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Ramification of truncated discrete valuation rings: a survey

By

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Abstract

A truncated discrete valuation ring is a commutative ring which is isomorphic to a quotient of finite length of a complete discrete valuation ring. We give an equivalence between the category of at most a-ramified finite separable extensions of a complete discrete valuation field K and the category of at most a-ramified finite extensions of the "length-a truncation" of the integer ring of K. This extends a theorem of Deligne in which he proved this fact assuming the residue field is perfect. Our theory depends heavily on Abbes-Saito's ramification theory.

§ 1. Introduction

A truncated discrete valuation ring (abbreviated as tdvr in the following) is a commutative ring which is isomorphic to a quotient of finite length of a discrete valuation ring (equivalently, it can be defined to be an Artinian local ring whose maximal ideal is generated by one element). This note is a short survey of the author's results of [9] and [10] (joint work with Y. Taguchi) which are concerned with the ramification of extensions of such rings.

One of the motivations for studying such objects is that many phenomena on objects over discrete valuation rings are often determined by their reduction modulo powers of the maximal ideal. Classical examples of such phenomena include Hensel's lemma (*cf.* [15], §2.2, Th. 1). In the case where the tdvr has a perfect residue field, Deligne ([5]) formulated this fact precisely as a categorical equivalence between the category of at most a-ramified finite separable extensions of a complete discrete valuation field K and the category of at most a-ramified finite extensions of the "length-a truncation"

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$\mathcal{O}_K/\mathfrak{m}_K^a$ of the integer ring of K. Our main result is a generalization of this theorem to the case of arbitrary residue field.

For a tdvr A of length $a \geqslant 1$ and each rational number m with $0 < m \leqslant a$, we shall construct a category $\mathcal{FFP}_A^{<m}$ of finite flat principal A-algebras[1] with ramification bounded by m (Def. 3.6 below). Next, let K be a cdvf and A a tdvr of length a. Suppose $\mathcal{O}_K/\mathfrak{m}_K^a$ and A are isomorphic as rings, where $\mathfrak{m}_K$ is the maximal ideal of $\mathcal{O}_K$. Then for each rational number m with $0 < m \leqslant a$, we construct a functor $T : \mathcal{FE}_K^{<m} \to \mathcal{FFP}_A^{<m}$ from the category $\mathcal{FE}_K^{<m}$ of finite étale K-algebras with ramification bounded by m (Def. 3.7). The object $T(L)$ for L in $\mathcal{FE}_K^{<m}$ is by definition the quotient ring $\mathcal{O}_L/\mathfrak{m}_K^a\mathcal{O}_L$, where $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in L. Our main theorem is the following, which generalizes Deligne's theorem ([5], Th. 2.8) to the case of imperfect residue fields, except that he uses a category of triples associated with the tdvr's in our $\mathcal{FFP}_A^{<m}$, which have *a priori* less information than the tdvr's themselves.

Theorem 1.1. *The functor $T : \mathcal{FE}_K^{<m} \to \mathcal{FFP}_A^{<m}$ is an equivalence of categories.*

Remark. (i) The case of $a = 1$ in the theorem is well-known (*cf.* [14], Chap. III, §5). Indeed, if $m \leqslant 1$, the objects of $\mathcal{FE}_K^{<m}$ are direct products of finite unramified extensions of K, and the objects of $\mathcal{FFP}_A^{<m}$ are étale over A (*cf.* Prop. 2.1). Thus our main interest is in the case $a > 1$.

(ii) Let $G_K = \mathrm{Gal}(\overline{K}/K)$ denote the absolute Galois group of K, and G_K^a its ath ramification subgroup defined by Abbes and Saito ([2], [3]). The category $\mathcal{FE}_K^{<m}$ is, and hence $\mathcal{FFP}_A^{<m}$ is also, a Galois category whose fundamental group is G_K/G_K^m by the very definition of the ramification filtration. Note that $\mathcal{FE}_K^{<m}$ is equivalent also to the category of coverings of $\mathrm{Spec}(\mathcal{O}_K)$ with ramification bounded by $\mathfrak{m}_K^m$ ([8], Def. 2.3); in the terminology of *op. cit.*, we have $\pi_1(\mathrm{Spec}(\mathcal{O}_K), \mathfrak{m}_K^m) = G_K/G_K^m$.

