RIMS Kôkyûroku Bessatsu
B19 (2010), 117–134

Algebraic interpretation of Kawashima relation for multiple zeta values

By

Tatsushi TANAKA*

Abstract

This is a survey article which is concerned with a recent remarkable work of Gaku Kawashima on a class of relations among multiple zeta values, and its applications to the quasi-derivation relation conjectured by Masanobu Kaneko and proven by the author and to the cyclic sum formula first proven by Michael Hoffman and Yasuo Ohno.

§ 1. Introduction

For each index $(k_1, \ldots, k_l)$ with $l \geq 1, k_1 > 1, k_2, \ldots, k_l \geq 1$, the multiple zeta value (MZV for short) is a real number defined by the convergent series

$$\zeta(k_1, k_2, \ldots, k_l) = \sum_{m_1 > m_2 > \cdots > m_l > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_l^{k_l}},$$

and the multiple zeta-star value (MZSV for short) is defined by the convergent series

$$\zeta^{\star}(k_1, k_2, \ldots, k_l) = \sum_{m_1 \geq m_2 \geq \cdots \geq m_l > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_l}}.$$

We call the number $k_1 + \cdots + k_l$ weight and l depth. When $l = 1$, MZV and MZSV coincide and are known as a special value of the Riemann zeta function. The properties of MZV's and MZSV's as 'values' such as irrationality, transcendency and linearly independency among them are rarely studied. (A few results appear in [1, 23, 28, ...].)

Received April 1, 2009.
2000 Mathematics Subject Classification(s): 11M41
Key Words: multiple zeta values, multiple zeta-star values, Kawashima relation, the derivation relation, Ohno relation, the quasi-derivation relation, the cyclic sum formula
*Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, Japan.
e-mail: `t.tanaka@math.kyushu-u.ac.jp`

In these days, MZV's and MZSV's appear in several aspects of mathematics and physics. One of the mathematical interests for MZV's and MZSV's is to find and prove explicit relations among them. It is well-known that each of MZSV's can be expressed as a $\mathbb{Q}$-linear combination of MZV's, and vice versa. For example,

$$\zeta^\star(2,1)=\sum_{m_1\geq m_2>0}\frac{1}{m_1^2m_2}=\sum_{m_1>m_2>0}\frac{1}{m_1^2m_2}+\sum_{m_1=m_2>0}\frac{1}{m_1^2m_2}=\zeta(2,1)+\zeta(3),$$

and so on. That means the $\mathbb{Q}$-vector space generated by MZSV's coincides with the $\mathbb{Q}$-vector space generated by MZV's. Zagier gives in [27] the following conjecture on the dimension of the space.

Dimension conjecture. *Let $\{d_k\}$ be a sequence given by $d_0=1, d_1=0, d_2=1, d_k=d_{k-2}+d_{k-3}$ $(k\geq 3)$. Then we have*

$$\dim_{\mathbb{Q}}\sum_{\substack{k_1+\cdots+k_l=k\\ l\geq 1,k_1>1,k_2,\dots,k_l\geq 1}}\mathbb{Q}\zeta(k_1,\dots,k_l)=d_k.$$

Goncharov and Terasoma showed independently in [8, 26] that the sequence d_k gives the upper bound of the dimension. The number of indices of weight k is 2^{k-2}, and hence there are at least $2^{k-2}-d_k$ linearly independent relations among MZV's.

Can we describe all $\mathbb{Q}$-linear relations among MZV's explicitly? A few classes of relations which are expected to give all $\mathbb{Q}$-linear relations among MZV's have been found so far. Ihara-Kaneko-Zagier [12] and other writers (Goncharov, Minh, Petitot, Boutet de Monvel, Écalle, Racinet, ...) proved the so-called regularized double shuffle relation, which is known to give $2^{k-2}-d_k$ linearly independent relations for $k\leq 20$ in [16]. Drinfel'd associator introduced in [3, 4, 6, ...] is a kind of generating functions of MZV's and satisfies the group-like property and 2-, 3- and 5-cycle relation. Relations of Drinfel'd associator can be translated into relations among MZV's and are also expected to give $2^{k-2}-d_k$ linearly independent relations. The following Kawashima relation also conjecturally gives all relations among MZV's.

From now on, we describe relations among MZV's by using the algebraic setup introduced by Hoffman [9]. Let $\mathfrak{H}=\mathbb{Q}\langle x,y\rangle$ denote the non-commutative polynomial algebra over $\mathbb{Q}$ in two indeterminates x and y, and let $\mathfrak{H}^1$ and $\mathfrak{H}^0$ denote the subalgebras $\mathbb{Q}+\mathfrak{H}y$ and $\mathbb{Q}+x\mathfrak{H}y$, respectively. We define the $\mathbb{Q}$-linear map $Z\colon \mathfrak{H}^0\to\mathbb{R}$ by $Z(1)=1$ and

$$Z(x^{k_1-1}yx^{k_2-1}y\cdots x^{k_n-1}y)=\zeta(k_1,k_2,\dots,k_n).$$

We also define the $\mathbb{Q}$-linear map $\overline{Z}\colon \mathfrak{H}^0\to\mathbb{R}$ by $\overline{Z}(1)=1$ and

$$\overline{Z}(x^{k_1-1}yx^{k_2-1}y\cdots x^{k_n-1}y)=\zeta^\star(k_1,k_2,\dots,k_n).$$

The degree (resp. degree with respect to y) of a word is the weight (resp. the depth) of the corresponding MZV or MZSV.

Let $z_k = x^{k-1}y$ for $k \geq 1$. The harmonic product $* : \mathfrak{H}^1 \times \mathfrak{H}^1 \to \mathfrak{H}^1$ is a $\mathbb{Q}$-bilinear map defined by the following rules:

i) For any $w \in \mathfrak{H}^1$, $1 * w = w * 1 = w$,

ii) For any $w, w' \in \mathfrak{H}^1$ and any $k, l \geq 1$,

$$z_k w * z_l w' = z_k(w * z_l w') + z_l(z_k w * w') + z_{k+l}(w * w').$$

This is, as shown in [9], an associative and commutative product on $\mathfrak{H}^1$. Another harmonic product $\overline{*} : \mathfrak{H}^1 \times \mathfrak{H}^1 \to \mathfrak{H}^1$ is a $\mathbb{Q}$-bilinear map defined by

i) For any $w \in \mathfrak{H}^1$, $1 \,\overline{*}\, w = w \,\overline{*}\, 1 = w$,

ii) For any $w, w' \in \mathfrak{H}^1$ and any $k, l \geq 1$,

$$z_k w \,\overline{*}\, z_l w' = z_k(w \,\overline{*}\, z_l w') + z_l(z_k w \,\overline{*}\, w') - z_{k+l}(w \,\overline{*}\, w').$$

The product $\overline{*}$ is also known to be associative and commutative. We denote

$$z_k w \,\dot{*}\, z_l w' = z_{k+l}(w * w'),\ z_k w \,\dot{\overline{*}}\, z_l w' = z_{k+l}(w \,\overline{*}\, w')$$

for any $k, l \geq 1$ and $w, w' \in \mathfrak{H}^1$.

