UNIMODULAR FOURIER MULTIPLIERS
ON MODULATION SPACES $M^{p,q}$ FOR $0 < p < 1$

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1. INTRODUCTION

In this note, we consider the boundedness of the Fourier multiplier operator $e^{i|D|^{\alpha}}$ on modulation spaces, where $\alpha > 0$ and $e^{i|D|^{\alpha}}$ is defined by

$$e^{i|D|^{\alpha}}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot \xi} e^{i|\xi|^{\alpha}} \hat{f}(\xi) d\xi.$$ 

In the case $\alpha = 2$, $u(t, x) = e^{it|D|^{2}}u_0(x)$ is the formal solution to the Schrödinger equation

$$\begin{cases}
    i\frac{\partial u}{\partial t}(t, x) = \Delta_x u(t, x) & (t > 0, x \in \mathbb{R}^n), \\
    u(0, x) = u_0(x) & (x \in \mathbb{R}^n).
\end{cases}$$

Modulation spaces $M_{s}^{p,q}$ were introduced by Feichtinger [3, 4] (see also Gröchenig [5]). We recall the definition of modulation spaces. Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$, and let $\psi \in S(\mathbb{R}^n)$ be such that

$$\text{supp } \psi \subset [-1, 1]^n \quad \text{and} \quad \sum_{k \in \mathbb{Z}^n} \psi(\xi - k) = 1 \quad \text{for all } \xi \in \mathbb{R}^n.$$ 

Then the modulation space $M_{s}^{p,q}(\mathbb{R}^n)$ consists of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f\|_{M_{s}^{p,q}} = \left( \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{sq} \|\psi(D - k)f\|_{L^p}^q \right)^{1/q} < \infty,$$

where $\psi(D - k)f = \mathcal{F}^{-1}[\psi(\cdot - k) \hat{f}]$. If $s = 0$, we simply write $M^{p,q}(\mathbb{R}^n)$ instead of $M_{0}^{p,q}(\mathbb{R}^n)$. We remark that $M_{s}^{2,2}$ coincides with the Sobolev space $W^{s,2}$.

It is known that $e^{i|D|^{2}}$ is bounded on $L^p$ if and only if $p = 2$ (Hörmander [7]). However, $e^{i|D|^{2}}$ is bounded on $M^{p,q}(\mathbb{R}^n)$ for all $1 \leq p, q \leq \infty$ (see Gröchenig-Heil [6], Toft [10], Wang-Zhao-Guo [11], Bényi-Gröchenig-Okoudjou-Rogers [1]). This is one of differences between $L^p$-spaces and modulation spaces. Bényi-Gröchenig-Okoudjou-Rogers ([1]) proved that if $0 \leq \alpha \leq 2$ then $e^{i|D|^{\alpha}}$ is bounded on $M^{p,q}(\mathbb{R}^n)$ for all $1 \leq p, q \leq \infty$. Furthermore, in the case $\alpha > 2$, Miyachi-Nicola-Rivetti-Tabacco-Tomita [9] showed that, for $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$, $e^{i|D|^{\alpha}}$ is bounded from $M_{s}^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ if and only if $s \geq (\alpha - 2)n|1/p - 1/2|$ (see [9] for more general results). In particular, this says that if $\alpha > 2$ and $p \neq 2$ then $e^{i|D|^{\alpha}}$ is not bounded on modulation spaces $M^{p,q}$.

The purpose of this note is to consider the case $0 < p < 1$, and our main result is the following:

**Theorem 1.1.** Let $0 < p < 1$, $0 < q \leq \infty$, $\alpha > n(1/p - 1)$ and $s \in \mathbb{R}$. Then $e^{i|D|^{\alpha}}$ is bounded from $M_{s}^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ if and only if $s \geq \max\{0, \alpha - 2\}n(1/p - 1/2)$.

We remark that Bényi-Okoudjou [2] considered the cases $0 \leq \alpha \leq 2$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$, and $\alpha \in \{1, 2\}$, $n/(n + 1) < p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. 

We end this section by explaining the organization of this note. In Section 2, we give the relation between \(L^p\)-boundedness and \(M^{p,q}\)-boundedness. In Section 3, we give the proof of Theorem 1.1.