A direct consequence of our theorem is:

Corollary 1.2. *Let F and K be two cdvf's, and a an integer $\geqslant 1$. Assume that $\mathcal{O}_F/\mathfrak{m}_F^a \simeq \mathcal{O}_K/\mathfrak{m}_K^a$ as a ring.*

(i) *The categories $\mathcal{FE}_F^{<m}$ and $\mathcal{FE}_K^{<m}$ are equivalent for any $m \leqslant a$.*

(ii) *There is a (non-canonical) isomorphism $\gamma : G_F/G_F^a \xrightarrow{\simeq} G_K/G_K^a$ such that $\gamma(G_F^m/G_F^a) = G_K^m/G_K^a$ for all $m \leqslant a$.*

Here, G_F^m etc. are the ramification subgroups of the absolute Galois group G_F of F etc. defined by Abbes and Saito ([2], [3]). This corollary holds because we have $\mathcal{FE}_F^{<m} \simeq \mathcal{FFP}_{\mathcal{O}_F/\mathfrak{m}_F^a}^{<m} \simeq \mathcal{FFP}_{\mathcal{O}_K/\mathfrak{m}_K^a}^{<m} \simeq \mathcal{FE}_K^{<m}$ as Galois categories from the above theorem. This

[1] We mean by a *principal* A-algebra an A-algebra of which every ideal is generated by one element. All algebras in this paper are commutative.

is particularly interesting when F and K have different characteristics. Note that, for any cdvf F of positive characteristic, there exists a cdvf K of characteristic 0 which satisfies the assumption $\mathcal{O}_F/\mathfrak{m}_F^a \simeq \mathcal{O}_K/\mathfrak{m}_K^a$ (Prop. 2.1). The larger a is, the larger absolute ramification index K must have. Thus the cdvf F of characteristic $p > 0$ may be thought of as a "limit" of cdvf's K of characteristic 0 (*cf.* [5]).

In view of the above results, we may define *the* Galois group G_A of A to be G_K/G_K^a (or equivalently, to be the fundamental group of the Galois category $\mathcal{FFP}_A^{<a}$) together with the ramification subgroups $G_A^m := G_K^m/G_K^a$, where K is any cdvf such that $A \simeq \mathcal{O}_K/\mathfrak{m}_K^a$. The filtered group G_A depends (up to inner automorphisms) only on the isomorphism class of A as a ring. It is natural to ask the converse:

Question. If A and A' are two tdvr's of length a and if there is an isomorphism $\gamma : G_A \to G_{A'}$ of groups such that $\gamma(G_A^m) = G_{A'}^m$ for all $m \leqslant a$, then is it true that $A \simeq A'$ as a ring?

This problem is a version of the Grothendieck conjecture in anabelian geometry. The Neukirch-Uchida theorem (= the Grothendieck conjecture on global fields) says that for global fields K and K', if we have an isomorphism $G_K \xrightarrow{\simeq} G_{K'}$ of the absolute Galois groups of K and K' then we have $K \simeq K'$ (*cf.* [16], Chap. XII, Sect. 3). In other words, any global field can be recovered from its absolute Galois group. However, it is known that the local analogue of this theorem dose not hold ([17], see also [11]). In [13], S. Mochizuki showed that any finite extension field over $\mathbb{Q}_p$ can be recovered from its absolute Galois group equipped with the filtration defined by the ramification groups in the upper numbering (See also [1] for the case of positive characteristic). This result can be thought of as the case of "$a = \infty$" and finite residue fields in the above Question. In general, it is necessary to assume that the residue fields of A and A' are either finite or of some "anabelian" nature. If we include the case $a = 1$, where the Question reduces to the usual Grothendieck conjecture on fields (the "birational" Grothendieck conjecture), the residue field should be a field for which the Grothendieck conjecture holds. For example, it should not be algebraically closed. However, if we look only at the case $a > 1$, then it is not clear if the question above holds even if the residue field is, say, algebraically closed.