Let φ, α be the automorphisms on $\mathfrak{H}$ characterized by $\varphi(x) = x + y, \varphi(y) = -y$ and $\alpha(x) = y, \alpha(y) = x$, and $\tilde{\alpha}$ the $\mathbb{Q}$-linear map on $\mathfrak{H}y$ satisfying $\tilde{\alpha}(wy) = \alpha(w)y$ $(w \in \mathfrak{H})$. Then Kawashima relation is stated as follows.

Theorem 1.1 (Kawashima relation [14]). *For any $m \geq 1$ and any $w, w' \in \mathfrak{H}y$, we have*

$$i)\ \sum_{\substack{p+q=m\\ p,q\geq 1}} Z(\varphi(w) \,\dot{*}\, y^p) Z(\varphi(w') \,\dot{*}\, y^q) = Z(\varphi(w * w') \,\dot{*}\, y^m),$$

$$ii)\ \sum_{\substack{p+q=m\\ p,q\geq 1}} \overline{Z}(\tilde{\alpha}(w) \,\dot{\overline{*}}\, y^p) \overline{Z}(\tilde{\alpha}(w') \,\dot{\overline{*}}\, y^q) = -\overline{Z}(\tilde{\alpha}(w \,\overline{*}\, w') \,\dot{\overline{*}}\, y^m).$$

He showed these formulas by studying an analytic property of certain Newton series, which is summarized in §2. Expanding the products of MZV's in the left-hand side of Kawashima relation by the iterated integral shuffle product (for the definition of the shuffle product, see [22, 12] for example), we find that Kawashima relation gives $2^{k-2} - d_k$ linearly independent relations for $k \leq 12$. (The calculation is due to Risa/asir, an open source general computer algebra system. Also see [24].)

For any $w \in \mathfrak{H}$, define the $\mathbb{Q}$-linear operator L_w on $\mathfrak{H}$ by $L_w(w') = ww'$ $(w' \in \mathfrak{H})$. We notice that $w \,\dot{*}\, y = xw = L_x(w)$ $(w \in \mathfrak{H}^1)$. We write

$$\mathfrak{H}y * \mathfrak{H}y = \{w * w' | w, w' \in \mathfrak{H}y\},$$

$$\mathfrak{H}y \,\overline{*}\, \mathfrak{H}y = \{w \,\overline{*}\, w' | w, w' \in \mathfrak{H}y\}.$$

Then, when $m = 1$, Theorem 1.1 reduces to

Corollary 1.2 (the linear part of Kawashima relation). *We have*

$$i)\ L_x\varphi(\mathfrak{H}y * \mathfrak{H}y) \subset \ker Z,$$
$$ii)\ L_x\tilde{\alpha}(\mathfrak{H}y \,\overline{*}\, \mathfrak{H}y) \subset \ker \overline{Z}.$$

Indeed, the statements i) and ii) in Corollary 1.2 are equivalent. The linear part of Kawashima relation looks so simple but contains many linearly independent relations and has some nice properties. Let τ be the anti-automorphism on $\mathfrak{H}$ such that $\tau(x) = y, \tau(y) = x$. Let σ_j $(j \geq 0)$ be the $\mathbb{Q}$-linear map on $\mathfrak{H}y$ such that

$$\sigma_j(z_{k_1} \cdots z_{k_l}) = \sum_{\substack{\varepsilon_1+\cdots+\varepsilon_l=j \\ \varepsilon_1,\ldots,\varepsilon_l \geq 0}} z_{k_1+\varepsilon_1} \cdots z_{k_l+\varepsilon_l}.$$

We note that $\sigma_0 = id$. Kawashima showed that Ohno relation ([19]), which reduces to the duality formula if $j = 0$, is contained in the linear part of Kawashima relation:

Theorem 1.3 ([14]). *For any $j \geq 0$, we have $\sigma_j(1-\tau)(\mathfrak{H}^0) \subset L_x\varphi(\mathfrak{H}y * \mathfrak{H}y)$.*

In [24], it is proven that the quasi-derivation relation, which is conjectured in Kaneko [15], is also contained in the linear part of Kawashima relation. In [25], it is proven that the cyclic sum formula, which is shown in Hoffman-Ohno [11] or Ohno-Wakabayashi [20], is also contained in the linear part of Kawashima relation. They are described in §3 and §4.

§ 2. Kawashima's work

We recall Kawashima's theory in the present section. Proofs can be seen in [14]. His first investigation is on some analytic properties of certain interpolation series called *Newton series.* Firstly, some fundamental properties of Newton series are explained in §2.1 (also see [7, 13, 14]). In §2.2, we discuss on the inversion sequence of certain multiple harmonic sums. In §2.3, we consider special Newton series made by interpolating the multiple harmonic sums. Such Newton series satisfy a functional equation. Then we describe that Taylor coefficients of the functional equation give us relations among MZV's, which we call Kawashima relation.

§ 2.1. Newton series

Let $a : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ be a complex-valued sequence. The Newton series for the sequence a is defined by

$$f_a(z) := \sum_{n=0}^{\infty} (-1)^n (\nabla a)(n) \binom{z}{n},$$

where $\binom{z}{n} = \frac{z(z-1)\cdots(z-n+1)}{n!}$, z is a complex variable, and ∇a denotes the inversion sequence of a:

$$(\nabla a)(m) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} a(i).$$

We note that the operator ∇ is an involution, and hence we find that $f_a(m) = a(m)$ holds for any $m \in \mathbb{Z}_{\geq 0}$. In this sense, we often denote $f_a(z)$ by $a(z)$.

The following properties are fundamental in the theory of the Newton series (see [7, 13, 14] for their proofs).

Proposition 2.1. *Let $a : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ be a sequence and $z \in \mathbb{C} \backslash \mathbb{Z}_{\geq 0}$. Then the Newton series*

$$\sum_{n=0}^{\infty} (-1)^n a(n) \binom{z}{n}$$

and the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{z+1}}$$

possess one and the same abscissa of convergence and absolute convergence.

Corollary 2.2. *Let $a : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ be a sequence and $\varepsilon \in \mathbb{R}$. If $a(n) = O(n^{\varepsilon})$ as $n \to \infty$, then the Newton series*

$$\sum_{n=0}^{\infty} (-1)^n a(n) \binom{z}{n}$$

converges absolutely for any $z \in \mathbb{C}$ with $\mathrm{Re}(z) > \varepsilon$.

Proposition 2.3. *Let $a : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ be a sequence and let the Newton series*

$$f(z) = \sum_{n=0}^{\infty} (-1)^n a(n) \binom{z}{n}$$

have the abscissa of convergence ξ. If there exists $N \in \mathbb{Z}_{\geq 0}$ such that $f(n) = 0$ for any integer $n \geq N$, then we have $f(z) = 0$ for any $\mathrm{Re}(z) > \xi$.

