T}^n)\oplus L^2(\mathbb{T}^n)$, $|\psi_0^\varepsilon| = 1$, *and* $(\psi_0^\varepsilon,\psi_1^\varepsilon)\to(\psi_0,0)$ *in* $H^1(\mathbb{T}^n)\oplus L^2(\mathbb{T}^n)$, $|\psi_0| = 1$, *and let* ψ^ε *be the corresponding weak solution of the modulated nonlinear Klein-Gordon equation* (4.39)–(4.40). *Then the weak limit* ψ *satisfies* $|\psi| = 1$ *a.e. and solves the wave map* (4.53)–(4.54). *Moreover, let* $\psi = e^{i\theta}$ *then the phase function* θ *satisfies the wave equation* (4.55).

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