Throughout this paper, K is a complete discrete valuation field with residual characteristic $p > 0$. We denote by $\mathcal{O}_K$ the valuation ring of K, $\mathfrak{m}_K$ the maximal ideal of $\mathcal{O}_K$, π_K a uniformizing element of K, and $\overline{K}$ a fixed separable closure of K. For any étale K-algebra L, we denote by $\mathcal{O}_L$ the integral closure of $\mathcal{O}_K$ in L. We use the following abbreviations:
dvr := discrete valuation ring,
cdvf := complete discrete valuation field,
cdvr := complete discrete valuation ring,

tdvr := truncated discrete valuation ring.

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§ 2. Truncated discrete valuation rings

First we recall some basic notions from [9]. A *tdvr* is an Artinian local ring whose maximal ideal is generated by one element. The *length* of a tdvr A is the length of A as an A-module. It is known that a tdvr A is principal, and any ideal is of the form $\mathfrak{m}_A^i$ for some $i \geqslant 0$. Any generator π_A of the maximal ideal $\mathfrak{m}_A$ is said to be a *uniformizer* of A. Any element x of a tdvr A of length a can be written as $x = u\pi_A^i$ with $u \in A^\times$, π_A a uniformizer of A, and $0 \leqslant i < a$ (with the convention $0^0 = 1$ if $a = 1$).

If K is a cdvf, then $\mathcal{O}_K/\mathfrak{m}_K^a$ is a tdvr for any integer $a \geqslant 1$. If L/K is a finite extension of cdvf's, then $B = \mathcal{O}_L/\mathfrak{m}_K^a\mathcal{O}_L$ is a finite extension of $A = \mathcal{O}_K/\mathfrak{m}_K^a$. Conversely, it is known that any tdvr is a quotient of a cdvr (*cf.* [12], Th. 3.3). More precisely, we have:

Proposition 2.1 ([9], Prop. 2.2). *Let A be a tdvr with residue field k of characteristic $p \geqslant 0$, and let a be the length of A. Then there exists a cdvr $\mathcal{O}$ such that A is isomorphic to $\mathcal{O}/\mathfrak{m}^a$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. If $pA = 0$, then this $\mathcal{O}$ can be taken to be the power series ring $k[\![\pi]\!]$; if $pA \neq 0$, then $\mathcal{O}$ as above must be finite over a Cohen p-ring ([6], $0_{\mathrm{IV}}, 19.8$) with residue field k. (If $pA = 0$ and $p \neq 0$, then both types of $\mathcal{O}$ are possible.)*

(ii) *Let K be a cdvf and let $A = \mathcal{O}_K/\mathfrak{m}_K^a$ with $a \geqslant 1$. For any finite extension B/A of tdvr's, there exist a finite separable extension L/K and an isomorphism $\psi : \mathcal{O}_L/\mathfrak{m}_K^a\mathcal{O}_L \to B$ such that the diagram*

$$\begin{tikzcd} \mathcal{O}_L/\mathfrak{m}_K^a\mathcal{O}_L \arrow[r, "\psi"] & B \\ \mathcal{O}_K/\mathfrak{m}_K^a \arrow[u] \arrow[r, equal] & A \arrow[u] \end{tikzcd}$$

is commutative, where the left vertical arrow is the one induced by $\mathcal{O}_K \hookrightarrow \mathcal{O}_L$.

By abuse of terminology, we may call a cdvr a tdvr of length ∞.

§ 3. Construction of the category $\mathcal{FFP}_A^{<m}$

In this section, we fix a tdvr A of length $a \leqslant \infty$ with residue field k of characteristic $p > 0$. We recall here the definition of the category of $\mathcal{FFP}_A^{<m}$ of finite flat principal A-algebras with "ramification $< m$", assuming for simplicity that m is a positive integer.