For any sequence $a : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ and any $n \in \mathbb{Z}_{\geq 0}$, we denote the difference of a by Δa. The m times composition of Δ is given by

$$(\Delta^m a)(n) = \sum_{i=n}^{m+n} (-1)^{i-n} \binom{m}{i-n} a(i)$$

for any $n \in \mathbb{Z}_{\geq 0}$. The inversion sequence ∇a is also written as $(\nabla a)(n) = (\Delta^n a)(0)$ for any $n \in \mathbb{Z}_{\geq 0}$.

Lemma 2.4. *Let $a : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ be a sequence. Let the abscissa of convergence of Newton series*

$$f(z) = \sum_{n=0}^{\infty} (-1)^n a(n) \binom{z}{n}$$

by ξ. Let $l \in \mathbb{Z}_{\geq 0}$. Then we have

$$(-1)^l \binom{z}{l} f(z) = \sum_{n=l}^{\infty} (-1)^n \binom{n}{l} (\Delta^l a)(n-l) \binom{z}{n}$$

for any $\mathrm{Re}(z) > \xi + l$.

Kawashima showed that Corollary 2.2 and Lemma 2.4 gives us the following proposition.

Proposition 2.5. *Let $a, b : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ be sequences. Suppose the Newton series*

$$f(z) = \sum_{n=0}^{\infty} (-1)^n a(n) \binom{z}{n}, \ g(z) = \sum_{n=0}^{\infty} (-1)^n b(n) \binom{z}{n}$$

have the abscissas of convergence ξ_a and ξ_b, respectively. Let $\varepsilon > 0$. We assume that the sequences $a, b : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ satisfy the following conditions.

i) The values $a(m), (\Delta^l b)(m)$ are non-negative for any $m, l \in \mathbb{Z}_{\geq 0}$,

ii) $\xi_a < 0$,

iii) For any $l \in \mathbb{Z}_{\geq 0}$, we have $(\Delta^l b)(m) = O(m^{-l-\varepsilon})$ as $m \to \infty$.

Then, the product $f(z)g(z)$ is expressed as a Newton series in the half-plane $\mathrm{Re}(z) > \max\{\xi_a, -\varepsilon\}$.

§ 2.2. Inversion sequence of multiple harmonic sums

For each word $w = z_{k_1} \cdots z_{k_l} \in \mathfrak{H}^1$, we define four kinds of multiple harmonic sums

$$s_w(m) = \sum_{m+1=m_1 \geq m_2 \geq \cdots \geq m_l > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_l^{k_l}},$$

$$a_w(m) = \sum_{m+1=m_1 > m_2 > \cdots > m_l > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_l^{k_l}},$$

$$S_w(m) = \sum_{m+1>m_1 \geq m_2 \geq \cdots \geq m_l > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_l^{k_l}},$$

$$A_w(m) = \sum_{m+1>m_1 > m_2 > \cdots > m_l > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_l^{k_l}}.$$

We consider that the subscripts of every harmonic sums are extended to any elements in $\mathfrak{H}^1$, that means the subscripts are defined $\mathbb{Q}$-linearly:

$$M_{pw+qw'}(m) = pM_w(m) + qM_{w'}(m)$$

for any $m \geq 0$ and any $p, q \in \mathbb{Q}$ and M which stands for any of s, a, S, A. By definition, we find that

$$\sum_{m=0}^{\infty} a_w(m) = \lim_{m\to\infty} A_w(m) = Z(w), \quad \sum_{m=0}^{\infty} s_w(m) = \lim_{m\to\infty} S_w(m) = \overline{Z}(w)$$

for $w \in \mathfrak{H}^0$. For any $m \geq 0$, each of multiple harmonic sums satisfies the corresponding harmonic product rule:

$$s_w(m)s_{w'}(m) = s_{w \dot{\overline{*}} w'}(m), \tag{2.1}$$

$$a_w(m)a_{w'}(m) = a_{w \dot{*} w'}(m), \tag{2.2}$$

$$S_w(m)S_{w'}(m) = S_{w \overline{*} w'}(m), \tag{2.3}$$

$$A_w(m)A_{w'}(m) = A_{w * w'}(m). \tag{2.4}$$

We also notice that the following property of Newton series holds. See [14] for the proof.

Proposition 2.6. *Let $a : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ be a sequence. We suppose the abscissa of convergence of Newton series*

$$f(z) = \sum_{n=0}^{\infty} (-1)^n a(n) \binom{z}{n}$$

is negative. Then we have

$$f(z) = a(0) + \sum_{m=1}^{\infty} (-1)^m \left\{ \sum_{n=0}^{\infty} a(n) a_{y^m}(n-1) \right\} z^m$$

in some neighborhood of 0.

Let γ be the automorphism on $\mathfrak{H}$ characterized by

$$\gamma(x) = x, \ \gamma(y) = x + y.$$

For any $w \in \mathfrak{H}$, we define the operator R_w on $\mathfrak{H}$ by $R_w(w') = w'w$ for any $w' \in \mathfrak{H}$. Let d be the $\mathbb{Q}$-linear map on $\mathfrak{H}^1$ given by $d(1) = 1$ and

$$d = R_y \gamma R_y^{-1}$$

on $\mathfrak{H}y$. The maps $Z, \overline{Z}$ defined in §1 satisfy $\overline{Z} = Zd$, which is known as a simple $\mathbb{Q}$-linear transformation between MZV's and MZSV's. Similarly, we find that

$$s_w(m) = a_{d(w)}(m),\ S_w(m) = A_{d(w)}(m) \tag{2.5}$$

hold for any $w \in \mathfrak{H}^1$ and any $m \geq 0$. It is also known that the identities

$$d(w \mathbin{\overline{*}} w') = d(w) * d(w'), \tag{2.6}$$

$$d(w \mathbin{\dot{\overline{*}}} w') = d(w) \mathbin{\dot{*}} d(w'), \tag{2.7}$$

hold for any $w, w' \in \mathfrak{H}^1$ (see [14, 17] for example). We note that the harmonic product rule (2.1) (resp. (2.3)) can be proven by the harmonic product rule (2.2) (resp. (2.4)) because of the identities (2.7) (resp. (2.6)) and (2.5), and vice versa.