2. Relation between \(L^p\)-boundedness and \(M^{p,q}\)-boundedness

Let \(S(\mathbb{R}^n)\) and \(S'(\mathbb{R}^n)\) be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform \(\mathcal{F}f\) and the inverse Fourier transform \(\mathcal{F}^{-1}f\) of \(f \in S(\mathbb{R}^n)\) by

\[
\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi) \, d\xi.
\]

For \(m \in S'(\mathbb{R}^n)\), we define the Fourier multiplier operator \(m(D)\) by

\[
m(D)f = \mathcal{F}^{-1}[m \hat{f}] = [\mathcal{F}^{-1}m]*f \quad \text{for all} \quad f \in S(\mathbb{R}^n).
\]

To avoid the fact that \(S(\mathbb{R}^n)\) is not dense in \(M^p_q(\mathbb{R}^n)\) if \(p = \infty\) or \(q = \infty\), we use the following definition of the boundedness of Fourier multiplier operators on modulation spaces: We say that \(m(D)\) is bounded from \(M^p_q(\mathbb{R}^n)\) to \(M^{p,q}(\mathbb{R}^n)\) if there exists a constant \(C > 0\) such that \(\|m(D)f\|_{M^{p,q}} \leq C\|f\|_{M^p_q}\) for all \(f \in S(\mathbb{R}^n)\), and set

\[
\|m(D)\|_{\mathcal{L}(M^p_q, M^{p,q})} = \sup\{\|m(D)f\|_{M^{p,q}} \mid f \in S(\mathbb{R}^n), \|f\|_{M^p_q} = 1\}.
\]

Similarly, we set

\[
\|m(D)\|_{\mathcal{L}(L^p, L^q)} = \sup\{\|m(D)f\|_{L^q} \mid f \in S(\mathbb{R}^n), \|f\|_{L^p} = 1\},
\]

and simply write \(\|m(D)\|_{\mathcal{L}(L^p, L^q)} = \|m(D)\|_{\mathcal{L}(L^p, L^q)}\) if \(p = q\).

The notations \(A \asymp B\) stands for \(C^{-1}A \leq B \leq CA\) for some positive constant \(C\) independent of \(A\) and \(B\). For \(1 \leq p \leq \infty, p'\) is the conjugate exponent of \(p\) (that is, \(1/p + 1/p' = 1\)). Throughout the rest of this note, \(\psi \in S(\mathbb{R}^n)\) is the same as in (1.1).

Lemma 2.1 ([8, Lemma 2.6]). Let \(0 < p < 1\), and let \(\Gamma\) be a compact subset of \(\mathbb{R}^n\). Then there exists a constant \(C > 0\) such that

\[
\|f * g\|_{L^p} \leq \|f\|_{L^p}\|g\|_{L^p}
\]

for all \(f, g \in L^p(\mathbb{R}^n)\) with \(\text{supp} \hat{f} \subset \xi + \Gamma\) and \(\text{supp} \hat{g} \subset \xi' + \Gamma\), where \(C > 0\) is independent of \(\xi, \xi' \in \mathbb{R}^n\).

The following is on the relation between \(L^p\)-boundedness and \(M^{p,q}\)-boundedness which is a slight modification of [9, Lemma 2.2]:

Lemma 2.2. Let \(0 < p, q \leq \infty, s \in \mathbb{R}\) and \(m \in S'(\mathbb{R}^n)\). Then \(m(D)\) is bounded from \(M^p_q(\mathbb{R}^n)\) to \(M^{p,q}(\mathbb{R}^n)\) if and only if there exists a constant \(C > 0\) such that

\[
(2.1) \quad \|\psi(D - k)m(D)f\|_{L^p} \leq C(1 + |k|)^s\|\psi(D - k)f\|_{L^p}
\]

for all \(k \in \mathbb{Z}^n\) and \(f \in S(\mathbb{R}^n)\).

