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On L-functions over function fields: Power-means of error-terms and distribution of L'/L-values

By

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§ 1. Introduction

Let K be a function field of one variable over a finite field $\mathbf{F}_q$. For a non-principal Dirichlet character χ on K, consider the L-function $L(s,\chi)$ and the partial L-function $L_P(s,\chi)$ associated to each finite set P of primes of K. Consider the differences

$$(1.1.1)\qquad \begin{cases} f_P(s,\chi) = \log L(s,\chi) - \log L_P(s,\chi) & (\log : \text{a suitable branch}) \\ f'_P(s,\chi) = \frac{L'}{L}(s,\chi) - \frac{L'_P}{L_P}(s,\chi) & \left(\frac{L'}{L}(s,\chi) := \frac{L'(s,\chi)}{L(s,\chi)}, \text{ etc.}\right) \end{cases}$$

on $\mathrm{Re}(s) > 1/2$. If $P = P_y = \{\mathfrak{p}; N(\mathfrak{p}) \le y\}$ and $y \mapsto \infty$, we know that each of $f_P(s,\chi), f'_P(s,\chi)$ tends to 0. But unless $\mathrm{Re}(s) > 1$, the convergence (say, for each fixed s) cannot be expected to be uniform in χ. The speed of convergence should depend on the size of the norm of the conductor of χ. We shall prove that, nevertheless, for each case of

$$(1.1.2)\qquad g_P(s,\chi) = f_P(s,\chi), \text{ or } = f'_P(s,\chi),$$

and for each positive integer k, the *average*

$$(1.1.3)\qquad \mathrm{Avg}_{\chi\,(\mathrm{mod}\,\mathbf{f})} |g_{P_y}(s,\chi)|^{2k}$$

tends to 0 as $y \mapsto \infty$ *uniformly* with respect to integral ideals $\mathbf{f}$ and to $s \in \mathbf{C}$ such that $\mathrm{Re}(s) \ge 1/2 + \epsilon$ (Theorem A, §2.2). Here, χ runs over the (suitably normalized)

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non-principal characters mod $\mathbf{f}$. The proof is based on the ideas and techniques used in [3] applied to the situation of the function field case.

As an application (of the case of $f'_P(s,\chi)$), we shall give a sharpened version of Theorem 7 of [1], to the effect that the function $M_\sigma(z)$ constructed there is, in fact, the density function for the distribution of values of $\{L'(s,\chi)/L(s,\chi)\}_\chi$ *in a strong sense.* Here, $s \in \mathbf{C}$ is fixed with $\sigma = \mathrm{Re}(s)$, and χ runs over a suitably normalized family of Dirichlet characters on K with prime conductors. The only conditions for σ is, now, $\sigma > 1/2$ (instead of $\sigma > 3/4$ as was in [1]). Also, the "too narrow" assumption in [1]Theorem 7 (i) for the test functions Φ is now considerably loosened (Theorem B in §2.3).

An application of the case of $f_P(s,\chi)$ to the study of distribution of values of $\{\log L(s,\chi)\}_\chi$ (including some number field cases) is left to the future publication.

In the Appendix (§5), for the sake of completeness and self-containedness, we shall provide proofs of function-field analogues of estimations of some basic arithmetic functions that are well-known in the number field case.

§ 2. The main results

§ 2.1. Preliminaries.

The basic notations are as follows.

K : a function field of one variable over a finite field $\mathbf{F}_q$,
$\mathfrak{p}_\infty$: a prime divisor of K.

These are fixed once and for all. The Landau and the Vinogradov symbols $\mathbf{O}$ and $\ll$ will usually depend on K and $\mathfrak{p}_\infty$, but these dependences will be suppressed from the notations.

$\mathbf{f}$: an integral divisor $\neq (1)$ on K which is coprime with $\mathfrak{p}_\infty$,
$I_\mathbf{f}$: the group of divisors of K coprime with $\mathbf{f}$,
$G_\mathbf{f} = I_\mathbf{f}/\langle\mathfrak{p}_\infty\rangle\{(\alpha); \alpha \equiv 1 (\mathrm{mod}\, \mathbf{f})\}$,
where $\langle\mathfrak{p}_\infty\rangle$ denotes the subgroup of $I_\mathbf{f}$ generated by $\mathfrak{p}_\infty$, and (α) for each $\alpha \in K^\times$ denotes the principal divisor generated by α.
$i_\mathbf{f} : I_\mathbf{f} \mapsto G_\mathbf{f}$: the projection,
$\hat{G}_\mathbf{f}$: the character group of $G_\mathbf{f}$, with the unit element χ_0.

A word about the role of the "infinite prime divisor" $\mathfrak{p}_\infty$. Recall that the principal divisors are all contained in the kernel of the degree-homomorphism $I_\mathbf{f} \mapsto \mathbf{Z}$ which is surjective; hence we must divide $I_\mathbf{f}$, not only by $\{(\alpha)\}$ but also by a cyclic subgroup generated by an element of degree > 0 such as $\mathfrak{p}_\infty$, to make the quotient finite. In terms of classfield theory, this corresponds to that the maximal abelian extension of K with conductor $\mathbf{f}$ is infinite because it contains all the constant field extensions but if we impose that a given prime $\mathfrak{p}_\infty$ should decompose completely, then the extension will be finite, with the Galois group $G_{\mathbf{f}}$.

For each $\chi \in \hat{G}_{\mathbf{f}}$ and an integral divisor D on K, we define $\chi(D) = \chi(i_{\mathbf{f}}(D))$ if $(D, \mathbf{f}) = 1$, and $\chi(D) = 0$ otherwise. In particular, we have $\chi(\mathfrak{p}_\infty) = 1$, and $\chi(\mathfrak{p}) = 0$ for all $\mathfrak{p} | \mathbf{f}$. We shall consider Dirichlet L-functions associated with each $\chi \in \hat{G}_{\mathbf{f}}$. A few words to explain our choice of notations. First, since L-functions with imprimitive characters will also be treated, we shall include $\mathbf{f}$ inside the symbols in order to indicate the precise modulus. Secondly, mainly for the sake of compatibility of notations with those of [1] (related to Theorem B), we shall use the basic L-symbols for *L-functions without the $\mathfrak{p}_\infty$-factor* $(1 - N(\mathfrak{p}_\infty)^{-s})^{-1}$. (As regards Theorem A, our concern is solely on the "difference" between the local and the global L-functions, so it does not matter whether we include or exclude one particular Euler factor from local or global L-functions, as long as we do it simultaneously. We shall exclude the $\mathfrak{p}_\infty$-factor from both.) Thus, we define, for each $\chi \in \hat{G}_{\mathbf{f}}$:

$$L(s, \chi, \mathbf{f}) = \prod_{\mathfrak{p} \neq \mathfrak{p}_\infty} (1 - \chi(\mathfrak{p}) N(\mathfrak{p})^{-s})^{-1}, \tag{2.1.1}$$

which converges absolutely on $\mathrm{Re}(s) > 1$ and extends to a meromorphic function on $\mathbf{C}$. Let $\mathbf{f}_\chi$ denote the conductor of χ, and χ^* the primitive character mod $\mathbf{f}_\chi$ associated with χ. Then $L(s, \chi, \mathbf{f})$ is obtained from $L(s, \chi^*, \mathbf{f}_\chi)$ by multiplying the product of $(1 - \chi^*(\mathfrak{p}) N(\mathfrak{p})^{-s})$ over those prime factors $\mathfrak{p}$ of $\mathbf{f}$ that do not divide $\mathbf{f}_\chi$. And by A. Weil [5], if χ is primitive and $\chi \neq \chi_0$, then $L(s, \chi, \mathbf{f}_\chi)(1 - N(\mathfrak{p}_\infty)^{-s})^{-1}$ is a polynomial of $u = q^{-s}$ of degree $2g - 2 + \deg \mathbf{f}_\chi$ (g: the genus of K), whose reciprocal roots have absolute values $q^{1/2}$. From these, it is clear that our $L(s, \chi, \mathbf{f})$ ($\chi \in \hat{G}_{\mathbf{f}} \setminus \{\chi_0\}$) is an entire function of s having zeros only on the vertical lines $\mathrm{Re}(s) = 1/2$ and $\mathrm{Re}(s) = 0$. In any case, it is holomorphic and non-vanishing on $\mathrm{Re}(s) > 1/2$. Finally, our choice of *the branch of* $\log L(s, \chi, \mathbf{f})$ on $\mathrm{Re}(s) > 1/2$ will be the unique holomorphic branch that tends to 0 when $\mathrm{Re}(s) \to +\infty$.

For any positive integral power y of q, set

$$P = P_y = \{\mathfrak{p} : prime\, divisors\, \neq \mathfrak{p}_\infty\, on\, K,\; N(\mathfrak{p}) \leq y\}, \tag{2.1.2}$$

and for each $\chi \in \hat{G}_{\mathbf{f}}$, define the local L-function by

$$L_P(s, \chi, \mathbf{f}) = \prod_{\mathfrak{p} \in P} (1 - \chi(\mathfrak{p}) N(\mathfrak{p})^{-s})^{-1}. \tag{2.1.3}$$

This is holomorphic and non-vanishing on $\mathrm{Re}(s) > 0$, and we define its logarithm by

$$\log L_P(s, \chi, \mathbf{f}) = -\sum_{\mathfrak{p} \in P} \log(1 - \chi(\mathfrak{p}) N(\mathfrak{p})^{-s}), \tag{2.1.4}$$

where the branch of log in each summand is chosen to be the principal branch.