Let m be a positive integer. For $n, k \geq 0$ and words $\mu = z_{\mu_1} \cdots z_{\mu_c}, \nu = z_{\nu_1} \cdots z_{\nu_d} \in \mathfrak{H}^1$ with $\mu_1 + \cdots + \mu_c = \nu_1 + \cdots + \nu_d = m$, we define

$$s_{\mu,\nu}(n,k) = \binom{n+k}{n}^{-1} \sum_{\substack{n=n_1 \geq \cdots \geq n_c \geq 0 \\ k=k_1 \geq \cdots \geq k_d \geq 0}} \frac{1}{(n_{i_1} + k_{j_1} + 1) \cdots (n_{i_m} + k_{j_m} + 1)},$$

where

$$(i_1, \ldots, i_m) = (\underbrace{1, \ldots, 1}_{\mu_1}, \ldots, \underbrace{c, \ldots, c}_{\mu_c}), (j_1, \ldots, j_m) = (\underbrace{1, \ldots, 1}_{\nu_1}, \ldots, \underbrace{d, \ldots, d}_{\nu_d}).$$

We consider that the subscripts of $s_{\mu,\nu}(n,k)$ are extended to any elements in $\mathfrak{H}^1$, that means the subscripts are defined $\mathbb{Q}$-bilinearly:

$$s_{p\mu+q\mu', u\nu+v\nu'}(n,k) = pus_{\mu,\nu}(n,k) + pvs_{\mu,\nu'}(n,k) + qus_{\mu',\nu}(n,k) + qvs_{\mu',\nu'}(n,k)$$

for any $n, k \geq 0$ and any $p, q, u, v \in \mathbb{Q}$. We note that $s_{\mu,\nu}(n,0) = s_\mu(n), s_{\mu,\nu}(0,k) = s_\nu(k)$.

We have the k times difference of s_w explicitly as follows.

Theorem 2.7 ([14]). *For any $w \in \mathfrak{H}^1$ and any $n, k \geq 0$, we have*

$$(\Delta^k s_w)(n) = s_{w,\tilde{\alpha}(w)}(n,k).$$

Since $(\nabla a)(n) = (\Delta^n a)(0)$ $(n \geq 0)$, we have the inversion sequence of s_w:

Corollary 2.8. *For any $w \in \mathfrak{H}^1$, we have $\nabla s_w = s_{\tilde{\alpha}(w)}$.*

Remark 1. The inversion formula of s_w described in Corollary 2.8 was proven also in Hoffman [10]. Here we present another different proof of the formula, which is due to the polylogarithm function as a generating function of s_w. For $w = z_{k_1} \cdots z_{k_l}$ with $k_1, \ldots, k_l \geq 1$ and a complex variable z with $|z| < 1$, the multiple polylogarithm functions $Li_w(z), \overline{Li}_w(z)$ are defined by

$$Li_w(z) = \sum_{m_1 > \cdots > m_n > 0} \frac{z^{m_1}}{m_1^{k_1} \cdots m_n^{k_n}}, \ \overline{Li}_w(z) = \sum_{m_1 \geq \cdots \geq m_n > 0} \frac{z^{m_1}}{m_1^{k_1} \cdots m_n^{k_n}}.$$

We consider that the subscripts of $Li_w(z)$ and $\overline{Li}_w(z)$ are extended to any elements in $\mathfrak{H}^1$, that means the subscripts are defined $\mathbb{Q}$-linearly:

$$Li_{pw+qw'}(z) = pLi_w(z) + qLi_{w'}(z), \ \overline{Li}_{pw+qw'}(z) = p\overline{Li}_w(z) + q\overline{Li}_{w'}(z)$$

for any $n, k \geq 0$ and any $p, q \in \mathbb{Q}$. We note that $\overline{Li}_w(z) = Li_{d(w)}(z)$. As a matter of fact, the inversion formula of s_w comes from Landen connection formula of multiple polylogarithm, which is described as follows (also see [18, 21]).

Differentiating $Li_{z_{k_1} \cdots z_{k_l}}(z)$ and $Li_{z_{k_1} \cdots z_{k_l}}(\frac{z}{z-1})$ by z, we obtain

$$\frac{d}{dz} Li_{z_{k_1} \cdots z_{k_l}}(z) = \begin{cases} \dfrac{1}{z} Li_{z_{k_1-1} z_{k_2} \cdots z_{k_l}}(z) & k_1 > 1, \\ \dfrac{1}{1-z} Li_{z_{k_2} \cdots z_{k_l}}(z) & k_1 = 1, l > 1, \\ \dfrac{1}{1-z} & k_1 = l = 1, \end{cases}$$

$$\frac{d}{dz} Li_{z_{k_1} \cdots z_{k_l}}\left(\frac{z}{z-1}\right) = \begin{cases} \left(\dfrac{1}{z} + \dfrac{1}{1-z}\right) Li_{z_{k_1-1} z_{k_2} \cdots z_{k_l}}\left(\dfrac{z}{z-1}\right) & k_1 > 1, \\ -\dfrac{1}{1-z} Li_{z_{k_2} \cdots z_{k_l}}\left(\dfrac{z}{z-1}\right) & k_1 = 1, l > 1, \\ -\dfrac{1}{1-z} & k_1 = l = 1. \end{cases}$$

According to these formulas together with the identity proven by Euler [5],

$$\sum_{i=1}^{n} (-1)^i \binom{n}{i} \frac{1}{i} = -\sum_{i=1}^{n} \frac{1}{i},$$

we have the Landen connection formula

$$Li_w(z) = Li_{\varphi(w)}\left(\frac{z}{z-1}\right),$$

where φ stands for the automorphism on $\mathfrak{H}$ given by $\varphi(x) = x + y, \varphi(y) = -y$. We find the identity

$$\varphi d = -d\tilde{\alpha}, \tag{2.8}$$

and hence we easily calculate
(2.9)
$$\overline{Li}_w(z) = Li_{d(w)}(z) = Li_{\varphi d(w)}\left(\frac{z}{z-1}\right) = -Li_{d\tilde{\alpha}(w)}\left(\frac{z}{z-1}\right) = -\overline{Li}_{\tilde{\alpha}(w)}\left(\frac{z}{z-1}\right).$$

Since $\overline{Li}_w(z) = \sum_{m=1}^{\infty} s_w(m-1)z^m$, we have

$$\frac{1}{1-z}\overline{Li}_w(z) = \sum_{m=1}^{\infty} s_w(m-1)\sum_{l=m}^{\infty} z^l = \sum_{i>0}\sum_{m=1}^{i} s_w(m-1)z^i = \sum_{i\geq 0}\big(\Sigma s_w\big)(i)z^{i+1},$$

where, for a sequence $a : \mathbb{Z}_{\geq 0} \to \mathbb{C}$, the sequence Σa stands for

$$(\Sigma a)(m) = \sum_{i=0}^{m} a(i).$$

Similarly, we can calculate

$$\frac{1}{1-z}\overline{Li}_w\left(\frac{z}{z-1}\right) = -\sum_{i\geq 0}\big(\nabla\Sigma^{-1}s_w\big)(i)z^{i+1},$$

where, for a sequence $a : \mathbb{Z}_{\geq 0} \to \mathbb{C}$, the sequence $\Sigma^{-1}a$ stands for

$$\big(\Sigma^{-1}a\big)(m) = \begin{cases} a(0) & m = 0, \\ a(m) - a(m-1) & m > 0. \end{cases}$$

By (2.9), we obtain

$$\Sigma s_w = \nabla\Sigma^{-1}s_{\tilde{\alpha}(w)}.$$

Since $\Sigma\nabla$ is an involution on $\mathbb{C}^{\mathbb{Z}_{\geq 0}}$, i.e. $\Sigma\nabla\Sigma\nabla a = a$, and $\Sigma\Sigma^{-1} = \Sigma^{-1}\Sigma = id$, we have $\nabla s_w = \Sigma\nabla\Sigma s_w = s_{\tilde{\alpha}(w)}$.