Proof. We assume that (2.1) holds for some constant \(C > 0\). Then, by our assumption, we have

\[
\|m(D)f\|_{M^{p,q}} = \left( \sum_{k \in \mathbb{Z}^n} \|\psi(D - k)m(D)f\|_{L^p}^q \right)^{1/q} \leq C \left( \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{sq}\|\psi(D - k)f\|_{L^p}^q \right)^{1/q} = C\|f\|_{M^{p,q}}
\]

for all \(f \in S\), and we obtain the boundedness of \(m(D)\) from \(M^p_q\) to \(M^{p,q}\).
We next assume that $0 < p < 1$ and $m(D)$ is bounded from $M^{p,q}_{s}$ to $M^{p,q}$. Let $\varphi \in S$ be such that $\varphi = 1$ on $\text{supp} \psi$, $\text{supp} \varphi \subset [-2,2]^n$ and $|\sum_{k \in \mathbb{Z}^n} \varphi(\xi-k)| \geq C > 0$ for all $\xi \in \mathbb{R}^n$. Note that $\|f\|_{M^{p,q}_{s}} = (\sum_{k \in \mathbb{Z}^n} (1 + |k|)^{sq} \|\varphi(D-k)f\|_{L^p}^q)^{1/q}$. Since $\psi = \varphi \psi$, $\text{supp} \psi(\cdot-(k+\ell)) \subset (k+\ell)+[-1,1]^n$ and $\text{supp} \psi(\cdot-k) \hat{f} \subset k+[-1,1]^n$ for all $k, \ell \in \mathbb{Z}^n$, we have by Lemma 2.1 and the boundedness of $m(D)$ from $M^{p,q}_{s}$ to $M^{p,q}$:

$$\|\psi(D-k)m(D)f\|_{L^p} = \|\varphi(D-k)(m(D)\psi(D-k)f)\|_{L^p} \leq (\sum_{\ell \in \mathbb{Z}^n} \|\varphi(D-\ell)(m(D)\psi(D-k)f)\|_{L^p}^q)^{1/q} \leq C \|m(D)(\psi(D-k)f)\|_{M^{p,q}_{s}} \leq C \|m(D)\|_{L(M^{p,q}_{s},M^{p,q})} \|\psi(D-k)f\|_{M^{p,q}_{s}} = C \|m(D)\|_{L(M^{p,q}_{s},M^{p,q})} (\sum_{|\ell| \leq 2\sqrt{n}} (1+|k+\ell|)^{sq} \|\mathcal{F}^{-1}\psi(D-(k+\ell))\|_{L^p}^q)^{1/q} \leq C \|m(D)\|_{L(M^{p,q}_{s},M^{p,q})} (\sum_{|\ell| \leq 2\sqrt{n}} (1+|k+\ell|)^{sq} \|\psi(D-k)f\|_{L^p}^q)^{1/q} \leq C (1+|k|)^{s} \|m(D)\|_{L(M^{p,q}_{s},M^{p,q})} \|\mathcal{F}^{-1}\psi\|_{L^p} \|\psi(D-k)f\|_{L^p}$$

for all $k \in \mathbb{Z}^n$ and $f \in S$, where we have used $(1+|k+\ell|)^{s} \leq (1+|k|)^{s}(1+|\ell|)^{|s|}$.

Hence, we obtain (2.1) with $0 < p < 1$. For $1 \leq p \leq \infty$, by using Young’s inequality ($\|f*g\|_{L^p} \leq \|fg\|$) instead of Lemma 2.1, we can prove (2.1) in the same way. $\square$

3. Proof of Theorem 1.1

The proof of the following lemma is based on that of [9, Lemma 3.1]:

**Lemma 3.1.** Let $0 < p < 1$, $N = \lceil n(1/p-1/2) \rceil + 1$ and $\alpha > n(1/p-1)$, where $\lceil n(1/p-1/2) \rceil$ stands for the largest integer $\leq n(1/p-1/2)$. If $m$ is a $C^N(\mathbb{R}^n \setminus \{0\})$ function with compact support satisfying $|\partial^\beta m(\xi)| \leq C_{\beta} |\xi|^{|\alpha|-|\beta|}$ for all $\xi \neq 0$ and $|\beta| \leq N$, then $\mathcal{F}^{-1}m \in L^p(\mathbb{R}^n)$.