We shall consider the differences between the global and the local functions

$$\begin{cases} f(s, \chi, \mathbf{f}, y) = \log L(s, \chi, \mathbf{f}) - \log L_{P_y}(s, \chi, \mathbf{f}), \\ f'(s, \chi, \mathbf{f}, y) = \frac{L'}{L}(s, \chi, \mathbf{f}) - \frac{L'_{P_y}}{L_{P_y}}(s, \chi, \mathbf{f}), \end{cases} \tag{2.1.5}$$

for $\mathrm{Re}(s) > 1/2$, and write as

$$g(s, \chi, \mathbf{f}, y) = \begin{cases} f'(s, \chi, \mathbf{f}, y) & \text{(Case 1)}, \\ f(s, \chi, \mathbf{f}, y) = \int_\infty^s f'(s, \chi, \mathbf{f}, y) ds & \text{(Case 2)}, \end{cases} \tag{2.1.6}$$

where the last integral is along the horizontal line from $+\infty$ to s (the initial point is $+\infty$, because of our choice of the branches of $\log L(s, \chi, \mathbf{f})$ and $\log L_{P_y}(s, \chi, \mathbf{f})$). In each case, $g(s, \chi, \mathbf{f}, y)$ is a holomorphic function of s on $\mathrm{Re}(s) > 1/2$. First let us pay attention to the following elementary estimations.

Proposition 2.1.7. *Let $\epsilon > 0$. Then*

(i) *For $\sigma = \mathrm{Re}(s) \geq 1/2 + \epsilon$,*

$$|g(s, \chi, \mathbf{f}, y)| \ll_\epsilon \begin{cases} (\log N(\mathbf{f})) y^{1/2 - \sigma} & \text{(Case 1)}, \\ (\log N(\mathbf{f})) y^{1/2 - \sigma} / \log y & \text{(Case 2)}. \end{cases}$$

(ii) *For $\sigma = \mathrm{Re}(s) \geq 1 + \epsilon$,*

$$|g(s, \chi, \mathbf{f}, y)| \ll_\epsilon \begin{cases} y^{1-\sigma} & \text{(Case 1)}, \\ y^{1-\sigma} / \log y & \text{(Case 2)}, \end{cases}$$

independently of $\mathbf{f}$ *and* χ.

The proof will be given in §3.2. Thus, $\lim_{y \to \infty} g(s, \chi, \mathbf{f}, y) = 0$ holds in each case, but the uniformity of convergence with respect to the conductor $\mathbf{f}$ is known only for $\sigma > 1$. (In fact, as an application of our second main result Theorem B, we can actually prove in Case 1 that the convergence is *not* uniform in χ when $\sigma \leq 1$; see Corollary 2.3.4 below.) Our first main result asserts that *the average* of powers of $|g(s, \chi, \mathbf{f}, y)|$

over non-trivial characters modulo $\mathbf{f}$ converges to 0 *uniformly*, i.e., independently of $\mathbf{f}$, and also of those s with $\sigma = \mathrm{Re}(s) \geq 1/2 + \epsilon$.

§ 2.2. The first main result.

We shall fix $0 < \epsilon < 1/2$, and a positive integer $k \in \mathbf{N}$. Consider only such $s \in \mathbf{C}$ that satisfies

$$\frac{1}{2} + \epsilon \leq \sigma = \mathrm{Re}(s). \tag{2.2.1}$$

Hereafter, the symbols $\ll$ and $\mathbf{O}$ will depend only on ϵ and k (in addition to K, $\mathfrak{p}_\infty$). Note that

$$\frac{1+\epsilon}{2} - \sigma \leq -\frac{\epsilon}{2} < 0. \tag{2.2.2}$$

Theorem A. *For any integral divisor* $\mathbf{f} \neq (1)$ *of* K *with* $(\mathbf{f}, \mathfrak{p}_\infty) = 1$, *any* y *which is a positive integral power of* q, *and for any* $s \in \mathbf{C}$ *with* $\sigma = \mathrm{Re}(s) \geq 1/2 + \epsilon$, *we have*

$$\left(\mathrm{Avg}_{\substack{\chi \in \hat{G}_{\mathbf{f}} \\ \chi \neq \chi_0}} |g(s, \chi, \mathbf{f}, y)|^{2k} \right)^{\frac{1}{2k}} \ll y^{\frac{1+\epsilon}{2} - \sigma} \times \begin{cases} \log y & \text{(Case 1)}, \\ 1 & \text{(Case 2)}, \end{cases} \tag{2.2.3}$$

where Avg *denotes the average over* $\chi \in \hat{G}_{\mathbf{f}} \setminus \{\chi_0\}$, *and* $\ll$ *depends only on* k, ϵ. *In particular, this average tends to* 0 *as* $y \to \infty$ *uniformly in* $\mathbf{f}$ *on* $\mathrm{Re}(s) \geq 1/2 + \epsilon$.

Remarks 2.2.4. (i) Since

$$\left(\frac{a_1^q + \cdots + a_n^q}{n} \right)^{1/q} \leq \left(\frac{a_1^p + \cdots + a_n^p}{n} \right)^{1/p}$$

holds for any $a_1, \cdots a_n \geq 0$ and $p > q > 0$, it follows that the exponent k in the above theorem may be replaced by any positive real number.

(ii) It is unlikely that the implicit constant in (2.2.3) can be chosen to be independent of k. If it were so, then (since the left hand side of (2.2.3) tends to

$$\mathrm{Max}_{\substack{\chi \in \hat{G}_{\mathbf{f}} \\ \chi \neq \chi_0}} |g(s, \chi, \mathbf{f}, y)|$$

as $k \mapsto \infty$), one would obtain the uniformity of convergence $g(s, \chi, \mathbf{f}, y) \to 0$ without averaging over χ.

(iii) When $\mathbf{f}$ is a prime divisor, we may replace $\chi \in \hat{G}_{\mathbf{f}}, \chi \neq \chi_0$ in Theorem A by $\chi \in \hat{G}_{\mathbf{f}}, \mathbf{f}_\chi = \mathbf{f}$. This can be checked easily by using the arguments in §3.5.

§ 2.3. The second main result.

By applying Theorem A for Case 1, we shall give a substantial improvement of Theorem 7 of [1]§6.1. Namely, let K and $\mathfrak{p}_\infty$ be as above, with an additional assumption $\deg(\mathfrak{p}_\infty) = 1$. Let $M_\sigma(z)$, $\tilde{M}_\sigma(z)$ ($\sigma > 1/2$, $z \in \mathbf{C}$) be the associated "M-function" and its Fourier dual, constructed in [1]. Let $\mathbf{f}$ run over the *prime* divisors $\neq \mathfrak{p}_\infty$ of K, and for each $\mathbf{f}$, let χ run over the Dirichlet characters on K with conductor $\mathbf{f}$ satisfying $\chi(\mathfrak{p}_\infty) = 1$. In other words, χ runs over $\hat{G}_{\mathbf{f}} \setminus \hat{G}_{(1)}$. (In [1], such a family of characters was called the "Case A family" in the function field case.) For each such χ, we write $L(s, \chi) = L(s, \chi, \mathbf{f})$ (and later, also $L_P(s, \chi) = L_P(s, \chi, \mathbf{f})$ for $P = P_y$) [2]. Define the weighted average Avg_χ, as in [1]§4.1. In this paper, we shall prove the following:

Theorem B. *The notations being as above, let $s \in \mathbf{C}$ be such that $\sigma = \mathrm{Re}(s) > 1/2$. Then the equality*

$$\mathrm{Avg}_\chi \Phi\left(\frac{L'}{L}(s, \chi)\right) = \int_{\mathbf{C}} M_\sigma(w)\Phi(w)|dw| \tag{2.3.1}$$

holds for any continuous function Φ on $\mathbf{C}$ with at most polynomial growth. In particular, the case $\Phi(w) = \psi_z(w) = \exp(i\mathrm{Re}(\bar{z}w))$ gives

$$\mathrm{Avg}_\chi \psi_z\left(\frac{L'}{L}(s, \chi)\right) = \tilde{M}_\sigma(z) \tag{2.3.2}$$

for any $\sigma > 1/2$ and $z \in \mathbf{C}$. Finally, the equality (2.3.1) holds also when Φ is the characteristic function of either a compact subset of $\mathbf{C}$ or the complement of such a subset.

Remarks 2.3.3. (i) In [1]§6 Theorem 7, our assumptions on σ and Φ were both more restrictive. The present improvement is in a sense along the line suggested in *loc.cit.* Remark 6.5.20. But it went beyond this; we shall not even need Fourier analysis developed in *loc.cit.* Chap. 5. With Theorem A at hand, it suffices to continue the naive argument of *loc.cit.* Chap. 4. We should add, however, that this stronger argument works only in the function field case where we can use the Weil Riemann Hypothesis for function fields. Another point to be added is that the result of [1]Theorem 7(iii), which dealt with a special case $\Phi(z) = \bar{z}^a z^b$ (for $\sigma > 1/2$), will be needed as a basis of the proof of the present Theorem B.

(ii) Theorem B does not hold when Φ is the characteristic function of an arbitrary measurable subset A of $\mathbf{C}$. Indeed, for each fixed s, the set $\{L'/L(s, \chi)\}_\chi$ is countable, and if we take as Φ the characteristic function of this set, then the left hand side of (2.3.1) is 1 while the right hand side is 0.

[2] In [1], we used a less traditional notation and wrote as $L(\chi, s)$, $L_P(\chi, s)$.

Corollary 2.3.4. *Fix $s \in \mathbf{C}$ such that $1/2 < \mathrm{Re}(s) \le 1$. Then* (i) *the point set*

$$\{\frac{L'}{L}(s,\chi)\}_\chi \tag{2.3.5}$$

is everywhere dense in $\mathbf{C}$; (ii) *the convergence*

$$\frac{L'_{P_y}}{L_{P_y}}(s,\chi) \to \frac{L'}{L}(s,\chi) \qquad (y \to \infty) \tag{2.3.6}$$

is not uniform in χ.