§ 2.3. A functional equation and Kawashima relation

Theorem 2.7 is important not only to lead the inversion sequence of s_w stated in Corollary 2.8 but also to prove the following proposition.

Proposition 2.9. *Let $m \geq 0$ and $w = z_{k_1}\cdots z_{k_l} \in \mathfrak{H}^1$. For any $\varepsilon > 0$, we have*

$$\big(\Delta^m s_w\big)(n) = O\left(\frac{1}{n^{m+k_1-\varepsilon}}\right) \quad (n \to \infty).$$

For $w \in \mathfrak{H}^1$, we define the Newton series

$$f_w(z) = \sum_{n=0}^{\infty}(-1)^n(\nabla s_w)(n)\binom{z}{n}, \; F_w(z) = \sum_{n=0}^{\infty}(-1)^n(\nabla S_w)(n)\binom{z}{n}.$$

Corollary 2.8 and Proposition 2.9 tell us the following proposition.

Proposition 2.10. *Let* $w = z_{k_1} \cdots z_{k_r} y^l$ $(k_r \geq 2, r, l \geq 0)$*. Then the abscissas of convergence of both Newton series* $f_w(z)$ *and* $F_w(z)$ *are* $-l-1$*.*

Then we obtain a functional equation as follows.

Theorem 2.11. *For any* $w, w' \in \mathfrak{H}^1$*, we have*

$$F_w(z)F_{w'}(z) = F_{w \bar{*} w'}(z)$$

for any $z \in \mathbb{C}$ *with* $\mathrm{Re}(z) > -1$*.*

Proof. The proof goes as follows. First, the function $(z+1)f_{L_y(w)}(z)$ $(w = z_{k_1} \cdots z_{k_l} \in \mathfrak{H}^1)$ is expanded to certain Newton series according to Lemma 2.4. This together with the formula

$$s_w(m) = S_{z_{k_2} \cdots z_{k_l}}(m+1) \frac{1}{(m+1)^{k_1}} \quad (m \geq 0)$$

and Proposition 2.3 tell us the identity

$$(z+1)f_{L_y(w)}(z) = F_w(z+1). \tag{2.10}$$

Proposition 2.10 shows that the Newton series $F_w(z)$ converges at least $\mathrm{Re}(z) > -1$, and hence the equation (2.10) makes sense for at least $\mathrm{Re}(z) > -2$. The product $f_{L_y(w)}(z) f_{L_y(w')}(z)$ is expanded to a Newton series by Proposition 2.5, and hence so does $F_w(z+1)F_{w'}(z+1)$. Because of the equation (2.3) and Proposition 2.3, we conclude the theorem. $\square$

Proposition 2.10 permits to expand $f_w(z)$ and $F_w(z)$ as Taylor series at $z = 0$. By Proposition 2.6 and Corollary 2.8, we have

$$F_w(z) = \sum_{m=1}^{\infty} (-1)^{m-1} \left\{ \sum_{n=1}^{\infty} s_{\tilde{\alpha}(w)}(n-1) a_{y^m}(n-1) \right\} z^m.$$

Using the equations (2.2), (2.5) and the formula (2.8),

$$\sum_{n=1}^{\infty} s_{\tilde{\alpha}(w)}(n-1) a_{y^m}(n-1) = \sum_{n=0}^{\infty} a_{d\tilde{\alpha}(w)}(n) a_{y^m}(n) = -\sum_{n=0}^{\infty} a_{\varphi d(w) \dot{*} y^m}(n).$$

This is equal to $-Z(\varphi d(w) \dot{*} y^m)$. Therefore we have

$$F_{d^{-1}(w)}(z) = \sum_{m=1}^{\infty} (-1)^m Z(\varphi(w) \dot{*} y^m) z^m. \tag{2.11}$$

According to the functional equation proven in Theorem 2.11 and the equation (2.11), we have a class of relations among MZV's stated in Theorem 1.1. This is what the author calls Kawashima relation.

§ 3. quasi-Derivation relation

The quasi-derivation relation is conjectured in Kaneko [15] as an extension of the derivation relation proven in Ihara-Kaneko-Zagier [12]. The derivation relation is proven [12] by reducing it to the regularized double shuffle relation. In [11, 12], the derivation relation is proven by reducing it to the Ohno relation appeared in [19]. But the quasi-derivation relation was never proven until the author proved it by reducing it to (the linear part of) the Kawashima relation in [24].

A derivation ∂ on $\mathfrak{H}$ is a $\mathbb{Q}$-linear endomorphism of $\mathfrak{H}$ satisfying the Leibniz rule

$$\partial(ww') = \partial(w)w' + w\partial(w')$$

for any $w, w' \in \mathfrak{H}$. Such a derivation is uniquely determined by its images of generators x and y. Let $z = x + y$. For each $n \geq 1$, the derivation $\partial_n : \mathfrak{H} \to \mathfrak{H}$ is defined by

$$\partial_n(x) = xz^{n-1}y,\ \partial_n(y) = -xz^{n-1}y.$$

It follows immediately that $\partial_n(\mathfrak{H}) \subset \mathfrak{H}^0$. Then we state the derivation relation for MZV's.

Theorem 3.1 (Derivation relation). *For any $n \geq 1$, we have $\partial_n(\mathfrak{H}^0) \subset \ker Z$.*

It is known that the operator ∂_n satisfies the identity

$$\partial_n = \frac{1}{(n-1)!}\mathrm{ad}(\theta)^{n-1}(\partial_1)$$

in [12], where θ stands for the derivation on $\mathfrak{H}$ defined by

$$\theta(x) = \frac{1}{2}(xz + zx),\ \theta(y) = \frac{1}{2}(yz + zy),$$

and $\mathrm{ad}(\theta)(\partial) = [\theta, \partial] = \theta\partial - \partial\theta$. Kaneko formulated the quasi-derivation operator $\partial_n^{(c)}$ by modifying this formula as follows.

Definition 3.2. *Let $c \in \mathbb{Q}$ and H the derivation on $\mathfrak{H}$ defined by $H(w) = \deg(w)w$ for any words $w \in \mathfrak{H}$. For each integer $n \geq 1$, the $\mathbb{Q}$-linear map $\partial_n^{(c)} : \mathfrak{H} \to \mathfrak{H}$, which is called the quasi-derivation (with respect to n and $\theta^{(c)}$ for the given $c \in \mathbb{Q}$) in [24], is defined by*

$$\partial_n^{(c)} = \frac{1}{(n-1)!}\mathrm{ad}(\theta^{(c)})^{n-1}(\partial_1),$$

where $\theta^{(c)} : \mathfrak{H} \to \mathfrak{H}$ *is the* $\mathbb{Q}$*-linear map defined by* $\theta^{(c)}(x) = \theta(x)$, $\theta^{(c)}(y) = \theta(y)$ *and the rule*

$$\theta^{(c)}(ww') = \theta^{(c)}(w)w' + w\theta^{(c)}(w') + c\partial_1(w)H(w') \tag{3.1}$$

for any $w, w' \in \mathfrak{H}$.