**Proof.** Assume that $\text{supp} \ m \subset \{ |\xi| \leq 2^j_0 \}$, where $j_0 \in \mathbb{Z}$. Let $\varphi \in S$ be such that supp $\varphi \subset \{ 1/2 \leq |\xi| \leq 2 \}$ and $\sum_{j \in \mathbb{Z}} \varphi(\xi/2^j) = 1$ for all $\xi \neq 0$. Since supp $\varphi(\cdot/2^j) \subset \{ 2^{j-1} \leq |\xi| \leq 2^{j+1} \}$, we see that

$$m(\xi) = \sum_{j = -\infty}^{j_0} \varphi(\xi/2^j) m(\xi) = \sum_{j = -\infty}^{j_0} m_j(\xi/2^j),$$

where $m_j(\xi) = \varphi(\xi)(2^j \xi)$. By using $p < 1$, we have

$$\|\mathcal{F}^{-1}m\|_{L^p}^p \leq \sum_{j = -\infty}^{j_0} \|2^jn(\mathcal{F}^{-1}m_j)(2^j \cdot)\|_{L^p}^p = \sum_{j = -\infty}^{j_0} 2^jn(p-1) \|\mathcal{F}^{-1}m_j\|_{L^p}^p.$$ 

Let $r$ be the conjugate exponent of $2/p$, and set $N = \lceil n/(pr) \rceil + 1$. Then $N = \lceil n/(p-1/2) \rceil + 1$. By Hölder’s inequality and Plancherel’s theorem,

$$\|\mathcal{F}^{-1}m_j\|_{L^p} = \|(1 + |\xi|)^{-N} (1 + |\xi|)^N \mathcal{F}^{-1}m_j\|_{L^p} \leq C \sum_{|\beta| \leq N} \|\partial^\beta m_j\|_{L^2}$$

for all $\xi \neq 0$ and $|\beta| \leq N$, where $\partial^\beta m_j(\xi) = \partial^\beta (m_j(\xi))$. By using $p < 1$, we have

$$\|\mathcal{F}^{-1}m\|_{L^p}^p \leq \sum_{j = -\infty}^{j_0} \|2^jn(\mathcal{F}^{-1}m_j)(2^j \cdot)\|_{L^p}^p = \sum_{j = -\infty}^{j_0} 2^jn(p-1) \|\mathcal{F}^{-1}m_j\|_{L^p}^p.$$ 

Let $r$ be the conjugate exponent of $2/p$, and set $N = \lceil n/(pr) \rceil + 1$. Then $N = \lceil n/(p-1/2) \rceil + 1$. By Hölder’s inequality and Plancherel’s theorem,

$$\|\mathcal{F}^{-1}m_j\|_{L^p} = \|(1 + |\xi|)^{-N} (1 + |\xi|)^N \mathcal{F}^{-1}m_j\|_{L^p} \leq C \sum_{|\beta| \leq N} \|\partial^\beta m_j\|_{L^2}$$

for all $\xi \neq 0$ and $|\beta| \leq N$, where $\partial^\beta m_j(\xi) = \partial^\beta (m_j(\xi))$. By using $p < 1$, we have

$$\|\mathcal{F}^{-1}m\|_{L^p}^p \leq \sum_{j = -\infty}^{j_0} \|2^jn(\mathcal{F}^{-1}m_j)(2^j \cdot)\|_{L^p}^p = \sum_{j = -\infty}^{j_0} 2^jn(p-1) \|\mathcal{F}^{-1}m_j\|_{L^p}^p.$$ 

Let $r$ be the conjugate exponent of $2/p$, and set $N = \lceil n/(pr) \rceil + 1$. Then $N = \lceil n/(p-1/2) \rceil + 1$. By Hölder’s inequality and Plancherel’s theorem,
for all $j \in \mathbb{Z}$, where we have used the fact $prN > n$. Since $\text{supp} \varphi \subset \{2^{-1} \leq |\xi| \leq 2\}$, we have by our assumption

$$
|\partial^\beta m_j(\xi)| = \left| \sum_{\beta_1 + \beta_2 = \beta} C_{\beta_1, \beta_2} (\partial^{\beta_1} \varphi)(\xi) 2^{j|\beta_1|}(\partial^{\beta_2} m)(2^j \xi) \right|
$$