A *polarization* $\boldsymbol{\alpha}$ of A is a pair $(S \overset{\alpha}{\to} A, (x))$ consisting of a surjective local ring homomorphism $\alpha : S \to A$, where S is a complete regular local ring with residue field k, and a principal ideal (x) of S such that $\alpha((x)) = \mathfrak{m}_A$. Two basic examples of polarizations are the following:

Example 3.1. (i) If A is a quotient of a cdvr S, it may be regarded as a polarization $(S \overset{\alpha}{\to} A, (x))$, where α is the quotient map $S \to A$ and x is a uniformizer of S.

(ii) Let W be a Cohen p-ring with residue field k, and $S = W[\![x]\!]$ the power series ring over W in one variable x. Then for any choice of a uniformizer π of A, there exists a unique continuous homomorphism $\alpha : S \to A$ which induces the identity map between the residue fields and maps x to π. The pair $(S \overset{\alpha}{\to} A, (x))$ is a polarization of A.

For the moment, we fix such an $\boldsymbol{\alpha}$. Then we view any A-algebra also as an S-algebra via α. We endow S with the $\mathfrak{m}_S$-adic topology. All S-algebra homomorphisms are assumed continuous with respect to the $\mathfrak{m}_S$-adic topology. Fix an algebraic closure $\bar{C}$ of the fraction field C of S and let $G_C = \mathrm{Gal}(\bar{C}/C)$ be the absolute Galois group of C. Let $\mathcal{S}_C$ be the category of finite G_C-sets. To define the notion of "ramification bounded by m" and the Hom sets in the category $\mathcal{FFP}_A^{<m}$, we define a contravariant functor $F_{\boldsymbol{\alpha}}^m$ from the category $\mathcal{FF}_A$ of finite flat algebras over A to $\mathcal{S}_C$. We say that an S-algebra $\mathbb{B}$ is *formally of finite type* over S if it is semi-local, $\mathfrak{m}_{\mathbb{B}}$-adically complete, Noetherian and the quotient $\mathbb{B}/\mathfrak{m}_{\mathbb{B}}$ is finite over k. Let B be a finite flat A-algebra and regard it as an S-algebra via α. We call a commutative diagram

$$\begin{tikzcd} \mathbb{B} \arrow[r, two heads, "\varepsilon"] & B \\ S \arrow[u] \arrow[r, "\alpha"] & A \arrow[u] \end{tikzcd}$$

an *embedding* of B over $\boldsymbol{\alpha}$ if $\mathbb{B}$ is an S-algebra which is formally of finite type and formally smooth over S and ε is a surjective homomorphism which induces an isomorphism $\mathbb{B}/\mathfrak{m}_{\mathbb{B}} \to B/\mathfrak{m}_B$. We may often call it briefly an $\boldsymbol{\alpha}$*-embedding* $\varepsilon : \mathbb{B} \to B$.

Let $\mathcal{E}_{\boldsymbol{\alpha}}(B)$ denote the category of which the objects are the $\boldsymbol{\alpha}$-embeddings of B

and the morphisms are the commutative diagrams

$$\begin{tikzcd} \mathbb{B} \arrow[rr, "\boldsymbol{f}"] \arrow[dr, "\varepsilon"'] & & \mathbb{B}' \arrow[dl, "\varepsilon'"] \\ & B & \end{tikzcd},$$

where $\varepsilon : \mathbb{B} \to B$ and $\varepsilon' : \mathbb{B}' \to B$ are objects of $\mathcal{E}_{\boldsymbol{\alpha}}(B)$ and $\boldsymbol{f} : \mathbb{B} \to \mathbb{B}'$ is an S-algebra homomorphism. The category $\mathcal{E}_{\boldsymbol{\alpha}}(B)$ is non-empty ([10], Lem. 2.1).