When $c = 0$, the quasi-derivation $\partial_n^{(c)}$ is reduced to the ordinary derivation ∂_n. If $c \neq 0$ and $n \geq 2$, the operator $\partial_n^{(c)}$ is no longer a derivation. Although the inclusion $\partial_n^{(c)}(\mathfrak{H}) \subset \mathfrak{H}^0$ does not hold in general, we have the inclusion $\partial_n^{(c)}(\mathfrak{H}^0) \subset \mathfrak{H}^0$.

One of the main theorem in [24] is that any two of the quasi-derivation operators commute with each other:

Theorem 3.3. *For any* $n, m \geq 1$ *and any* $c, c' \in \mathbb{Q}$, *we have* $[\partial_n^{(c)}, \partial_m^{(c')}] = 0$.

The proof is not so simple but certain kind of induction works. This theorem also helps us to find the Connes-Moscovici's Hopf algebra structure of the quasi-derivation operators (see [2]).

Our aim is to prove the inclusion

$$\partial_n^{(c)}(\mathfrak{H}^0) \subset L_x\varphi(\mathfrak{H}y * \mathfrak{H}y) \tag{3.2}$$

for any $n \geq 1$ and any $c \in \mathbb{Q}$, which equals to show that $\varphi L_x^{-1}\partial_n^{(c)}(\mathfrak{H}^0)$ is an element in $\mathfrak{H}y * \mathfrak{H}y$. Actually we can show the following identity.

Theorem 3.4. *Let* $\chi_x = \tau L_x \varepsilon$. *For any* $n \geq 1$ *and any* $c \in \mathbb{Q}$, *there exists an element* $w = w(n, c) \in \mathfrak{H}y$ *such that* $\partial_n^{(c)}\chi_x = \chi_x\mathcal{H}_w$ *on* $\mathfrak{H}^1$. *In other words, the following diagram commutes:*

$$\begin{tikzcd} \mathfrak{H}^1 \arrow[r, "\mathcal{H}_w"] \arrow[d, "\chi_x"'] & \mathfrak{H}^1 \arrow[d, "\chi_x"] \\ \mathfrak{H}^0 \arrow[r, "\partial_n^{(c)}"'] & \mathfrak{H}^0 \end{tikzcd}$$

For the proof of Theorem 3.4, we use Theorem 3.3. The proof is also by certain kind of induction.

Theorem 3.4 is the key property to show the inclusion (3.2). The proof goes as follows. Since φ is an automorphism on $\mathfrak{H}$ with $\varphi(y) = -y$ and τ an anti-automorphism on $\mathfrak{H}$ given by $\tau(x) = y$ and $\tau(y) = x$, we find

$$\chi_x(\mathfrak{H}y) = x\mathfrak{H}y.$$

By Theorem 3.4, there exists $w = w(n, c) \in \mathfrak{H}y$ satisfying

$$\partial_n^{(c)}\chi_x = \chi_x\mathcal{H}_w.$$

We therefore have

$$\partial_n^{(c)}(x\mathfrak{H}y) = \partial_n^{(c)}\chi_x(\mathfrak{H}y) = \chi_x\mathcal{H}_w(\mathfrak{H}y) \subset \chi_x(\mathfrak{H}y * \mathfrak{H}y).$$

This is included $L_x\varphi(\mathfrak{H}y * \mathfrak{H}y)$ because

$$\chi_x(\mathfrak{H}y*\mathfrak{H}y) = (1-(1-\tau))L_x\varphi(\mathfrak{H}y*\mathfrak{H}y) \subset L_x\varphi(\mathfrak{H}y*\mathfrak{H}y) - (1-\tau)(\mathfrak{H}^0) \subset L_x\varphi(\mathfrak{H}y*\mathfrak{H}y).$$

The last inclusion is by Thorem 1.3. Since $\mathfrak{H}^0 = \mathbb{Q} + x\mathfrak{H}y$ and $\partial_n^{(c)}(\mathbb{Q}) = \{0\} \subset L_x\varphi(\mathfrak{H}y * \mathfrak{H}y)$, we conclude the inclusion (3.2).

Another extension of the derivation operator ∂_n is discussed in [24].

Definition 3.5. *Let $c \in \mathbb{Q}$ and H the same operator as in Definition 3.2. For each integer $n \geq 1$, the $\mathbb{Q}$-linear map $\widehat{\partial}_n^{(c)} : \mathfrak{H} \to \mathfrak{H}$ is defined by*

$$\widehat{\partial}_n^{(c)} = \frac{1}{(n-1)!}\mathrm{ad}(\widehat{\theta}^{(c)})^{n-1}(\partial_1)$$

where $\widehat{\theta}^{(c)}$ is the $\mathbb{Q}$-linear map defined by $\widehat{\theta}^{(c)}(x) = \theta(x)$, $\widehat{\theta}^{(c)}(y) = \theta(y)$ and the rule

$$\widehat{\theta}^{(c)}(ww') = \widehat{\theta}^{(c)}(w)w' + w\widehat{\theta}^{(c)}(w') + cH(w)\partial_1(w') \tag{3.3}$$

for any $w, w' \in \mathfrak{H}$.

The operator $\widehat{\partial}_n^{(c)}$ gives another quasi-derivation operator (with respect to n and $\widehat{\theta}^{(c)}$ for the given $c \in \mathbb{Q}$). The only difference between $\theta^{(c)}$ and $\widehat{\theta}^{(c)}$ is the order of H and ∂_1 appearing in the right-hand side of (3.1) and (3.3).

In fact, the quasi-derivation $\widehat{\partial}_n^{(c)}$ satisfies

Proposition 3.6. *For any $n \geq 1$ and any $c \in \mathbb{Q}$, we have $\widehat{\partial}_n^{(c)} \in \mathbb{Q}[\partial_1^{(-c)}, \ldots, \partial_n^{(-c)}]$.*

Example 3.7. The polynomials in Proposition 3.6 can be constructed explicitly. For example,

$$\begin{aligned}
\widehat{\partial}_2^{(c)} &= \partial_2^{(-c)} + c\partial_1^2,\\
\widehat{\partial}_3^{(c)} &= \partial_3^{(-c)} + 2c\partial_1\partial_2^{(-c)} + c^2\partial_1^3,\\
\widehat{\partial}_4^{(c)} &= \partial_4^{(-c)} + \frac{7}{3}c\partial_1\partial_3^{(-c)} + \frac{2}{3}c{\partial_2^{(-c)}}^2 + 3c^2\partial_1^2\partial_2^{(-c)} + c^3\partial_1^4.
\end{aligned}$$

Because of Theorem 3.3 and Proposition 3.6, we see that

$$[\partial_n^{(c)}, \widehat{\partial}_m^{(c')}] = 0,\ [\widehat{\partial}_n^{(c)}, \widehat{\partial}_m^{(c')}] = 0$$

for any $n, m \geq 1$ and any $c, c' \in \mathbb{Q}$. Moreover, we obtain the following identity.