(3.3)

$$
\leq \sum_{\beta_1 + \beta_2 = \beta} C_{\beta_1, \beta_2} |(\partial^{\beta_1} \varphi)(\xi)| 2^{j|\beta_2|}(C_{\beta_2} 2^j |\alpha - |\beta_2||) \leq C_\beta 2^{j\alpha}
$$

for all $j \in \mathbb{Z}$ and $|\beta| \leq N$. On the other hand, $\text{supp} m_j \subset \{2^{-1} \leq |\xi| \leq 2\}$ for all $j \in \mathbb{Z}$. Therefore, by (3.1)-(3.3),

$$
\|\mathcal{F}^{-1} m\|_{L^p}^p \leq \sum_{j=-\infty}^{j_0} 2^{jn(p-1)} \|\mathcal{F}^{-1} m_j\|_{L^p}^p
$$

$$
\leq C \sum_{j=-\infty}^{j_0} 2^{-jn(1/p-1)p} \sum_{|\beta| \leq N} \|\partial^\beta m_j\|_{L^2}^p \leq C \sum_{j=-\infty}^{j_0} 2^{j(\alpha - n(1/p-1))p} < \infty.
$$

The proof is complete. $\square$

For $\alpha > 0$ and $k \in \mathbb{Z}^n$, we set

$$
(3.4) \quad \sigma_\alpha(\xi) = |\xi|^\alpha \quad \text{and} \quad \tau_{\alpha,k}(\xi) = \sigma_\alpha(\xi + k) - \sigma_\alpha(k) - (\nabla \sigma_\alpha)(k) \cdot \xi.
$$

**Lemma 3.2.** Let $0 < p < 1$ and $\alpha > n(1/p - 1)$. Then there exists a constant $C > 0$ such that

$$
\|\psi(D-k)e^{i\sigma_\alpha(D)}f\|_{L^p} \leq C\|\psi(D-k)f\|_{L^p}
$$

for all $|k| < 4\sqrt{n}$ and $f \in \mathcal{S}(\mathbb{R}^n)$.

**Proof.** Let $\eta$ be a Schwartz function with compact support. Then

$$
|\partial^\beta [\eta(\xi)(e^{i\sigma_\alpha(\xi)} - 1)]| \leq C_\beta |\xi|^{\alpha - |\beta|}
$$

for all $\xi \neq 0$ and $\beta$. Hence, it follows from Lemma 3.1 that

$$
\mathcal{F}^{-1}[\eta e^{i\sigma_\alpha}] = \mathcal{F}^{-1}[\eta(e^{i\sigma_\alpha} - 1)] + \mathcal{F}^{-1}\eta \in L^p.
$$

Take $\varphi \in \mathcal{S}$ such that $\text{supp} \varphi$ is compact and $\varphi = 1$ on supp $\psi$. Then, by Lemma 2.1 and the first part of this proof with $\eta = \varphi(\cdot - k)$, for $|k| < 4\sqrt{n},$

$$
\|\psi(D-k)e^{i\sigma_\alpha(D)}f\|_{L^p} = \|\varphi(D-k)e^{i\sigma_\alpha(D)}\psi(D-k)f\|_{L^p}
$$

$$
\leq C \|\mathcal{F}^{-1} [\varphi(\cdot - k)e^{i\sigma_\alpha}] \|_{L^p} \|\psi(D-k)f\|_{L^p}
$$

(3.5)

$$
\leq C \|\psi(D-k)f\|_{L^p}
$$

for all $f \in \mathcal{S}$. This completes the proof. $\square$

**Lemma 3.3.** Let $0 < p < 1$. Then there exists a constant $C > 0$ such that

$$
\|\psi(D-k)e^{i\sigma_\alpha(D)}f\|_{L^p} \leq C |k|^{\max\{0, \alpha - 2\}n(1/p - 1/2)} \|\psi(D-k)f\|_{L^p}
$$

for all $|k| \geq 4\sqrt{n}$ and $f \in \mathcal{S}(\mathbb{R}^n)$. 4
Proof. Throughout this proof, we assume that $|k| \geq 4\sqrt{n}$ and $f \in S$. Let $\varphi \in S$ be such that $\text{supp} \, \varphi \subset [-2, 2]^n$ and $\varphi = 1$ on supp $\psi$. Then, by Lemma 2.1,
\[
\|\psi(D-k)e^{i\sigma_\alpha(D)}f\|_{L^p} = \|\varphi(D-k)e^{i\sigma_\alpha(D)}\psi(D-k)f\|_{L^p} 
\]
\[
= \|\mathcal{F}^{-1}[\varphi(\cdot-k)e^{i\sigma_\alpha}] * \psi(D-k)f\|_{L^p} 
\]
\[
\leq C\|\mathcal{F}^{-1}[\varphi(\cdot-k)e^{i\sigma_\alpha}]\|_{L^p}\|\psi(D-k)f\|_{L^p} 
\]
\[
= C\|\mathcal{F}^{-1}[\varphi e^{i\sigma_\alpha(k)}]\|_{L^p}\|\psi(D-k)f\|_{L^p}.
\]
(3.6)