Proof (i) By Theorem B, it suffices to show that when $1/2 < \sigma = \mathrm{Re}(s) \le 1$,

$$\int_{|z-z_0|\le r} M_\sigma(z)|dz| > 0 \tag{2.3.7}$$

holds for any $z_0 \in \mathbf{C}$ and $r > 0$, or equivalently, that the spectrum of the measure $M_\sigma(z)|dz|$ is the whole complex plane.[3] Now, with the notations of [1]§2, $M_{\sigma,P_y}(z)$ converges uniformly to $M_\sigma(z)$ (*ibid.* Theorem 2); hence the general argument in [2] Theorem 3 shows that this spectrum is equal to the set-theoretic limit of the spectrum of $M_{\sigma,P_y}(z)|dz|$. By [1]§2.1, the latter consists of all those points of $\mathbf{C}$ that can be expressed as a sum over $\mathfrak{p} \in P_y$ of points on the circle $|z - c_{\sigma,\mathfrak{p}}| = r_{\sigma,\mathfrak{p}}$, where $c_{\sigma,\mathfrak{p}} = -(\log N(\mathfrak{p}))/(N(\mathfrak{p})^{2\sigma} - 1)$ and $r_{\sigma,\mathfrak{p}} = N(\mathfrak{p})^\sigma|c_{\sigma,\mathfrak{p}}|$. Since $\sum_\mathfrak{p} r_{\sigma,\mathfrak{p}} = \infty$ for $\sigma \le 1$ (and $\sum_\mathfrak{p} c_{\sigma,\mathfrak{p}} < \infty$ for $\sigma > 1/2$), this limit set must be the whole complex plane. This settles the proof of (i).

(ii) In particular, $|L'/L(s,\chi)|$ is unbounded. But since $|L'_{P_y}/L_{P_y}(s,\chi)|$ for each fixed y (and s) is bounded, the difference

$$|\frac{L'_{P_y}}{L_{P_y}}(s,\chi) - \frac{L'}{L}(s,\chi)|$$

is unbounded. In particular, the convergence (2.3.6) cannot be uniform in χ. □

To establish the validity of the log-case analogues of Theorem B and Corollary 2.3.4, it "only" remains to carry out constructions and establish main properties of the "M-functions" for the log-case, which will be done in a forthcoming paper.

[3] We can actually show, by the same argument as in [2](Remark after Theorem 9), a slightly stronger result that when $1/2 < \sigma \le 1$, the *support* of $M_\sigma(z)$ is also the whole complex plane. But this is not needed here.

§ 3. Proof of Theorem A

§ 3.1. The integral expression.

Let $\chi \in \hat{G}_{\mathbf{f}} \setminus \{\chi_0\}$ and $y = q^m$ $(m \in \mathbf{N})$. Recall that $g(s, \chi, \mathbf{f}, y)$ denotes either one of

$$f'(s,\chi,\mathbf{f},y) = \frac{L'}{L}(s,\chi,\mathbf{f}) - \frac{L'_{P_y}}{L_{P_y}}(s,\chi,\mathbf{f}) \qquad \text{(Case 1)}, \tag{3.1.1}$$

$$f(s,\chi,\mathbf{f},y) = \log L(s,\chi,\mathbf{f}) - \log L_{P_y}(s,\chi,\mathbf{f}) \qquad \text{(Case 2)}. \tag{3.1.2}$$

In each case, $g(s, \chi, \mathbf{f}, y)$ is a holomorphic function on $\mathrm{Re}(s) > 1/2$. And being a function of q^{-s}, it is vertically periodic.

Now, when $\mathrm{Re}(s) > 1$, we obtain directly from the absolutely convergent Euler product expansions (2.1.1) for $L(s, \chi, \mathbf{f})$ and (2.1.3) for $L_{P_y}(s, \chi, \mathbf{f})$ (and from our choice of the branches of their logarithms), the following absolutely convergent Dirichlet series expansions; first,

$$f(s,\chi,\mathbf{f},y) = \sum_{\substack{N(\mathfrak{p})>y,\, \mathfrak{p}\neq\mathfrak{p}_\infty \\ r\geq 1}} \frac{\chi(\mathfrak{p}^r)}{rN(\mathfrak{p}^r)^s},$$

and then, by differentiation,

$$f'(s,\chi,\mathbf{f},y) = \sum_{\substack{N(\mathfrak{p})>y,\, \mathfrak{p}\neq\mathfrak{p}_\infty \\ r\geq 1}} \frac{-\chi(\mathfrak{p}^r)\log N(\mathfrak{p})}{N(\mathfrak{p}^r)^s}.$$

Rewrite these expansions in the form

$$g(s,\chi,\mathbf{f},y) = \sum_D \chi(D)\alpha(D,y)N(D)^{-s} \qquad (\mathrm{Re}(s) > 1), \tag{3.1.3}$$

where D runs only over the integral divisors $\neq (1)$ of K such that $(D, \mathbf{f}) = 1$, and

$$\alpha(D,y) = \begin{cases} -\log N(\mathfrak{p}) & \text{(Case 1)}, \\ 1/r & \text{(Case 2)}, \end{cases} \tag{3.1.4}$$

when D is of the form $D = \mathfrak{p}^r$ $(\mathfrak{p} \neq \mathfrak{p}_\infty,\ N(\mathfrak{p}) > y,\ \ r \geq 1)$, and $\alpha(D, y) = 0$ otherwise. Note that

$$\alpha(D,y) = 0 \qquad (\text{if } N(D) \leq y). \tag{3.1.5}$$

Note also that the series (3.1.3) is absolutely convergent on $\mathrm{Re}(s) > 1$, while if we collect all terms with the same norm $N(D)$, the series thus obtained, which is a power series of q^{-s}, is absolutely convergent on $\mathrm{Re}(s) > 1/2$, being holomorphic on $|q^{-s}| < q^{-1/2}$.

Now let $X \geq 1$ be a real parameter to be fixed later.

Proposition 3.1.6. (i) *On the domain* $\mathrm{Re}(s) \geq 1/2 + \epsilon$*, one can express* $g(s, \chi, \mathbf{f}, y)$ *as the difference*

$$g(s, \chi, \mathbf{f}, y) = Int_+ - Int_- \tag{3.1.7}$$

of two holomorphic functions

$$Int_+ = Int_+(s, \chi, \mathbf{f}, y, X) = \frac{1}{2\pi i} \int_{\mathrm{Re}(w)=c} \Gamma(w) g(s+w, \chi, \mathbf{f}, y) X^w dw, \tag{3.1.8}$$

where c is any positive real number satisfying $c > \mathrm{Max}(0, 1-\sigma)$*, and*

$$Int_- = Int_-(s, \chi, \mathbf{f}, y, X) = \frac{1}{2\pi i} \int_{\mathrm{Re}(w)=-\epsilon/2} \Gamma(w) g(s+w, \chi, \mathbf{f}, y) X^w dw. \tag{3.1.9}$$

(ii) *Int_+ has a Dirichlet series expansion*

$$Int_+ = \sum_D \chi(D) \alpha(D, y) \exp(-\frac{N(D)}{X}) N(D)^{-s} \tag{3.1.10}$$

over the integral ideals D, which is absolutely convergent for any $\chi \in \hat{G}_{\mathbf{f}}$ and any $s \in \mathbf{C}$.

Proof First, we claim that

$$g(s, \chi, \mathbf{f}, y) = \frac{1}{2\pi i} \int_B \Gamma(w) g(s+w, \chi, \mathbf{f}, y) X^w dw, \tag{3.1.11}$$

where B is the positively oriented rectangle bordering

$$-\epsilon/2 \leq \mathrm{Re}(w) \leq c, \qquad |\mathrm{Im}(w)| \leq T \tag{3.1.12}$$

($T > 0$). This is clear, because the integrand is holomorphic in w on (3.1.12) except for a simple pole at $w = 0$ with the residue $g(s, \chi, \mathbf{f}, y)$. (In fact, since $\epsilon < 1/2$, the only pole of $\Gamma(w)$ on (3.1.12) is $w = 0$, and since $\mathrm{Re}(s+w) \geq \mathrm{Re}(s) - \epsilon/2 \geq 1/2 + \epsilon/2 > 1/2$, $g(s+w, \chi, \mathbf{f}, y)$ is holomorphic on (3.1.12).)

To prove (i), let us estimate the integrand on $-\epsilon/2 \leq \mathrm{Re}(w) \leq c$; $|\mathrm{Im}(w)| \geq T$. First, $|X^w| \leq X^c$ (because $X \geq 1$); secondly, $g(s+w, \chi, \mathbf{f}, y)$ is holomorphic and vertically periodic, hence bounded; thirdly,

$$|\Gamma(w)| \ll |\mathrm{Im}(w)|^{c-1/2} \exp(-\frac{\pi}{2}|\mathrm{Im}(w)|)$$

for $|\mathrm{Im}(w)| \geq 1$. Now (i) follows directly from these by letting $T \to \infty$ in (3.1.11).

(ii) By (3.1.3), the Dirichlet series expansion

$$g(s+w,\chi,\mathbf{f},y) = \sum_D \chi(D)\alpha(D,y)N(D)^{-s-w} \tag{3.1.13}$$

is absolutely convergent on $\mathrm{Re}(w) = c$, and the convergence is uniform with respect to $\mathrm{Im}(w)$ (note here that $\sigma + c > 1$). Therefore,

$$\begin{aligned} Int_+ &= \frac{1}{2\pi i}\int_{\mathrm{Re}(w)=c} \Gamma(w)\left(\sum_D \chi(D)\alpha(D,y)N(D)^{-s-w}\right) X^w dw \\ &= \sum_D \chi(D)\alpha(D,y)N(D)^{-s}\left(\frac{1}{2\pi i}\int_{\mathrm{Re}(w)=c}\Gamma(w)N(D)^{-w}X^w dw\right). \end{aligned} \tag{3.1.14}$$

But since

$$\frac{1}{2\pi i}\int_{\mathrm{Re}(u)=c}\Gamma(u)a^{-u}du = e^{-a} \qquad (a,\, c>0), \tag{3.1.15}$$

we obtain the desired Dirichlet series expansion (3.1.10). Because of the exponential factor, this converges absolutely for any $s \in \mathbf{C}$ and any $\chi \in \hat{G}_{\mathbf{f}}$. This can be seen easily by noting that $\alpha(D,y) \ll \log N(D)$, and that the number of D with $N(D) = q^n$ is $\ll q^n$ (cf. §5.1). □

We are going to estimate

$$\mathrm{Avg}_{\substack{\chi\in\hat{G}_{\mathbf{f}}\\ \chi\neq\chi_0}} |g(s,\chi,\mathbf{f},y)|^{2k}$$

by estimating each of

$$\mathrm{Avg}_{\substack{\chi\in\hat{G}_{\mathbf{f}}\\ \chi\neq\chi_0}} |Int_-|^{2k}, \qquad \mathrm{Avg}_{\substack{\chi\in\hat{G}_{\mathbf{f}}\\ \chi\neq\chi_0}} |Int_+|^{2k}.$$

As for the former, in our function field case where the Weil Riemann Hypothesis is valid, we do not need to average over χ but a direct estimation of $|Int_-|$ for each χ by using Proposition 2.1.7(i) will suffice. As for the latter, we shall use Proposition 3.1.6(ii) and the orthogonality relation for characters.