Proposition 3.8. *For any $c \in \mathbb{Q}$, we have* $\widehat{\partial}_n^{(c)} = -\tau \partial_n^{(-c)} \tau$.

By Proposition 3.8, we have $\widehat{\partial}_n^{(c)}(\mathfrak{H}^0) \subset L_x\varphi(\mathfrak{H}y * \mathfrak{H}y)$ for any $n \geq 1$ and any $c \in \mathbb{Q}$. Therefore the quasi-derivation $\widehat{\partial}_n^{(c)}$ also gives a class of relations among MZV's, which is in fact the same class induced by the quasi-derivation $\partial_n^{(c)}$.

§ 4. Cyclic sum formula

For $k_1, \ldots, k_l \geq 1$ with some $k_q > 1$, the cyclic sum formula (CSF for short) for MZV's

(4.1)
$$\sum_{j=1}^{l} \sum_{i=1}^{k_j-1} \zeta(k_j - i + 1, k_{j+1}, \ldots, k_l, k_1, \ldots, k_{j-1}, i) = \sum_{j=1}^{l} \zeta(k_j + 1, k_{j+1}, \ldots, k_l, k_1, \ldots, k_{j-1})$$

is proven in Hoffman-Ohno [11] by means of partial fraction expansions and CSF for MZSV's

$$\sum_{j=1}^{l} \sum_{i=1}^{k_j-1} \zeta^{\star}(k_j - i + 1, k_{j+1}, \ldots, k_l, k_1, \ldots, k_{j-1}, i) = k\zeta(k+1), \tag{4.2}$$

where $k = k_1 + \cdots + k_l$, in Ohno-Wakabayashi [20] in a similar way. Hoffman and Ohno also introduced an algebraic expression of CSF for MZV's. They formulated CSF using 'cyclic derivatives' on $\mathfrak{H}$ (see [11, 25] for details).

In the present section, using the formulation of CSF in [25], we give another proof by showing that CSF is contained in the linear part of Kawashima relation stated in Corollary 1.2.

Let $n \geq 1$. We denote an action of $\mathfrak{H}$ on $\mathfrak{H}^{\otimes(n+1)}$ by "$\diamond$", which is defined by

$$a \diamond (w_1 \otimes \cdots \otimes w_{n+1}) = w_1 \otimes \cdots \otimes w_n \otimes a w_{n+1},$$
$$(w_1 \otimes \cdots \otimes w_{n+1}) \diamond b = w_1 b \otimes w_2 \otimes \cdots \otimes w_{n+1}.$$

The action $\diamond$ is a $\mathfrak{H}$-bimodule structure on $\mathfrak{H}^{\otimes(n+1)}$. We define the $\mathbb{Q}$-linear map $\mathcal{C}_n \colon \mathfrak{H} \to \mathfrak{H}^{\otimes(n+1)}$ by

$$\mathcal{C}_n(x) = x \otimes z^{\otimes(n-1)} \otimes y, \ \mathcal{C}_n(y) = -(x \otimes z^{\otimes(n-1)} \otimes y),$$

where $z = x + y$, and

$$\mathcal{C}_n(ww') = \mathcal{C}_n(w) \diamond w' + w \diamond \mathcal{C}_n(w') \tag{4.3}$$

for any $w, w' \in \mathfrak{H}$. We find that $\mathcal{C}_n(1) = 0$ by putting $w = w' = 1$ in (4.3).

Let $M_n \colon \mathfrak{H}^{\otimes(n+1)} \to \mathfrak{H}$ denote the multiplication map, i.e.,

$$M_n(w_1 \otimes \cdots \otimes w_{n+1}) = w_1 \cdots w_{n+1},$$

and let $\rho_n = M_n \mathcal{C}_n$ $(n \geq 1)$. Then we have

Theorem 4.1. *For $n \geq 1$, we have $\rho_n(\check{\mathfrak{H}}^1) \subset L_x\varphi(\mathfrak{H}y * \mathfrak{H}y)$.*

Hence, by Corollary 1.2, we obtain

Corollary 4.2. *For $n \geq 1$, we have $\rho_n(\check{\mathfrak{H}}^1) \subset \ker Z$.*

When $n = 1$ and $w = z_{k_1} \cdots z_{k_l}$, we have

$$\rho_1(w) = \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} z_{k_j-i+1} z_{k_{j+1}} \cdots z_{k_l} z_{k_1} \cdots z_{k_{j-1}} z_i - \sum_{j=1}^{l} x z_{k_{j+1}} \cdots z_{k_l} z_{k_1} \cdots z_{k_j}.$$

This is evaluated to CSF for MZV's (4.1) by the map Z.

Theorem 4.1 immediately follows by two key lemmas below. Let $A_0 = 1$ and $A_j = z^{j-1}y$ for $j \geq 1$ and let $\check{\mathfrak{H}}^1$ be a subvector space of $\mathfrak{H}^1$ generated by words of $\mathfrak{H}^1$ except for powers of y.

Lemma 4.3. *The set $\{A_{k_1+\cdots+k_l} - A_{k_1} \cdots A_{k_l} \mid k_1, \ldots, k_l \geq 1, l \geq 1\}$ is a set of bases of $\check{\mathfrak{H}}^1$.*

Lemma 4.4. *For $k_1 \geq n \geq 1, k_2, \ldots, k_l \geq 1$, we have*

$$\begin{aligned}\varphi L_x^{-1} \rho_n(&A_{k_1+\cdots+k_l-n+1} - A_{k_1-n+1} A_{k_2} \cdots A_{k_l}) \\ &= \sum_{m=2}^{l} \frac{(-1)^{l-m}}{m} \sum_{j=1}^{l} \sum_{\substack{\alpha_1+\cdots+\alpha_m=l \\ \alpha_1,\ldots,\alpha_m \geq 1}} H(j, \alpha_1, \ldots, \alpha_m).\end{aligned}$$

Here, $H(j, \alpha_1, \ldots, \alpha_m)$ is given by

$$H(j, \alpha_1, \ldots, \alpha_m) = (z_{k_j} \cdots z_{k_{\alpha_1+j-1}}) * \cdots * (z_{k_{\alpha_1+\cdots+\alpha_{m-1}+j}} \cdots z_{k_{\alpha_1+\cdots+\alpha_m+j-1}}),$$

where the subscripts of k's of the right-hand side are viewed as numbers modulo l ($\in \{1, \ldots, l\}$).

Lemma 4.4 is proven by certain combinatorial argument (see [25]).

According to Lemma 4.4, we see

$$\varphi L_x^{-1} \rho_n(A_{k_1+\cdots+k_l-n+1} - A_{k_1-n+1} A_{k_2} \cdots A_{k_l}) \in \mathfrak{H}y * \mathfrak{H}y.$$

This together with Lemma 4.3 concludes Theorem 4.1.