Let $\varepsilon : \mathbb{B} \to B$ be an object of $\mathcal{E}_{\boldsymbol{\alpha}}(B)$ and I_ε its kernel. For any positive integer $m < a$, we define a C-algebra $\mathbb{B}^m_{\varepsilon,C}$ by $\mathbb{B}^m_{\varepsilon,C} = \mathbb{B}[I_\varepsilon/x^m]^\wedge \otimes_S C$, where $\wedge$ means the x-adic completion. Define[2]

$$\pi_0(\mathbb{B}^m_{\varepsilon,\bar{C}}) := \varprojlim_{C'} \pi_0(\mathbb{B}^m_{\varepsilon,C} \otimes_C C')$$

where C' runs through the finite separable extensions of C contained in $\bar{C}$. The sets $\pi_0(\mathbb{B}^m_{\varepsilon,\bar{C}})$ form a projective system when the embeddings ε vary, which is in fact constant (*op. cit.*, Cor. 3.6). Put

$$F^m_{\boldsymbol{\alpha}}(B) := \varprojlim_{\varepsilon \in \mathcal{E}_{\boldsymbol{\alpha}}(B)} \pi_0(\mathbb{B}^m_{\varepsilon,\bar{C}}).$$

Then $F^m_{\boldsymbol{\alpha}}(B)$ is a finite G_C-set. This correspondence $B \mapsto F^m_{\boldsymbol{\alpha}}(B)$ is functorial, and thus we obtain a contravariant functor $F^m_{\boldsymbol{\alpha}} : \mathcal{FF}_A \to \mathcal{S}_C$.

We can show that the functor $F^m_{\boldsymbol{\alpha}}$ does not depend on the polarization $\boldsymbol{\alpha}$ in the following sense: Let $\mathcal{P}(A)$ be the category of polarizations of A, in which a *morphism* $\gamma : (S \overset{\alpha}{\to} A, (x)) \to (T \overset{\beta}{\to} A, (y))$ is a commutative diagram

$$\begin{tikzcd} S \arrow[rr, "\gamma"] \arrow[dr, "\alpha"'] & & T \arrow[dl, "\beta"] \\ & A & \end{tikzcd}$$

such that $\gamma : S \to T$ is a surjective ring homomorphism and $\gamma((x)) = (y)$. Suppose there is a morphism γ of polarizations from $\boldsymbol{\alpha} = (S \overset{\alpha}{\to} A, (x))$ to $\boldsymbol{\beta} = (T \overset{\beta}{\to} A, (y))$. Let C and K be the fraction fields of S and T, respectively. Put $\mathfrak{p} := \mathrm{Ker}\,(\gamma : S \to T)$, and let $D_{\mathfrak{p}}$ be a decomposition group for $\mathfrak{p}$; thus it is the subgroup of G_C consisting of elements σ such that $\sigma(\overline{\mathfrak{p}}) = \overline{\mathfrak{p}}$ for a choice of a prime ideal $\overline{\mathfrak{p}}$ lying above $\mathfrak{p}$ in the integral closure of S in C. Let $\mathcal{S}_{C,\mathfrak{p}}$ denote the category of finite $D_{\mathfrak{p}}$-sets. There are

[2]In the case where S is a cdvr $\mathcal{O}_K$ with fraction field $C = K$, the set $\pi_0(\mathbb{B}^m_{\varepsilon,\overline{K}})$ is in fact identical with the set $\pi_0(X^m_{\overline{K}})$ of geometric connected components of the affinoid variety $X^m = \mathrm{Sp}(\mathbb{B}^m_{\varepsilon,K})$ considered in [3], [7] and [9] by Proposition 9.1.8 of [4]. Thus the above definition of $F^m_{\boldsymbol{\alpha}}$ coincides with that of F^m in [9].