As for the choice of the parameter X, the larger (resp. smaller) the better as regards the estimation of the former (resp. the latter). The choice $X = N(\mathbf{f})^\beta$, with $\beta > 0$ will suffice for the former, and with $\beta < 1/2k$ for the latter, as we shall see.

§ 3.2. Estimation of $|Int_-|$.

In what follows, we shall write

$$(3.2.1) \qquad \ell(y) = \begin{cases} \log y & \text{(Case 1)}, \\ 1 & \text{(Case 2)}. \end{cases}$$

Lemma 3.2.2. *Let $\sigma = \mathrm{Re}(s) \geq 1/2 + \epsilon$. Then*

$$(3.2.3) \qquad |Int_-| \ll X^{-\epsilon/2}(\log N(\mathbf{f}))y^{\frac{1+\epsilon}{2}-\sigma}(\log y)^{-1}\ell(y).$$

Proof By definition,

$$(3.2.4) \qquad Int_- = \frac{1}{2\pi i}\int_{\mathrm{Re}(w)=-\epsilon/2} \Gamma(w)g(s+w,\chi,\mathbf{f},y)X^w dw.$$

But when $\mathrm{Re}(w) = -\epsilon/2$,

$$(3.2.5) \qquad \Gamma(w) \ll \begin{cases} \exp(-\frac{\pi}{2}|\mathrm{Im}(w)|) & (|\mathrm{Im}(w)| \geq 1), \\ 1 & (|\mathrm{Im}(w)| \leq 1). \end{cases}$$

Hence

$$(3.2.6) \qquad \int_{\mathrm{Re}(w)=-\epsilon/2} |\Gamma(w)|dw \ll 1.$$

As for $g(s+w,\chi,\mathbf{f},y)$, since $\mathrm{Re}(s+w) = \sigma - \epsilon/2$ ($\geq (1+\epsilon)/2$), by Proposition 2.1.7 (i) (to be proved below) we have

$$(3.2.7) \qquad |g(s+w,\chi,\mathbf{f},y)| \ll (\log N(\mathbf{f}))y^{\frac{1+\epsilon}{2}-\sigma}(\log y)^{-1}\ell(y).$$

So, Lemma 3.2.2 is reduced to Proposition 2.1.7 (i).

Proof of Proposition 2.1.7 (i) (Case 1) Let $\chi^* \in \hat{G}_{\mathbf{f}_\chi}$ be the primitive character associated with χ. By [1] Lemma 6.5.2, we have

$$(3.2.8) \qquad |f'(s,\chi^*,\mathbf{f}_\chi,y)| \ll_\epsilon (\log N(\mathbf{f}_\chi)+1)y^{1/2-\sigma} \ll (\log N(\mathbf{f}))y^{1/2-\sigma}.$$

(In fact, when $N(\mathfrak{p}_\infty) \leq y$, the left hand side of [1](6.5.4) is equal to that of (3.2.8). When $N(\mathfrak{p}_\infty) > y$, their difference is $\ll (\log N(\mathfrak{p}_\infty))N(\mathfrak{p}_\infty)^{-\sigma} \ll N(\mathfrak{p}_\infty)^{-\sigma} \ll y^{-\sigma}$.) So, it suffices to prove that the difference $|f'(s,\chi,\mathbf{f},y) - f'(s,\chi^*,\mathbf{f}_\chi,y)|$ is also bounded by the quantity on the right most side of (3.2.8). But by definition,

$$(3.2.9) \qquad f'(s,\chi,\mathbf{f},y) - f'(s,\chi^*,\mathbf{f}_\chi,y) = \sum_{\substack{\mathfrak{p}|\mathbf{f},\,\nmid\mathbf{f}_\chi \\ N(\mathfrak{p})>y}} \frac{\chi^*(\mathfrak{p})\log N(\mathfrak{p})}{N(\mathfrak{p})^s - \chi^*(\mathfrak{p})}.$$

(Primarily, this equality is for $\mathrm{Re}(s) > 1$, but the right hand side being a finite sum and hence holomorphic on $\mathrm{Re}(s) > 0$, this must hold on $\mathrm{Re}(s) > 1/2$.) Therefore,

$$|f'(s,\chi,\mathbf{f},y) - f'(s,\chi^*,\mathbf{f}_\chi,y)| \le \sum_{\substack{\mathfrak{p}|\mathbf{f},\,\nmid\mathbf{f}_\chi \\ N(\mathfrak{p})>y}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^\sigma - 1} \ll \sum_{\mathfrak{p}|\mathbf{f}, N(\mathfrak{p})>y} N(\mathfrak{p})^{1/2-\sigma}$$
$$\ll y^{1/2-\sigma} \sum_{\mathfrak{p}|\mathbf{f}} 1 \ll (\log N(\mathbf{f}))y^{1/2-\sigma},$$

the last $\ll$ being by e.g. [1] Sublemma 3.10.5. This settles Case 1.

(Case 2) This case follows directly from Case 1 by integration. In fact,

$$f(s,\chi,\mathbf{f},y) = \int_\infty^s f'(s,\chi,\mathbf{f},y)ds = -\int_0^\infty f'(s+u,\chi,\mathbf{f},y)du; \tag{3.2.10}$$

hence

$$|f(s,\chi,\mathbf{f},y)| \le \int_0^\infty |f'(s+u,\chi,\mathbf{f},y)|du$$
$$\ll (\log N(\mathbf{f}))y^{1/2-\sigma} \int_0^\infty y^{-u}du = \frac{(\log N(\mathbf{f}))y^{1/2-\sigma}}{\log y},$$

as desired.

(ii)(Case 1) For $\sigma \ge 1+\epsilon$,

$$|f'(s,\chi,\mathbf{f},y)| \le \sum_{N(\mathfrak{p})>y} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^\sigma - 1} \ll \int_y^\infty y^{-\sigma}dy = \frac{y^{1-\sigma}}{\sigma-1} \ll_\epsilon y^{1-\sigma},$$

as desired. (As for the justification of the estimation using the integral, which is standard in the number field case but may not be so in the function field case, use §5.2(5.2.7).)

(Case 2) This follows from Case 1 in the same manner as in (i). □

§ 3.3. Estimation of $\mathrm{Avg}\,|Int_+|^{2k}$.

We are going to prove the following

Lemma 3.3.1. *Let $\sigma = \mathrm{Re}(s) \ge 1/2+\epsilon$. Then*

$$\mathrm{Avg}_{\substack{\chi\in\hat{G}_\mathbf{f} \\ \chi\neq\chi_0}} |Int_+|^{2k} \ll \left((qy)^{(1-2\sigma)k} + (\log N(\mathbf{f}))N(\mathbf{f})^{-1}y^{-2k\sigma}X^{2k}\right)\ell(y)^{2k}. \tag{3.3.2}$$

This proof will be carried through in §3.3-3.5. First, recall (Proposition 3.1.6 (ii)):

$$Int_+ = Int_+(s,\chi,\mathbf{f},y,X) = \sum_D \chi(D)\alpha(D,y)\exp(-\frac{N(D)}{X})N(D)^{-s}, \tag{3.3.3}$$

which is absolutely convergent for any $\chi \in \hat{G}_{\mathbf{f}}$ and any $s \in \mathbf{C}$. *Define* $Int_+(s,\chi,\mathbf{f},y,X)$ *also for* $\chi = \chi_0$ *by this series.* First, let us consider the average over all $\chi \in \hat{G}_{\mathbf{f}}$ including χ_0. Then the orthogonality relation for characters gives directly:

$$S := \mathrm{Avg}_{\chi \in \hat{G}_{\mathbf{f}}} |Int_+(s,\chi,\mathbf{f},y,X)|^{2k} = \sum_{c \in G_{\mathbf{f}}} \Big| \sum_{\substack{(D,\mathbf{f})=1 \\ i_{\mathbf{f}}(D)=c}} A_k(D,y) N(D)^{-s} \Big|^2, \tag{3.3.4}$$

where

$$A_k(D,y) = \sum_{D = D_1 \cdots D_k} \alpha(D_1,y) \cdots \alpha(D_k,y) \exp\left(-\frac{N(D_1) + \cdots + N(D_k)}{X}\right). \tag{3.3.5}$$

Sublemma 3.3.6. *Put*

$$\alpha_k(D,y) = \sum_{D = D_1 \cdots D_k} |\alpha(D_1,y) \cdots \alpha(D_k,y)|. \tag{3.3.7}$$

Then

(i)

$$|A_k(D,y)| \le \alpha_k(D,y) \exp\left(-\frac{kN(D)^{1/k}}{X}\right). \tag{3.3.8}$$

(ii) $\alpha_k(D,y) = 0$ *if* $N(D) < (qy)^k$, *and for general* D,

$$\alpha_k(D,y) \ll \begin{cases} (\log N(D))^k & \text{(Case 1)}, \\ 1 & \text{(Case 2)}. \end{cases} \tag{3.3.9}$$

Proof (i) Since the arithmetic mean is no less than the geometric mean, we have $\sum_{i=1}^k N(D_i) \ge kN(D)^{1/k}$; hence (i) is obvious.

