Lifting Galois representations over arbitrary number fields: A resume

By

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Let \( k \) be a finite field of characteristic \( p > 5 \). Let \( K \) be a number field of finite degree over \( \mathbb{Q} \) and \( G_K \) its absolute Galois group \( \text{Gal}(\overline{K}/K) \). Let \( \bar{\rho} : G_K \rightarrow \text{GL}_2(k) \) be a continuous representation and let \( W(k) \) be the ring of Witt vectors of \( k \). We consider the following question:

**Question 0.1.** Is there a continuous representation \( \rho : G_K \rightarrow \text{GL}_2(W(k)) \) satisfying \( \bar{\rho} = \rho \mod p \)?

This question has been motivated by a conjecture of Serre ([S]) saying that all odd absolutely irreducible continuous representations \( \bar{\rho} : G_\mathbb{Q} \rightarrow \text{GL}_2(k) \) are modular. This implies the existence of a lift to characteristic zero. This conjecture was proved by Khare and Wintenberger in [KW1, KW2].

In [K], Khare proved the existence of a lift to \( W(k) \) for any \( \bar{\rho} : G_K \rightarrow \text{GL}_2(k) \) which is reducible. Thus we may assume that \( \bar{\rho} \) is irreducible.

For a place \( v \) of \( K \), let \( K_v \) be the completion of \( K \) at \( v \), and let \( G_v \) be its absolute Galois group \( \text{Gal}(\overline{K_v}/K_v) \). Let \( \text{Ad}^0 \bar{\rho} \) be the \( k \)-vector space of all trace zero two-by-two matrices over \( k \) on which \( G_K \) acts by conjugation. Our main result is the following:

**Theorem 0.2.** Assume that \( H^2(G_v, \text{Ad}^0 \bar{\rho}) = 0 \) for each place \( v \mid p \). Then \( \bar{\rho} \) lifts to a continuous representation \( \rho : G_K \rightarrow \text{GL}_2(W(k)) \) which is unramified outside a finite set of places of \( K \).

For \( K = \mathbb{Q} \), Ramakrishna proved under very general conditions on \( \bar{\rho} \) that there exist lifts to \( W(k) \) in [R1, R2]. Gee ([G]) and Manoharmayum ([M]) proved, independently, that there exist lifts to \( W(k) \) for \( K \) satisfying \( [K(\mu_p) : K] > 2 \), where \( \mu_p \) is the group of \( p \)-th roots of unity.

Our method used in the proof of the Theorem is essentially that of Ramakrishna [R1, R2], but we follow the more axiomatic treatment presented in [Ta]. We denote by \( S \)
a finite set of places of $K$ containing the places above $p$, the infinite places and the places at which $\bar{\rho}$ is ramified. Let $K_S$ denote the maximal algebraic extension of $K$ unramified outside $S$ and put $G_{K,S} = \text{Gal}(K_S/K)$. Thus $\bar{\rho}$ factors through $G_{K,S}$. The existence of a lifting of $\bar{\rho}$ follows from the triviality of a certain class of $H^2(G_{K,S}, \text{Ad}^0 \bar{\rho})$. Since it is difficult to calculate it directly, we reduce the calculations of global obstructions to those of local obstructions, which are much better understood. By Taylor [Ta], it boils down to showing the vanishing of a certain Selmer group. We can prove the triviality of this group by extending $S$ by a suitably chosen finite set $Q$. With the choice of $Q$ of Ramakrishna and Gee, they could not prove the triviality of the Selmer group for an arbitrary number field. Our choice of $Q$ is different from Ramakrishna’s and Gee’s. For more details, see our preprint [To].

References

[KW1] C. Khare and J.-P. Wintenberger, Serre’s modularity conjecture (I), preprint