In the case of CSF for MZSV's, we define another map $\overline{\rho}_n$ as follows. We first notice that γ^{-1} is also the automorphism on $\mathfrak{H}$ given by

$$\gamma^{-1}(x) = x,\ \gamma^{-1}(y) = y - x.$$

We define the $\mathbb{Q}$-linear map $\overline{\mathcal{C}}_n : \mathfrak{H} \to \mathfrak{H}^{\otimes(n+1)}$ by

$$\overline{\mathcal{C}}_n(x) = x \otimes y^{\otimes n},\ \overline{\mathcal{C}}_n(y) = -(x \otimes y^{\otimes n})$$

and

$$\overline{\mathcal{C}}_n(ww') = \overline{\mathcal{C}}_n(w) \diamond \gamma^{-1}(w') + \gamma^{-1}(w) \diamond \overline{\mathcal{C}}_n(w')$$

for any $w, w' \in \mathfrak{H}$. Let $\overline{\rho}_n = M_n\overline{\mathcal{C}}_n$ $(n \geq 1)$. Then we find the identity $\rho_n = d\overline{\rho}_n$ for any $n \geq 1$. By the identity (2.8), we also obtain the following equivalence.

Proposition 4.5. *For any $n \geq 1$, we have*

$$\overline{\rho}_n(\check{\mathfrak{H}}^1) \subset L_x\widetilde{\alpha}(\mathfrak{H}y \,\overline{*}\, \mathfrak{H}y) \iff \rho_n(\check{\mathfrak{H}}^1) \subset L_x\varphi(\mathfrak{H}y * \mathfrak{H}y).$$

Theorem 4.1 and Proposition 4.5 conclude

Corollary 4.6. *For $n \geq 1$, we have $\overline{\rho}_n(\check{\mathfrak{H}}^1) \subset L_x\tilde{\alpha}(\mathfrak{H}y \,\overline{*}\, \mathfrak{H}y)$.*

Therefore the map $\overline{\rho}_n$ $(n \geq 1)$ gives relations among MZSV's. The identity

$$\overline{\rho}_1(\gamma(z_{k_1} \cdots z_{k_l}) - x^{k_1+\cdots+k_l}) = \sum_{j=1}^{l}\sum_{i=1}^{k_j-1} z_{k_j-i+1}z_{k_{j+1}} \cdots z_{k_l}z_{k_1} \cdots z_{k_{j-1}}z_i - kz_{k+1},$$

where $k = k_1 + \cdots + k_l$, is evaluated to the formula (4.2) by the map $\overline{Z}$.

Acknowledgements. The author thanks to Dr Kentaro Ihara for his helpful comments.

References

[1] R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$*, Astérisque, 61 (1979), 11-13.

[2] A. Connes and H. Moscovici, *Rankin-Cohen brackets and the Hopf algebra of transverse geometry*, Moscow Mathematical Journal 4 (2004), 111–130.

[3] P. Deligne, *Le groupe fondamental de la droite projective moins trois points*, Galois groups over Q (Berkeley, CA, 1987), 79–297, Math. S. Res. Inst. Publ., 16, Springer, New York-Berlin, 1989.

[4] V. G. Drinfel'd, *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. 2 (1991), no. 4, 829–860.

[5] L. Euler, *Demonstratio insignis theorematis numerici circa unicias potestatum binomialium*, Nova Acta Academiae Scientarum Imperialis Petropolitinae 15, (1799/1802), 33–43.

[6] H. Furusho, *p-adic multiple zeta values II – tannakian interpretations*, Amer. J. Math, Vol 129, No 4, (2007), 1105–1144.

[7] A. Gel'fond, *Calculus of Finite Differences*, Hindustan, 1971.

[8] A. Goncharov, *Multiple ζ-values, Galois groups, and geometry of modular varieties*, European Congress of Mathematics, Vol. I (Barcelona, 2000), 361–392, Progr. Math., 201, Birkhauser, Basel, 2001.
[9] M. Hoffman, *The algebra of multiple harmonic series*, J. Algebra 194 (1997), 477–495.
[10] M. Hoffman, *Quasi-symmetric functions and mod p multiple harmonic sums*, preprint, arXiv.math/0401.319.
[11] M. Hoffman and Y. Ohno, *Relations of multiple zeta values and their algebraic expression*, J. Algebra 262 (2003), 332–347.
[12] K. Ihara, M. Kaneko and D. Zagier, *Derivation and double shuffle relations for multiple zeta values*, Compos. Math. 142 (2006), no. 2, 307–338.
[13] S. Izumi, *Teisa-hô*, Iwanami, 1934 (in Japanese).
[14] G. Kawashima, *A class of relations among multiple zeta values*, J. Number Theory 129 (4), 2009, 755–788.
[15] M. Kaneko, *On an extension of the derivation relation for multiple zeta values*, The Conference on L-Functions, 89–94, World Sci. Publ., Hackensack, NJ, 2007.
[16] M. Kaneko, M. Noro and K. Tsurumaki, *On a conjecture for the dimension of the space of the multiple zeta values*, Software for algebraic geometry, 47–58, IMA Vol. Math. Appl., 148, Springer, New York, 2008.
[17] G. Kawashima and T. Tanaka, *Newton series and extended derivation relations for multiple L-values*, arXiv:math.NT/08013062.
[18] L. Lewin, editor, *Structural properties of polylogarithms*, Vol. 37 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1991.
[19] Y. Ohno, *A generalization of the duality and sum formulas on the multiple zeta values*, J. Number Theory 74 (1999), 39–43.
[20] Y. Ohno and N. Wakabayashi, *Cyclic sum of multiple zeta values*, Acta Arith. 123 (2006), 289–295.
[21] J. Okuda and K. Ueno, *Relations for multiple zeta values and Mellin transforms of multiple polylogarithms*, Publ. Res. Inst. Math. Sci. 40 (2004), 537–564.
[22] C. Reutenauer, *Free Lie Algebras*, Oxford Science Publications, 1993.
[23] T. Rivoal, *La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs*, C. R. Acad. Sci. Paris Sér. I Math. 331 (2000), no. 4, 267–270.
[24] T. Tanaka, *On the quasi-derivation relation for multiple zeta values*, to appear in J. Number Theory.
[25] T. Tanaka and N. Wakabayashi, *An algebraic proof of cyclic sum formula for multiple zeta values*, preprint, arXiv.math/0902.2723.
[26] T. Terasoma, *Mixed Tate motives and multiple zeta values*, Invent. Math. 149 (2002), no. 2, 339–369.
[27] D. Zagier, *Values of zeta functions and their applications*, First European Congress of Mathematics, Vol. II (Paris, 1992), 497–512, Progr. Math., 120, Birkhäuser, Basel, 1994.
[28] W. Zudilin, *One of the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational*, Uspekhi Mat. Nauk 56 (2001), no. 4(340), 149–150; translation in Russian Math. Surveys 56 (2001), no. 4, 774–776.