(ii) The first statement is because if $N(D) < (qy)^k$ and $D = D_1 \ldots D_k$ then $N(D_i) < qy$ for at least one i, but since y is an integral power of q this means $N(D_i) \le y$; hence $\alpha(D_i,y) = 0$ by (3.1.5). The inequality (3.3.9) for Case 1 is given in [1] §3.8. In Case 2, let $D = \prod_{i=1}^h \mathfrak{p}_i^{n_i}$ be the prime factorization. We may assume that $h \le k$ and that $N(\mathfrak{p}_i) > y$ for all i, for otherwise $\alpha_k(D,y) = 0$. Then, by definition, $\alpha_k(D,y)$ is nothing but the coefficient of $\prod_{i=1}^h x_i^{n_i}$ in the power series

$$\Big(-\sum_{i=1}^h \log(1 - x_i)\Big)^k \tag{3.3.10}$$

on h independent variables $x_1, ..., x_h$. Since k is fixed, the number of possible values of h is limited. So, it suffices to see that for each $k \geq 1$ the coefficients in the power series

$$(\sum_{n=1}^{\infty} \frac{x^n}{n})^k \tag{3.3.11}$$

are bounded. But since

$$\sum_{\substack{\mu,\nu\geq 1\\ \mu+\nu=n}} (\mu\nu)^{-1} = \frac{2}{n}\sum_{\mu=1}^{n-1} \mu^{-1} < \frac{2}{n}(\log n + 1),$$

(as is shown in [4][4]) it follows directly by induction on $k \geq 1$ that the coefficient of x^n in (3.3.11) is $\leq (2\log n + 2)^{k-1}/n \ll_k 1$. □

Now rewrite (3.3.4) as

$$S = \sum_{c\in G_{\mathbf{f}}} \Bigg| \sum_{\substack{i_{\mathbf{f}}(D)=c\\ N(D)<N(\mathbf{f})}} A_k(D,y)N(D)^{-s} + \sum_{\substack{i_{\mathbf{f}}(D)=c\\ N(D)\geq N(\mathbf{f})}} A_k(D,y)N(D)^{-s}\Bigg|^2. \tag{3.3.12}$$

Here and in what follows, in order to simplify indications under the summation sign, we shall omit writing $(D, \mathbf{f}) = 1$ when the other conditions include "$i_{\mathbf{f}}(D) = c$". The former is considered automatic under the latter. Now, in (3.3.12), the first inner sum over $\{D;\ i_{\mathbf{f}}(D) = c, N(D) < N(\mathbf{f})\}$ has *at most one* term $A_k(D_c, y)N(D_c)^{-s}$ by Proposition 3.3.16(iii) below. Here, when such a term exists for a given class c (c: *small* in the sense of [1]§6.8), D_c denotes the unique integral divisor satisfying $i_{\mathbf{f}}(D_c) = c$ and $N(D_c) < N(\mathbf{f})$. This gives

$$S \leq 2(S_1 + S_2), \tag{3.3.13}$$

with

$$S_1 = \sum_{c:\,small} |A_k(D_c, y)|^2 N(D_c)^{-2\sigma} = \sum_{N(D)<N(\mathbf{f})} |A_k(D, y)|^2 N(D)^{-2\sigma}, \tag{3.3.14}$$

$$S_2 = \sum_{c\in G_{\mathbf{f}}} (\sum_{\substack{i_{\mathbf{f}}(D)=c\\ N(D)\geq N(\mathbf{f})}} |A_k(D, y)|N(D)^{-\sigma})^2. \tag{3.3.15}$$

We shall estimate S_1, S_2 separately, using Sublemma 3.3.6 and the following

[4] Incidentally, or rather, accidentally, the same inequality was used in [4] for a different purpose.

Proposition 3.3.16. *Let n be any positive integer. Then:*
(i) *The number of integral divisors D of K with $N(D) \leq q^n$ is $\mathbf{O}_K(q^n)$.*
(ii) *Let c be any fixed element of $G_{\mathbf{f}}$. Then the number of integral divisors D satisfying $N(D) = q^n$ and $i_{\mathbf{f}}(D) = c$ cannot exceed* $\mathrm{Max}(1, q^{n+1}/N(\mathbf{f}))$.
(iii) *There is at most one integral divisor D coprime with $\mathfrak{p}_\infty$ satisfying $i_{\mathbf{f}}(D) = c$ and $N(D) < N(\mathbf{f})$.*

The proof will be given in the Appendix. We shall also need the formula for the cardinality of $G_{\mathbf{f}}$:

$$|G_{\mathbf{f}}| = \deg(\mathfrak{p}_\infty) h_K \frac{N(\mathbf{f})}{q-1} \prod_{\mathfrak{p}|\mathbf{f}} \left(1 - \frac{1}{N(\mathfrak{p})}\right) \tag{3.3.17}$$

(h_K: the class number of K), and its consequence

$$\frac{N(\mathbf{f})}{\log N(\mathbf{f})} \ll |G_{\mathbf{f}}| \ll N(\mathbf{f}). \tag{3.3.18}$$

(As regards (3.3.17), the product of the first two factors on the right hand side gives the index of the subgroup of $G_{\mathbf{f}}$ represented by principal divisors, and the rest gives the index of the multiplicative group $\mathbf{F}_q^\times \langle \alpha \equiv 1 (\mathrm{mod}\, \mathbf{f}) \rangle$ in the group of all elements of $K^\times$ that are coprime with $\mathbf{f}$. As for the estimations (3.3.18), the second $\ll$ is obvious, because we have fixed K and $\mathfrak{p}_\infty$; the first follows from the estimation

$$\prod_{N(\mathfrak{p}) \leq y} \left(1 - \frac{1}{N(\mathfrak{p})}\right)^{-1} \ll \log y, \tag{3.3.19}$$

which is standard at least in the number field case (see (5.2.4) below)).

§ 3.4. Estimations of S_1, S_2.

Estimation of S_1. By the definition of S_1 and by Sublemma 3.3.6, we obtain a simplified bound

$$S_1 \leq \sum_D \alpha_k(D, y)^2 N(D)^{-2\sigma}, \tag{3.4.1}$$

irrelevant of $N(\mathbf{f})$ and X. (This may look "too rough", because what characterized the partial sum S_1 was the condition $N(D) < N(\mathbf{f})$. But once we have used the strong "at most one term" property mentioned above, what remains is only to drop the condition $N(D) < N(\mathbf{f})$ in order to obtain an estimation independent of $\mathbf{f}$. Also, X is irrelevant here. We only use $\exp(-kN(D)^{1/k}/X) < 1$ to derive $|A_k(D, y)| \leq \alpha_k(D, y)$.)

Therefore, by putting $N(D) = q^n$ and using Proposition 3.3.16(i) and Sublemma 3.3.6 (ii), we obtain

$$S_1 \ll_k \sum_{q^n \geq (qy)^k} \ell_n^{2k} q^{(1-2\sigma)n}, \tag{3.4.2}$$

where $\ell_n = n$ (Case 1), $= 1$ (Case 2). From this follows easily that

$$S_1 \ll_{k,\epsilon} (qy)^{(1-2\sigma)k} \ell(y)^{2k}. \tag{3.4.3}$$

Indeed, if we write $(qy)^k = q^N$, the right hand side of (3.4.2) is

$$\begin{aligned} \ell_N^{2k} q^{(1-2\sigma)N} \sum_{i=0}^{\infty} (\ell_{N+i}/\ell_N)^{2k} q^{(1-2\sigma)i} &\leq \ell_N^{2k} q^{(1-2\sigma)N} \sum_{i=0}^{\infty} (1+i)^{2k} q^{-2\epsilon i} \\ &\ll_{k,\epsilon} \ell(y)^{2k} (qy)^{(1-2\sigma)k}. \end{aligned}$$

Estimation of S_2. We shall first estimate the quantity

$$S'_c = \sum_{\substack{i_{\mathbf{f}}(D)=c \\ N(D) \geq N(\mathbf{f})}} |A_k(D,y)| N(D)^{-\sigma} \tag{3.4.4}$$

for each $c \in G_{\mathbf{f}}$. If we write $N(D) = q^n$, then $A_k(D,y) = 0$ for $q^n < (qy)^k$, and $|A_k(D,y)| \ll \ell_n^k \exp(-kq^{n/k}/X)$ for any n, by Sublemma 3.3.6. By Proposition 3.3.16(ii), the number of D satisfying both $N(D) = q^n$ and $i_{\mathbf{f}}(D) = c$ is $\ll q^n/N(\mathbf{f})$. Therefore,

$$S'_c \ll N(\mathbf{f})^{-1} S', \tag{3.4.5}$$

where

$$\begin{aligned} S' &= \sum_{q^n \geq (qy)^k} q^n \ell_n^k \exp(-kq^{n/k}/X) q^{-n\sigma} \\ &\ll \sum_{q^n \geq (qy)^k} (q^n - q^{n-1}) q^{-n\sigma} \exp(-kq^{n/k}/X) \ell_n^k \\ &\ll \sum_{q^n \geq (qy)^k} \int_{q^{n-1}}^{q^n} t^{-\sigma} \exp(-kt^{1/k}/X) \ell(t)^k dt \\ &\leq \int_{y^k}^{\infty} t^{-\sigma} \exp(-kt^{1/k}/X) \ell(t)^k dt, \end{aligned} \tag{3.4.6}$$

where, as before, $\ell(t) = \log t$ (Case 1), $= 1$ (Case 2). Now we shall show that

$$t^{-\sigma} \ell(t)^k \ll y^{-k\sigma} \ell(y)^k \qquad (t \geq y^k). \tag{3.4.7}$$

In Case 2 where $\ell(t) = 1$, this is obvious. In Case 1 where $\ell(t) = \log t$, the derivative of $t^{-\sigma}\ell(t)^k$ is $(k - \sigma \log t)(\log t)^{k-1}t^{-\sigma-1}$, and at the zero of this derivative, the value of $t^{-\sigma}\ell(t)^k$ is $e^{-k}(k/\sigma)^k$. Therefore, when $\log(y^\sigma) \geq 1$, $t^{-\sigma}\ell(t)^k$ is monotone decreasing on $t \geq y^k$, and hence (3.4.7) holds. When $\log(y^\sigma) < 1$, then the maximal possible value of $t^{-\sigma}\ell(t)^k$ is $e^{-k}(k/\sigma)^k \ll 1$, while in this case $y^{-k\sigma}\ell(y)^k > e^{-k}\ell(y)^k \geq (e^{-1}\log q)^k \gg 1$. Therefore, (3.4.7) holds in all cases.