Let us estimate $\|\mathcal{F}^{-1}[\varphi e^{i\sigma_\alpha(k)}]\|_{L^p}$. Since
\[
\mathcal{F}^{-1}[\varphi e^{i\sigma_\alpha(k)}](x) = e^{i\sigma_\alpha(k)}(x + (\nabla \sigma_\alpha)(k)),
\]
where $\tau_{\alpha,k}$ is defined by (3.4), we see that
\[
\|\mathcal{F}^{-1}[\varphi e^{i\sigma_\alpha(k)}]\|_{L^p} = \|\mathcal{F}^{-1}[\varphi e^{i\tau_{\alpha,k}}]\|_{L^p}.
\]
By Taylor’s formula,
\[
\tau_{\alpha,k}(\xi) = 2 \sum_{|\beta|=2} \xi^\beta \int_0^1 (1-t)(\partial^\beta \sigma_\alpha)(k+t\xi) \, dt.
\]
(3.8)

If $\xi \in [-2, 2]^n$, then $|k + t\xi| \approx |k|$ for all $0 \leq t \leq 1$. Since $|\partial^\gamma \sigma_\alpha(\eta)| \leq C_\gamma |\eta|^{|\alpha|-|\gamma|}$, we have by (3.8)
\[
|\partial^\gamma \tau_{\alpha,k}(\xi)| \approx \sum_{|\beta|=2} \sum_{\gamma_1+\gamma_2=\gamma} C_{\beta,\gamma_1,\gamma_2} |\partial^{\gamma_1}(\xi^\beta)| \int_0^1 (1-t)^{|\gamma_2|}(\partial^{\beta+\gamma_2} \sigma_\alpha)(k+t\xi) \, dt 
\]
\[
\leq C_{\beta} \sum_{|\beta|=2} \sum_{\gamma_1+\gamma_2=\gamma} |\partial^{\gamma_1}(\xi^\beta)| \int_0^1 |k + t\xi|^{|\alpha|-|\beta|-|\gamma_2|} \, dt \leq C_{\beta} |k|^{|\alpha|-2}
\]
for all multi-indices $\gamma$. Hence, by noting $|k|^{\max\{0,|\alpha|-2\}} \geq 1$, we have
\[
|\partial^\gamma (\varphi(\xi)e^{i\tau_{\alpha,k}(\xi)})| 
\]
\[
= \sum_{|\gamma|} \sum_{N=0} |\mu|+\nu_1+\cdots+\nu_N=\gamma \|\partial^\mu \varphi\|_{L^\infty}(C_{\nu_1}|k|^{|\alpha|-2}) \cdots (C_{\nu_N}|k|^{|\alpha|-2}) 
\]
\[
\leq C_{\gamma} \sum_{|\gamma|} |\partial^\mu \varphi|_{L^\infty}(C_{\nu_1}|k|^{|\alpha|-2}) \cdots (C_{\nu_N}|k|^{|\alpha|-2}) 
\]
\[
\leq C_{\gamma} |k|^{\max\{0,|\alpha|-2\} n}. \]