natural functors $d_{\mathfrak{p}} : \mathcal{S}_C \to \mathcal{S}_{C,\mathfrak{p}}$ and $r_{\mathfrak{p}} : \mathcal{S}_K \to \mathcal{S}_{C,\mathfrak{p}}$, corresponding respectively to the group homomorphisms $D_{\mathfrak{p}} \hookrightarrow G_C$ and $D_{\mathfrak{p}} \twoheadrightarrow G_K$. Recall that we have defined two functors $F^m_{\boldsymbol{\alpha}} : \mathcal{FF}_A \to \mathcal{S}_C$ and $F^m_{\boldsymbol{\beta}} : \mathcal{FF}_A \to \mathcal{S}_K$. These functors all together form the diagram

$$\begin{tikzcd} & \mathcal{S}_C \arrow[dr, "d_{\mathfrak{p}}"] & \\ \mathcal{FF}_A \arrow[ur, "F^m_{\boldsymbol{\alpha}}"] \arrow[dr, "F^m_{\boldsymbol{\beta}}"'] & & \mathcal{S}_{C,\mathfrak{p}} \\ & \mathcal{S}_K \arrow[ur, "r_{\mathfrak{p}}"'] & \end{tikzcd} \tag{3.1}$$

One of the main results in [10] is the following Proposition:

Proposition 3.2 (*op. cit.*, Prop. 3.1). *The diagram (3.1) is commutative.*

More precisely, this means that there exists a natural isomorphism $r_{\mathfrak{p}} F^m_{\boldsymbol{\beta}} \to d_{\mathfrak{p}} F^m_{\boldsymbol{\alpha}}$ of functors. Thus, intuitively speaking, the two functors $F^m_{\boldsymbol{\alpha}}$ and $F^m_{\boldsymbol{\beta}}$ may be identified by means of reduction modulo $\mathfrak{p}$ (or, extension of scalars by $\gamma : S \to T$). If we forget about the Galois action and regard $F^m_{\boldsymbol{\alpha}}$ simply as a functor to the category $\mathcal{S}$ of finite sets, then there exists a natural isomorphism $F^m_{\boldsymbol{\alpha}} \to F^m_{\boldsymbol{\beta}}$ of functors.

Then we define a functor $F^m_A : \mathcal{FF}_A \to \mathcal{S}$ by

$$F^m_A(B) := \varinjlim_{\boldsymbol{\alpha} \in \mathcal{P}(A)} F^m_{\boldsymbol{\alpha}}(B).$$

We can prove the lemma below ([9]).

Lemma 3.3. *For any finite flat principal A-algebra B and $0 < m \leqslant a$, we have $\# F^m_A(B) \leqslant \operatorname{rank}_A(B)$.*

Proof. It is enough to show the inequality $\# F^m_{\boldsymbol{\alpha}}(B) \leqslant \operatorname{rank}_A(B)$ for some polarization $\boldsymbol{\alpha} = (S \overset{\alpha}{\to} A, (x))$. By Proposition 3.2, we may assume that S is a cdvr. Let T be a finite flat S-algebra such that $T/\mathfrak{m}_S^a T \simeq B$; then we have $\operatorname{rank}_A(B) = \operatorname{rank}_S(T)$ (Prop. 2.1). By Hattori's lemma below, we have $F^m_{\boldsymbol{\alpha}}(B) \simeq F^m_A(B) = F^m_S(T)$. Since $F^m_S(T)$ is naturally identified with a quotient of $F_S(L) := \operatorname{Hom}_S(T, \bar{C}) \simeq \pi_0(T \otimes_S \bar{C})$, we have $\sharp F^m_S(T) \leqslant \operatorname{rank}_S(T)$, and hence $\# F^m_{\boldsymbol{\alpha}}(B) \leqslant \operatorname{rank}_A(B)$. $\square$

Lemma 3.4 ([7], Lem. 1). *Let $S \to A$ be a surjective local ring homomorphism, where S is a cdvr with residue field k. For any finite flat S-algebra T, put $B = T/\mathfrak{m}_S^a T$. Then we have $F^m_A(B) = F^m_S(T)$ as an object of $\mathcal{S}$ the category of finite sets for any rational number $0 < m \leqslant a$. In particular, the ramification of B is bounded by m if and only if so is T.*

Definition 3.5. Let B be a finite flat principal A-algebra. We say that the *ramification of B is bounded by m* if $\#F_A^m(B) = \mathrm{rank}_A(B)$.