Therefore,

$$S' \ll y^{-k\sigma}\ell(y)^k \int_0^\infty \exp(-kt^{1/k}/X)dt. \tag{3.4.8}$$

But since the integral in (3.4.8) is $k^{1-k}\Gamma(k)X^k \ll X^k$, we obtain

$$S' \ll y^{-k\sigma}\ell(y)^k X^k. \tag{3.4.9}$$

Therefore,

$$\begin{aligned} S_2 = \sum_{c \in G_{\mathbf{f}}} (S'_c)^2 &\leq |G_{\mathbf{f}}|(N(\mathbf{f})^{-1}S')^2 \\ &\ll N(\mathbf{f})^{-1}S'^2 \ll N(\mathbf{f})^{-1}y^{-2k\sigma}X^{2k}\ell(y)^{2k}. \end{aligned} \tag{3.4.10}$$

§ 3.5. Proof of Lemma 3.3.1.

Now by (3.3.13),(3.4.3),(3.4.10), we obtain

$$S := \mathrm{Avg}_{\chi \in \hat{G}_{\mathbf{f}}} |Int_+|^{2k} \ll \left((qy)^{(1-2\sigma)k} + N(\mathbf{f})^{-1}y^{-2k\sigma}X^{2k}\right)\ell(y)^{2k}. \tag{3.5.1}$$

So, it remains to verify that

$$\begin{aligned} \Delta := \mathrm{Avg}_{\substack{\chi \in \hat{G}_{\mathbf{f}} \\ \chi \neq \chi_0}} |Int_+|^{2k} - \mathrm{Avg}_{\chi \in \hat{G}_{\mathbf{f}}} |Int_+|^{2k} \\ \ll (\log N(\mathbf{f}))N(\mathbf{f})^{-1}y^{-2k\sigma}X^{2k}\ell(y)^{2k}. \end{aligned} \tag{3.5.2}$$

This $(\log N(\mathbf{f}))$-factor comes from the possible difference between $N(\mathbf{f})$ and $|G_{\mathbf{f}}|$ when $\mathbf{f}$ contains many prime factors. To check (3.5.2), note first that

$$\Delta \ll |G_{\mathbf{f}}|^{-1}\mathrm{Max}_{\chi \in \hat{G}_{\mathbf{f}}} |Int_+|^{2k}. \tag{3.5.3}$$

This and (3.3.18) give

$$\Delta \ll (\log N(\mathbf{f}))N(\mathbf{f})^{-1}\mathrm{Max}_{\chi \in \hat{G}_{\mathbf{f}}} |Int_+|^{2k}. \tag{3.5.4}$$

Hence it remains to prove

$$|Int_+| \ll y^{-\sigma}X \cdot \ell(y). \tag{3.5.5}$$

But by Propositions 3.1.6(ii), 3.3.16(i) and by Sublemma 3.3.6 (for $k=1$), we have

$$|Int_+| \le \sum_D |\alpha(D,y)| \exp(-N(D)/X) N(D)^{-\sigma} \ll \sum_{q^n \ge qy} \ell_n q^{n-n\sigma} \exp(-q^n/X). \tag{3.5.6}$$

This last quantity is nothing but S' for $k=1$; hence (3.4.9) gives (3.5.5). This settles the proof of Lemma 3.3.1.

§ 3.6. The final stage.

Finally, since $|g(s,\chi,\mathbf{f},y)|^{2k} = |Int_+ - Int_-|^{2k} \ll_k |Int_+|^{2k} + |Int_-|^{2k}$, we obtain from Lemmas 3.2.2, 3.3.1,

$$\mathrm{Avg}_{\substack{\chi\in\hat{G}_{\mathbf{f}}\\ \chi\neq\chi_0}} |g(s,\chi,\mathbf{f},y)|^{2k} \ll (I + II + III) \times \ell(y)^{2k}, \tag{3.6.1}$$

where

$$\begin{cases} I = \left(X^{-\epsilon}(\log N(\mathbf{f}))^2 y^{1+\epsilon-2\sigma}(\log y)^{-2}\right)^k; \\ II = (qy)^{(1-2\sigma)k}; \\ III = (\log N(\mathbf{f}))N(\mathbf{f})^{-1} y^{-2k\sigma} X^{2k}. \end{cases} \tag{3.6.2}$$

Now choose X by the equality

$$X^{2k+\epsilon} = N(\mathbf{f}). \tag{3.6.3}$$

Then, clearly, $I, II \ll y^{(1+\epsilon-2\sigma)k}$, and

$$III = (\log N(\mathbf{f}))N(\mathbf{f})^{-\epsilon/(2k+\epsilon)} y^{-2k\sigma} \ll y^{-2k\sigma} \ll y^{(1+\epsilon-2\sigma)k}. \tag{3.6.4}$$

Therefore,

$$\mathrm{Avg}_{\substack{\chi\in\hat{G}_{\mathbf{f}}\\ \chi\neq\chi_0}} |g(s,\chi,\mathbf{f},y)|^{2k} \ll \ell(y)^{2k} y^{(1+\epsilon-2\sigma)k}. \tag{3.6.5}$$

This settles the proof of Theorem A.

§ 4. Proof of Theorem B

§ 4.1.

We shall apply Theorem A for Case 1 to prove Theorem B.

First, consider the case where the test function Φ on $\mathbf{C}$ belongs to class C^1 (as a function of two real variables) and has a compact support. Then clearly,

$$\int |\Phi(z)||dz| < \infty, \tag{4.1.1}$$

$$|\Phi(z_1) - \Phi(z_2)| \ll |z_1 - z_2|. \tag{4.1.2}$$

(Here and in what follows, the integral will be over the whole complex plane $\mathbf{C}$ unless otherwise specified.) Now, an alternative version of Theorem A given in Remarks 2.2.4 (iii), for Case 1 for $k = 1$, and the Schwarz inequality, give

$$\lim_{y\to\infty} \mathrm{Avg}_{\substack{\chi\in\hat{G}_{\mathbf{f}}\\ \mathrm{f}_\chi=\mathrm{f}}} \left| \frac{L'_{P_y}}{L_{P_y}}(s,\chi) - \frac{L'}{L}(s,\chi)\right| = 0 \qquad \text{(uniformly in } \mathbf{f}\text{);} \tag{4.1.3}$$

hence by (4.1.2),

$$\lim_{y\to\infty} \mathrm{Avg}_{\substack{\chi\in\hat{G}_{\mathbf{f}}\\ \mathrm{f}_\chi=\mathrm{f}}} \Phi\left(\frac{L'_{P_y}}{L_{P_y}}(s,\chi)\right) = \mathrm{Avg}_{\substack{\chi\in\hat{G}_{\mathbf{f}}\\ \mathrm{f}_\chi=\mathrm{f}}} \Phi\left(\frac{L'}{L}(s,\chi)\right) \qquad \text{(uniformly in } \mathbf{f}\text{).} \tag{4.1.4}$$

Therefore, by the definition of $\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}$ ([1]§4.1), we also obtain immediately the uniform convergence in m; namely,

$$\lim_{y\to\infty} \mathrm{Avg}_{N(\mathbf{f}_\chi)\le m} \Phi\left(\frac{L'_{P_y}}{L_{P_y}}(s,\chi)\right) = \mathrm{Avg}_{N(\mathbf{f}_\chi)\le m} \Phi\left(\frac{L'}{L}(s,\chi)\right) \qquad \text{(uniformly in } m\text{).} \tag{4.1.5}$$

This is the first (and the main) point. Secondly, we already know ([1](4.4.3)) that

$$\lim_{m\to\infty} \mathrm{Avg}_{N(\mathbf{f}_\chi)\le m} \Phi\left(\frac{L'_{P_y}}{L_{P_y}}(s,\chi)\right) = \int M_{\sigma,P_y}(w)\Phi(w)|dw| \tag{4.1.6}$$

holds for each $y > 1$, and thirdly, since $\lim_{y\to\infty} M_{\sigma,P_y}(w) = M_\sigma(w)$ uniformly on $\mathbf{C}$ (*ibid* Theorem 2 (i)), and since Φ satisfies (4.1.1), we have

$$\lim_{y\to\infty} \int M_{\sigma,P_y}(w)\Phi(w)|dw| = \int M_\sigma(w)\Phi(w)|dw|. \tag{4.1.7}$$

Therefore,

$$\begin{aligned} \lim_{m\to\infty} \mathrm{Avg}_{N(\mathbf{f}_\chi)\le m} \Phi\left(\frac{L'}{L}(s,\chi)\right) &= \lim_{m\to\infty}\lim_{y\to\infty} \mathrm{Avg}_{N(\mathbf{f}_\chi)\le m} \Phi\left(\frac{L'_{P_y}}{L_{P_y}}(s,\chi)\right) \\ &= \lim_{y\to\infty}\lim_{m\to\infty} \mathrm{Avg}_{N(\mathbf{f}_\chi)\le m} \Phi\left(\frac{L'_{P_y}}{L_{P_y}}(s,\chi)\right) = \int M_\sigma(w)\Phi(w)|dw|, \end{aligned} \tag{4.1.8}$$

which settles the proof of Theorem B for this case.

§ 4.2.