Then, setting $\varphi_{\alpha,k}(\xi) = \varphi(\xi)e^{i\tau_{\alpha,k}(\xi)}$, we have
\[
|\partial^\gamma (\varphi_{\alpha,k}(\xi)/|k|^{\max\{0,|\alpha|-2\}})| \leq C_{\gamma} \chi_{[-2|k|^{\max\{0,|\alpha|-2\}},2|k|^{\max\{0,|\alpha|-2\}}]^n}(\xi)
\]
for all multi-indices $\gamma$, where $\chi_A$ denote the characteristic function of $A$. Therefore, by (3.2), (3.6), (3.7) and (3.9),
\[
\|\psi(D-k)e^{i\sigma_\alpha(D)}f\|_{L^p} \leq C\|\mathcal{F}^{-1}[\varphi_{\alpha,k}]\|_{L^p}\|\psi(D-k)f\|_{L^p} 
\]
\[
= C|k|^{\max\{0,|\alpha|-2\} n(1/p-1)}\|\mathcal{F}^{-1}[\varphi_{\alpha,k}(\cdot)/|k|^{\max\{0,|\alpha|-2\}}]\|_{L^p}\|\psi(D-k)f\|_{L^p} 
\]
\[
\leq C|k|^{\max\{0,|\alpha|-2\} n(1/p-1)} \left( \sum_{|\gamma| \leq N} \|\partial^\gamma [\varphi_{\alpha,k}(\cdot)/|k|^{\max\{0,|\alpha|-2\}}]\|_{L^2} \right) \|\psi(D-k)f\|_{L^p} 
\]
5
\[ \leq C|k|^\max\{0, \alpha-2\}n(1/p-1/2)\|\psi(D - k)f\|_{L^p}, \]

where \( N = [n(1/p - 1/2)] + 1 \) and \( C > 0 \) is independent of \( k \) satisfying \( |k| \geq 4\sqrt{n} \). The proof is complete. \( \square \)

Before proving Theorem 1.1, we give the following remark on the case \( 0 \leq \alpha \leq 2 \):

**Remark 3.4.** Let \( 0 \leq \alpha \leq 2 \) and \( 1 \leq p \leq \infty \). In this case, \( e^{i\sigma_\alpha(D)} \) is bounded from \( M^{p,q}_s(\mathbb{R}^n) \) to \( M^{p,q}_t(\mathbb{R}^n) \) only if \( s \geq 0 \).

We first consider the case \( p = 2 \). By Plancherel’s theorem,

\[ \|e^{i\sigma_\alpha(D)} f\|_{M^{2,q}} = \left( \sum_{k \in \mathbb{Z}^n} \|\psi(D - k)e^{i\sigma_\alpha(D)} f\|_{L^2}^q \right)^{1/q} \]

\[ = \left\{ \sum_{k \in \mathbb{Z}^n} \left( \frac{1}{(2\pi)^{n/2}}\|\psi(\cdot - k)e^{i\sigma_\alpha}\hat{f}\|_{L^2} \right)^q \right\}^{1/q} \]

\[ = \left\{ \sum_{k \in \mathbb{Z}^n} \left( \frac{1}{(2\pi)^{n/2}}\|\psi(\cdot - k)\hat{f}\|_{L^2} \right)^q \right\}^{1/q} = \|f\|_{M^{2,q}}. \]

Hence, the boundedness of \( e^{i\sigma_\alpha(D)} \) from \( M^{2,q}_s \) to \( M^{2,q}_t \) implies the embedding \( M^{2,q}_s \hookrightarrow M^{2,q}_t \). Therefore, \( e^{i\sigma_\alpha(D)} \) is bounded from \( M^{2,q}_s \) to \( M^{2,q}_t \) only if \( s \geq 0 \).

We next consider the case \( 1 \leq p \leq \infty \) and \( p \neq 2 \). If \( m(D) \) is bounded from \( M^{p,q}_s \) to \( M^{p,q}_t \), then \( m(D) \) is also bounded from \( M^{p,q}_s \) to \( M^{p,q}_t \). This follows from the facts that \( m(D) \) is bounded from \( M^{p,q}_s \) to \( M^{p,q}_t \) if and only if \( \sup_{k \in \mathbb{Z}^n} (1 + |k|)^{-s}\| \psi(D - k)m(D)\|_{L(L^p)} < \infty \) (see [9, Lemma 2.2]) and \( \|\psi(D - k)m(D)\|_{L(L^p)} = \|\psi(D - k)m(D)\|_{L(L^{p'})}. \) Then, by interpolation, if \( e^{i\sigma_\alpha(D)} \) is bounded from \( M^{p,q}_s \) to \( M^{p,q}_t \) for some \( s < 0 \), then \( e^{i\sigma_\alpha(D)} \) is also bounded from \( M^{p,q}_s \) to \( M^{p,q}_t \). Therefore, \( e^{i\sigma_\alpha(D)} \) is bounded from \( M^{p,q}_s \) to \( M^{p,q}_t \) only if \( s \geq 0 \).