Definition 3.6. For each rational number m with $0 < m \leqslant a$, define $\mathcal{FFP}_A^{<m}$ to be the category whose objects are finite flat principal A-algebras with ramification bounded by m in which we define $\mathrm{Hom}_{\mathcal{FFP}_A^{<m}}(B, B')$ to be the quotient set of $\mathrm{Hom}_A(B, B')$ by the equivalence relation $\overset{m}{\approx}$ defined as follows: For $f, g \in \mathrm{Hom}_A(B, B')$,

$$f \overset{m}{\approx} g \overset{\text{def.}}{\Longleftrightarrow} F_A^m(f) = F_A^m(g) \quad \text{in } \mathrm{Hom}_{\mathcal{S}}(F_A^m(B'), F_A^m(B)),$$

where $\mathcal{S}$ is the category of finite sets.

Thus the set $\mathrm{Hom}_{\mathcal{FFP}_A^{<m}}(B, B')$ may be identified with the image of
$\mathrm{Hom}_A(B, B') \to \mathrm{Hom}_{\mathcal{S}}(F_A^m(B'), F_A^m(B))$.

Definition 3.7. Define[3] $\mathcal{FE}_K^{<m} := \mathcal{FFP}_{\mathcal{O}_K}^{<m}$.

Choose a polarization $\boldsymbol{\alpha} = (\mathcal{O}_K \overset{\alpha}{\to} A, (\pi))$ of A. Then the truncation functor

$$T : \mathcal{FE}_K^{<m} \to \mathcal{FFP}_A^{<m}$$

is defined by $T(\mathcal{O}) := \mathcal{O}/\mathfrak{m}_K^a\mathcal{O}$. The image is in $\mathcal{FFP}_A^{<m}$ by Hattori's lemma.

Finally, we prove Theorem 1.1. The essential surjectivity of T follows from (ii) of Proposition 2.1 and Lemma 3.4, since any object of $\mathcal{FFP}_A^{<m}$ is a direct product of finite extensions of A. To prove the full-faithfulness, let $\mathcal{O}$ and $\mathcal{O}'$ be two objects in $\mathcal{FFP}_{\mathcal{O}_K}^{<m}$, and let $B = T(\mathcal{O})$ and $B' = T(\mathcal{O}')$. Since the functor $F_{\mathcal{O}_K}^m$ gives an anti-equivalence of the Galois category $\mathcal{FE}_K^{<m}$ with a full-subcategory of $\mathcal{S}_K$, we have

$$\mathrm{Hom}_{\mathcal{FFP}_{\mathcal{O}_K}^{<m}}(\mathcal{O}, \mathcal{O}') \ \simeq \ \mathrm{Hom}_{\mathcal{S}_K}(F_{\mathcal{O}_K}^m(\mathcal{O}'), F_{\mathcal{O}_K}^m(\mathcal{O})).$$

By Lemma 3.4, we have

$$\mathrm{Hom}_{\mathcal{S}_K}(F_{\mathcal{O}_K}^m(\mathcal{O}'), F_{\mathcal{O}_K}^m(\mathcal{O})) \ = \ \mathrm{Hom}_{\mathcal{S}_K}(F_A^m(B'), F_A^m(B)).$$

It follows from our definition of Hom in $\mathcal{FFP}_{\mathcal{O}_K}^{<m}$ and $\mathcal{FFP}_A^{<m}$ that

$$\mathrm{Hom}_{\mathcal{FFP}_{\mathcal{O}_K}^{<m}}(\mathcal{O}, \mathcal{O}') \ = \ \mathrm{Hom}_{\mathcal{FFP}_A^{<m}}(B, B').$$

This completes the proof of the Theorem.

[3]In [9], the category $\mathcal{FE}_K^{<m}$ is defined to be finite étale K-algebras with ramification bounded by m (*op. cit.*, Def. 3.1). The correspondence $L \mapsto \mathcal{O}_L$ gives an equivalence of categories from this to $\mathcal{FFP}_{\mathcal{O}_K}^{<m}$. Thus the two definitions coincide.

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