Now we consider the case where Φ belongs to C^1 and has at most polynomial growth, i.e., when

$$|\Phi(z)| \ll |z|^k \qquad (|z| \geq 1) \tag{4.2.1}$$

holds for some $k \geq 1$. Let

$$1 = \sum_{r=1}^{\infty} E_r(z) \tag{4.2.2}$$

be a partition of unity by C^1-functions $E_r(z)$ on $\mathbf{C}$ satisfying $0 \leq E_r(z) \leq 1$ and

$$\mathrm{Supp}(E_r) \subseteq \{r-1 \leq |z| \leq r+1\}, \tag{4.2.3}$$

for the support of $E_r(z)$. (A word of caution: this expression may give an impression that the point $z = 0$ should lie on the boundary of $\mathrm{Supp}(E_1)$, but it is not; the condition for $r = 1$ is simply $\mathrm{Supp}(E_1) \subseteq \{|z| \leq 2\}$.) For any $R \in \mathbf{N}$, put

$$E^{(R)} = \sum_{r=1}^{R} E_r \qquad (\leq 1), \tag{4.2.4}$$

$$\Phi_r = \Phi \cdot E_r, \qquad \Phi^{(R)} = \sum_{r=1}^{R} \Phi_r = \Phi \cdot E^{(R)}. \tag{4.2.5}$$

Thus,

$$\Phi(z) = \sum_{r=1}^{\infty} \Phi_r(z) = \lim_{R\to\infty} \Phi^{(R)}(z). \tag{4.2.6}$$

Since each $\Phi^{(R)}(z)$ belongs to C^1 and is compactly supported, the result of §4.1 can be applied which gives

$$\lim_{m\to\infty} \mathrm{Avg}_{N(\mathbf{f}_\chi)\leq m} \Phi^{(R)}\left(\frac{L'}{L}(s,\chi)\right) = \int M_\sigma(z)\Phi^{(R)}(z)|dz| \tag{4.2.7}$$

for any $\sigma = \mathrm{Re}(s) > 1/2$. We claim now that

$$\lim_{R\to\infty} \int M_\sigma(z)\Phi^{(R)}(z)|dz| = \int M_\sigma(z)\Phi(z)|dz|. \tag{4.2.8}$$

To prove this, first note that $|\Phi^{(R)}(z)| \leq |\Phi(z)|$. Also,

$$\mathrm{Supp}(\Phi - \Phi^{(R)}) \subseteq \{z \in \mathbf{C}; |z| \geq R\}. \tag{4.2.9}$$

Therefore,

$$\left| \int M_\sigma(z)(\Phi(z) - \Phi^{(R)}(z))|dz| \right| \leq 2 \int_{|z| \geq R} M_\sigma(z)|\Phi(z)||dz| \ll \int_{|z| \geq R} M_\sigma(z)|z|^k |dz| \tag{4.2.10}$$

by our assumption (4.2.1). But since $M_\sigma(z) = O(|z|^{-N})$ for any N ([1] Theorem 2 (iii)), in particular for $N = k+3$, this tends to 0 as $R \mapsto \infty$. This proves (4.2.8).

Now we shall prove that

$$\lim_{R \to \infty} \mathrm{Avg}_{N(\mathbf{f}_\chi) \leq m} \Phi^{(R)} \left(\frac{L'}{L}(s, \chi) \right) = \mathrm{Avg}_{N(\mathbf{f}_\chi) \leq m} \Phi \left(\frac{L'}{L}(s, \chi) \right) \qquad \text{(uniformly in } m), \tag{4.2.11}$$

which, together with (4.2.7)(4.2.8), proves

$$\lim_{m \to \infty} \mathrm{Avg}_{N(\mathbf{f}_\chi) \leq m} \Phi \left(\frac{L'}{L}(s, \chi) \right) = \int M_\sigma(z)\Phi(z)|dz| \tag{4.2.12}$$

in the present case of Φ. (First apply $\lim_{m \to \infty}$ to (4.2.11), note that the order of two limits can be changed, then use (4.2.7), and then (4.2.8).)

To prove (4.2.11), observe first that (4.2.1) and (4.2.9) give

$$|\Phi(z) - \Phi^{(R)}(z)| \ll \mathrm{ch}^{(R)}(z)|z|^k, \tag{4.2.13}$$

where $\mathrm{ch}^{(R)}$ denotes the characteristic function of $|z| \geq R$. Moreover, we have

$$\mathrm{Avg}_{N(\mathbf{f}_\chi) \leq m} \mathrm{ch}^{(R)} \left(\frac{L'}{L}(s, \chi) \right) \ll R^{-2k} \qquad \text{(uniformly in } m). \tag{4.2.14}$$

In fact, since

$$\lim_{m \to \infty} \mathrm{Avg}_{N(\mathbf{f}_\chi) \leq m} \left| \frac{L'}{L}(s, \chi) \right|^{2k} = \mu_\sigma^{(k,k)} \ll 1 \tag{4.2.15}$$

([1] Theorem 7(iii) for $\sigma > 1/2$ (fixed) and $a = b = k$), we have

$$\mathrm{Avg}_{N(\mathbf{f}_\chi) \leq m} \left| \frac{L'}{L}(s, \chi) \right|^{2k} \ll 1; \qquad \text{(uniformly in } m). \tag{4.2.16}$$

On the other hand, we have the obvious inequality

$$\mathrm{ch}^{(R)} \left(\frac{L'}{L}(s, \chi) \right) R^{2k} \leq \left| \frac{L'}{L}(s, \chi) \right|^{2k}. \tag{4.2.17}$$

Therefore,

$$\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\mathrm{ch}^{(R)}\left(\frac{L'}{L}(s,\chi)\right)R^{2k} \le \mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\left|\frac{L'}{L}(s,\chi)\right|^{2k} \ll 1 \qquad \text{(uniformly in } m\text{)},$$

whence (4.2.14).

Therefore, by (4.2.13)(4.2.14)(4.2.16) (noting also that $\mathrm{ch}_R^2 = \mathrm{ch}_R$) and the Schwarz inequality we obtain

$$\begin{aligned}
&\left|\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\left(\Phi\left(\frac{L'}{L}(s,\chi)\right) - \Phi^{(R)}\left(\frac{L'}{L}(s,\chi)\right)\right)\right| \\
&\quad \ll \mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\left(\mathrm{ch}^{(R)}\left(\frac{L'}{L}(s,\chi)\right)\times\left|\frac{L'}{L}(s,\chi)\right|^{k}\right) \\
&\quad \le \left(\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\mathrm{ch}^{(R)}\left(\frac{L'}{L}(s,\chi)\right)\times \mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\left|\frac{L'}{L}(s,\chi)\right|^{2k}\right)^{1/2} \ll R^{-k}.
\end{aligned}$$

Since this estimation is uniform in m, this settles the proof of (4.2.11), and hence also of (4.2.12) in the present case of Φ.

§ 4.3. The general case.

Now let Φ be any continuous function satisfying $|\Phi(z)| \ll |z|^k$ ($|z| \ge 1$) with some $k \ge 1$. Then, for any $\epsilon > 0$, there exists a C^1-function Φ_1 satisfying $|\Phi - \Phi_1| < \epsilon$ everywhere. For such Φ_1, we have

$$\left|\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\left(\Phi\left(\frac{L'}{L}(s,\chi)\right) - \Phi_1\left(\frac{L'}{L}(s,\chi)\right)\right)\right| < \epsilon \qquad \text{(any } m\text{)}, \tag{4.3.1}$$

$$\int M_\sigma(z)|\Phi(z)-\Phi_1(z)||dz| < \epsilon\int M_\sigma(z)|dz| = \epsilon. \tag{4.3.2}$$

Hence

$$\begin{aligned}
&\left|\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\Phi\left(\frac{L'}{L}(s,\chi)\right) - \int M_\sigma(z)\Phi(z)|dz|\right| \\
&\quad < 2\epsilon + \left|\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\Phi_1\left(\frac{L'}{L}(s,\chi)\right) - \int M_\sigma(z)\Phi_1(z)|dz|\right| < 3\epsilon
\end{aligned} \tag{4.3.3}$$

for m sufficiently large, by §4.2. Therefore, (4.3.3), which is independent of the choice of Φ_1, must be 0; hence Theorem B is proved also in this general case.

It remains to deal with the case where Φ is the characteristic function of either a compact set or of its complement, and clearly it suffices to deal with the former. Let A be any compact subset of $\mathbf{C}$ and ch_A denote its characteristic function. Then for any $\epsilon > 0$, there exist two continuous real valued functions ϕ_1, ϕ_2 on $\mathbf{C}$ with compact supports such that

$$0 \leq \phi_1 \leq \mathrm{ch}_A \leq \phi_2 \leq 1 \tag{4.3.4}$$

and that the support of $\phi_2 - \phi_1$ has volume $< \epsilon$. Put $C_\sigma = \mathrm{Max}_{z \in \mathbf{C}} M_\sigma(z)$ $(< \infty)$. Then clearly,

$$\int M_\sigma(w)(\mathrm{ch}_A(w) - \phi_1(w))|dw| \qquad \textit{and} \qquad \int M_\sigma(w)(\phi_2(w) - \mathrm{ch}_A(w))|dw| \qquad < C_\sigma \epsilon.$$

Therefore, by Theorem B for $\Phi = \phi_1$, ϕ_2 and (4.3.4) we obtain

$$\begin{aligned} \int_A M_\sigma(w)|dw| - C_\sigma\epsilon \leq \int M_\sigma(w)\phi_1(w)|dw| &= \lim_{m\to\infty} \mathrm{Avg}_{N(\mathbf{f}_\chi)\leq m}\phi_1\left(\frac{L'}{L}(s,\chi)\right) \\ \leq \underline{\lim}\, \mathrm{Avg}_{N(\mathbf{f}_\chi)\leq m}\mathrm{ch}_A\left(\frac{L'}{L}(s,\chi)\right) &\leq \overline{\lim}\, \mathrm{Avg}_{N(\mathbf{f}_\chi)\leq m}\mathrm{ch}_A\left(\frac{L'}{L}(s,\chi)\right) \\ \leq \lim_{m\to\infty} \mathrm{Avg}_{N(\mathbf{f}_\chi)\leq m}\phi_2\left(\frac{L'}{L}(s,\chi)\right) = \int M_\sigma(w)\phi_2(w)|dw| &\leq \int_A M_\sigma(w)|dw| + C_\sigma\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the equality

$$\lim_{m\mapsto\infty} \mathrm{Avg}_{N(\mathbf{f}_\chi)\leq m}\mathrm{ch}_A\left(\frac{L'}{L}(s,\chi)\right) = \int_A M_\sigma(w)|dw| \tag{4.3.5}$$

must hold.

This completes the proof of Theorem B.