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( 0 < p < 1, 0 < q \leq \infty, \alpha > n(1/p - 1/2) \) and \( s \in \mathbb{R} \).

We first assume that \( s \geq \max\{0, \alpha - 2\}n(1/p - 1/2) \). By Lemmas 3.2 and 3.3,

\[ \|\psi(D - k)e^{i\sigma_\alpha(D)} f\|_{L^p} \leq C(1 + |k|^\max\{0, \alpha-2\}n(1/p-1/2))\|\psi(D - k)f\|_{L^p} \]

\[ \leq C(1 + |k|^\alpha)\|\psi(D - k)f\|_{L^p} \]

for all \( k \in \mathbb{Z}^n \) and \( f \in \mathcal{S} \). Hence, by Lemma 2.2, we have the boundedness of \( e^{i\sigma_\alpha(D)} \) from \( M^{p,q}_s \) to \( M^{p,q}_t \).

We next assume that \( e^{i\sigma_\alpha(D)} \) is bounded from \( M^{p,q}_s \) to \( M^{p,q}_t \). By Lemma 2.2, we may assume \( q \geq 1 \). We note that \( e^{i\sigma_\alpha(D)} \) is bounded on \( M^{2,q}_s \) (see Remark 3.4). Hence, it follows from interpolation with the boundedness on \( M^{2,q}_s \) that, if \( s < \max\{0, \alpha - 2\}n(1/p - 1/2) \), then \( e^{i\sigma_\alpha(D)} \) is bounded from \( M^{p,q}_s \) to \( M^{p,q}_{\tilde{s}} \), where \( 1 < \tilde{p} < 2 \) and \( \tilde{s} < \max\{0, \alpha - 2\}n(1/\tilde{p} - 1/2) \). However, in the case \( 1 < \tilde{p} < 2 \), \( e^{i\sigma_\alpha(D)} \) is bounded from \( M^{p,q}_s \) to \( M^{p,q}_{\tilde{s}} \) only if \( \tilde{s} \geq \max\{0, \alpha - 2\}n(1/\tilde{p} - 1/2) \) (see Remark 3.4 and [9]). Therefore, \( s \) must satisfy \( s \geq \max\{0, \alpha - 2\}n(1/p - 1/2) \).

We end this note by giving the following remark on the case \( 1 \leq p \leq \infty \) and \( 0 < q < 1 \):

**Remark 3.5.** Let \( \alpha \geq 0, 1 \leq p \leq \infty \) and \( s \in \mathbb{R} \). Lemma 2.2 says that \( e^{i\sigma_\alpha(D)} \) is bounded from \( M^{p,q}_s(\mathbb{R}^n) \) to \( M^{p,q}_s(\mathbb{R}^n) \) for some \( 0 < q < \infty \) if and only if \( e^{i\sigma_\alpha(D)} \) is bounded from \( M^{p,q}_s(\mathbb{R}^n) \) to \( M^{p,q}_s(\mathbb{R}^n) \) for all \( 0 < q \leq \infty \). In particular, the boundedness of \( e^{i\sigma_\alpha(D)} \) from \( M^{p,q}_s(\mathbb{R}^n) \) to \( M^{p,q}_s(\mathbb{R}^n) \) with \( 0 < q < 1 \) is equivalent to that with \( 1 \leq q \leq \infty \). On the other
hand, by [1, 9] and Remark 3.4, $e^{i\sigma_\alpha(D)}$ is bounded from $M_{s}^{p,q}(\mathbb{R}^n)$ to $M_{s}^{p,q}(\mathbb{R}^n)$ if and only if $s \geq \max\{0, \alpha - 2\} n|1/p - 1/2|$, where $1 \leq q \leq \infty$. Combining these facts, we see that $e^{i\sigma_\alpha(D)}$ is bounded from $M_{s}^{p,q}(\mathbb{R}^n)$ to $M_{s}^{p,q}(\mathbb{R}^n)$ if and only if $s \geq \max\{0, \alpha - 2\} n|1/p - 1/2|$, where $0 < q < 1$.

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