§ 5. Appendix: Function-field analogues of well-known estimations of some basic arithmetic functions.

Here we shall supply proofs for the estimates of some relevant arithmetic functions that are well-known in the number field case but not necessarily so in our function field case. Very probably, each of them had been proved and used somewhere in some past literature, but since we could not find suitable references, we shall provide their proofs. (Among them, Proposition 5.3.1 is not used in this text but will be used in a forthcoming paper.)

§ 5.1. Number of integral divisors in a given class having a given norm.

We shall here give a proof of Proposition 3.3.16 (copied below with a new numbering):

Proposition 5.1.1. *Let n be any positive integer. Then:*
(i) *The number of integral divisors D of K with $N(D) \leq q^n$ is $\mathbf{O}_K(q^n)$.*
(ii) *Let c be any fixed element of $G_{\mathbf{f}}$. Then the number of integral divisors D satisfying $N(D) = q^n$ and $i_{\mathbf{f}}(D) = c$ cannot exceed* $\mathrm{Max}(1, q^{n+1}/N(\mathbf{f}))$.
(iii) *There is at most one integral divisor D coprime with $\mathfrak{p}_\infty$ satisfying $i_{\mathbf{f}}(D) = c$ and $N(D) < N(\mathbf{f})$.*

Proof (i) For any positive integer n, denote by A_n (resp. B_n) the number of integral (resp. prime) divisors of K will degree n. Since the values of norms are restricted to integral powers of q, the statement (i) is equivalent (only in the function field case!) to that the number of integral divisors of K with norm $= q^n$ is $\mathbf{O}(q^n)$, i.e., to

$$A_n = \mathbf{O}(q^n). \tag{5.1.2}$$

This proof is very simple. Let $\zeta_K(s)$ be the (congruence) zeta function of K. Then, as a formal power series of $u = q^{-s}$ over $\mathbf{Z}$,

$$\zeta_K(s) = \sum_{n=1}^{\infty} A_n u^n = \prod_{m=1}^{\infty} (1 - u^m)^{-B_m} = \frac{P(u)}{(1-u)(1-qu)}, \tag{5.1.3}$$

where $P(u)$ is a polynomial. Since the coefficient of u^n in the power series expansion of $((1-u)(1-qu))^{-1}$ is $\ll q^n$, and the polynomial $P(u)$ depends only on the field K, (5.1.2) follows immediately.

(ii) Suppose that c contains at least one such divisor D_0. Then any integral divisor D satisfying $N(D) = q^n$ and $i_{\mathbf{f}}(D) = c$ must be of the form $(\alpha)D_0$, with some $\alpha \in K^\times$ satisfying the congruence $\alpha \equiv 1 \pmod{\mathbf{f}}$. (Since DD_0^{-1} has norm 1, its $\langle \mathfrak{p}_\infty \rangle$-component is trivial.) Such an element α is uniquely determined by its divisor and hence by D, because the group of units in K is $\mathbf{F}_q^\times$ and hence the only unit congruent to 1 (mod $\mathbf{f}$) is 1. Put $\beta = \alpha - 1$, so that $(\beta) \succeq \mathbf{f}$. The integrality condition for D in terms of α is $(\alpha) \succeq D_0^{-1}$, which is equivalent to $(\beta) \succeq D_0^{-1}$, because D_0 is integral. Therefore, the condition for D is $(\beta) \succeq \mathbf{f}D_0^{-1}$. But such β form a linear space over $\mathbf{F}_q$ of dimension at most $\mathrm{Max}(0, \deg D_0 - \deg \mathbf{f} + 1)$ (cf. e.g. [5] p.7, Prop. 4).

(iii) Two integral divisors coprime with $\mathfrak{p}_\infty$ belonging to the same class c must have the equal norm (because the norm of any principal divisor is 1). Therefore, (iii) is an immediate consequence of (ii). □

§ 5.2. Sums over prime divisors.

Here, we list some basic estimates related to sums over primes with restricted norms (in terms of restricted degrees) that are more or less relevant. The Landau symbol $\mathbf{O}$

below depends on K, $\sim$ means that the limit of the ratio as $n \mapsto \infty$ tends to 1, and log is the natural logarithm (not the one with the base q).

$$\sum_{i=1}^{n} iB_i \sim (1-q^{-1})^{-1}q^n, \tag{5.2.1}$$

$$\sum_{i=1}^{n} B_i \sim (1-q^{-1})^{-1}(q^n/n), \tag{5.2.2}$$

$$\sum_{i=1}^{n} B_i/q^i = \log n + \mathbf{O}(1), \tag{5.2.3}$$

$$\prod_{i=1}^{n} (1-q^{-i})^{-B_i} \ll n. \tag{5.2.4}$$

To prove these, we need to know more about the zeros of $P(u)$. As in the number field case where these 4 formulas correspond to the well-known estimates of $\psi(x)$, $\pi(x)$, $\sum_{N(\mathfrak{p})\leq x} N(\mathfrak{p})^{-1}$ and $\prod_{N(\mathfrak{p})\leq x}(1-N(\mathfrak{p})^{-1})^{-1}$ respectively, we do not need as strong as the Riemann hypothesis. But let us use the Weil Riemann Hypothesis for function fields to make the arguments much simpler. It asserts that

$$P(u) = \prod_{\nu=1}^{g} (1-\pi_\nu u)(1-\bar{\pi}_\nu u) \tag{5.2.5}$$

(g: the genus), with

$$|\pi_\nu| = |\bar{\pi}_\nu| = q^{1/2} \qquad (1 \leq \nu \leq g). \tag{5.2.6}$$

The key basic formula for all the above estimates is

$$B_m = \frac{q^m}{m} + \mathbf{O}(q^{m/2}). \tag{5.2.7}$$

To prove (5.2.7), put

$$N_m = \sum_{d|m} dB_d = q^m + 1 - \sum_{\nu=1}^{g} (\pi_\nu^m + \bar{\pi}_\nu^m). \tag{5.2.8}$$

By the first defining equality, N_m gives the number of $\mathbf{F}_{q^m}$-rational points of the corresponding curve, and the second equality is obtained from the last equality in (5.1.3) by taking the logarithmic derivative with respect to u and by comparing the coefficients of u^{m-1} (cf. e.g. [5]). Now, (5.2.8) and (5.2.6) give

$$N_m = q^m + \mathbf{O}(q^{m/2}) = \mathbf{O}(q^m), \tag{5.2.9}$$

while the Möbius inversion formula gives

$$mB_m = \sum_{d|m} \mu(d) N_{m/d}; \tag{5.2.10}$$

hence

$$|mB_m - N_m| \leq \sum_{d|m,\, d\geq 2} N_{m/d} \ll \sum_{d=2}^{m} q^{m/d} \leq mq^{m/2}. \tag{5.2.11}$$

Hence by combining with (5.2.9), we obtain $|mB_m - q^m| \ll mq^{m/2}$, i.e., (5.2.7).

This decomposition (5.2.7) of B_m reduces the proof of each formula above to elementary calculus.

§ 5.3. The number of factors of D.

For each integral divisor D, let $S(D)$ denote the number of distinct factors of D; namely, $S(D) = \prod_{\mathfrak{p}}(r_{\mathfrak{p}} + 1)$ when D has the prime factorization $D = \prod_{\mathfrak{p}} \mathfrak{p}^{r_{\mathfrak{p}}}$.

Proposition 5.3.1.

$$S(D) \ll_{\epsilon'} N(D)^{\epsilon'} \tag{5.3.2}$$

holds for any $\epsilon' > 0$.

Proof Write $D = \prod_{i=1}^{s} \mathfrak{p}_i^{r_i}$ $(r_1, ..., r_s \geq 1)$. Then by [1] Sublemma 3.10.5, we have

$$s \leq s_0^\star := C_0 \frac{\log N(D)}{\log\log N(D)} \tag{5.3.3}$$

for some positive constant C_0 for $N(D) > 3$. Since $r_i + 1 \leq C_1(\log N(\mathfrak{p}_i)) r_i$ (say, for $C_1 = 2/\log 2$), we have

$$\sum_{i=1}^{s}(r_i + 1) \leq C_1 \sum_{i=1}^{s} r_i \log N(\mathfrak{p}_i) = C_1 \log N(D). \tag{5.3.4}$$

But since

$$S(D) = \prod_{i=1}^{s}(r_i + 1) \leq \left(\frac{1}{s}\sum_{i=1}^{s}(r_i + 1)\right)^s, \tag{5.3.5}$$

we obtain

$$\log S(D) \leq s(\log\log N(D) + \log C_1 - \log s). \tag{5.3.6}$$

Now consider

$$f(s) = s(\log\log N(D) + \log C_1 - \log s) \tag{5.3.7}$$

as a function of a free *real variable* s and look for its maximal value on the region $1 \le s \le s_0^\star$. Since its derivative is

$$f'(s) = \log\log N(D) + \log C_1 - \log s - 1, \tag{5.3.8}$$

we see that $f'(1) > 0$ if $N(D)$ is sufficiently large, and that $f'(s)$ is monotone decreasing with the limit $-\infty$ at $s \to +\infty$. Its unique zero $s_0 > 0$ is given by $s_0 = C_1 e^{-1} \log N(D)$, which is greater than $s_0^\star$ if $N(D)$ is sufficiently large. Therefore, $f'(s) > 0$ for $1 \le s \le s_0^\star$. Therefore, on this region, we have

$$\begin{aligned} f(s) \le f(s_0^\star) &= s_0^\star(\log\log N(D) + \log C_1 - \log s_0^\star) \\ &= C_0 \frac{\log N(D)}{\log\log N(D)}(\log C_1 - \log C_0 + \log\log\log N(D)). \end{aligned} \tag{5.3.9}$$

Therefore,

$$\begin{aligned} \log S(D) &\le C_0 \frac{\log\log\log N(D) + \log(C_1/C_0)}{\log\log N(D)} \log N(D) \\ &\le \epsilon' \log N(D) \end{aligned} \tag{5.3.10}$$

for $N(D) \gg_{\epsilon'} 1$. This proves Proposition 5.3.1.

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