Symmetric Solution of the Painlevé III and its Linear Monodromy

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1 Introduction

The Painlevé equation is obtained from the monodromy preserving deformation of a linear equation [2] [4]. We call the monodromy data of the linear equation as the *linear monodromy* of the Painlevé transcendent.

Jimbo[5] gave an explicit formula which specifies the asymptotic behavior of the Painlevé transcendent at a fixed critical point from the linear monodromy [5].

Nevertheless it is generically difficult to calculate the linear monodromy itself for a given Painlevé transcendent. Our interest is in the Painlevé transcendents whose corresponding linear monodromy can be determined exactly. In this paper, we call such Painlevé functions *monodromy solvable*.

One example of the monodromy solvable solutions is a classical solution found by Umemura [12]. For any Umemura's classical solution, we can calculate the linear monodromy. Here, we remark that the classical solutions exist only for some special parameters. The class of monodromy solvable solutions is broader than the class of Umemura's classical solutions, and contains many important solutions.

R. Fuchs is the first who found a non-classical monodromy solvable solution [3]. He calculates the linear monodromy of Picard's solution [11], which satisfies the sixth Painlevé equation with a special parameter. His work is found again in [8] recently.

Another example of the monodromy solvable solution is a symmetric solution. For the first, second and fourth Painlevé equations, there exist symmetric solutions which are invariant under some action of the cyclic groups. The symmetric solutions are studied by A. V. Kitaev [7] for P_I and P_{II}, and by Kaneko [6] for P_{IV}. In their cases, the corresponding linear equations become the confluent hypergeometric equations. The symmetric solutions exist for the general parameters.

In this paper, we will construct a new example of non-classical monodromy solvable solutions for the third Painlevé equation:

$$(P_{III}) \qquad \frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}\left(\alpha y^2 + \beta\right) + \gamma y^3 + \frac{\delta}{y}.$$

The third Painlevé equation is invariant under the transformation $(t,y) \mapsto (-t,-y)$. There exist two symmetric solutions around the origin:

$$y = -\frac{\delta}{\beta}t + \frac{\alpha\delta^3}{\beta^2(\beta^2+4\delta)}t^3 + O(t^5), \tag{1}$$

$$y = -\frac{\alpha}{\gamma}t^{-1} - \frac{\beta\gamma}{\alpha^2-4\gamma}t + O(t^3). \tag{2}$$

These solutions are transformed each other by the Bäcklund transformations. In this paper, we will show (1) is monodromy solvable.

Since the third Painlevé equation has a singularity at the origin, its solutions also have a singularity at the origin in most cases. Nevertheless the solution (1) is holomorphic at the origin.

The solution (1) exists for the general parameters, and thus it is not a classical solution in most cases. However, for some special parameters, our solution becomes a classical solution.

In this paper, we will calculate the linear monodromy of the holomorpic symmetric solution (1). The associated linear equation can be reduced to Whittaker's equation, and we can calculate the linear monodromy.

2 Symmetric Solution of P_{III}

By some rescaling, we can normalize the parameters of P_{III} as:

$$\alpha = -4\theta_\infty, \qquad \beta = 4(1+\theta_0), \qquad \gamma = 4, \qquad \delta = -4.$$

P_{III} is equivalent to the Painlevé system:

$$\begin{aligned} t\frac{dy}{dt} &= 4y^2z - 2ty^2 - (2\theta_0+1)y + 2t, \\ t\frac{dz}{dt} &= -4yz^2 + (4ty + 2\theta_0 + 1)z - (\theta_0+\theta_\infty)t, \end{aligned} \tag{3}$$

which is the Hamiltonian system

$$\frac{dy}{dt} = \frac{\partial H_{III}}{\partial z}, \qquad \frac{dz}{dt} = -\frac{\partial H_{III}}{\partial y},$$

with the Hamiltonian function

$$tH_{III} = 2y^2z^2 - (2ty^2 + (2\theta_0 + 1)y - 2t)z + (\theta_0 + \theta_\infty)ty.$$

Since the system (3) is the Briot-Bouquet type [1], there exists a unique solution with initial data $y(0) = 0$ and $z(0) = 0$. Thus we obtain the following theorem.

Theorem 2.1 *The solution of the Painlevé system* (3) *with initial data* $y(0) = 0$ *and* $z(0) = 0$ *is holomorphic in a neighborhood of* $t = 0$*:*

$$\begin{aligned} y &= \frac{1}{\theta_0 + 1}t + \frac{\theta_\infty}{\theta_0(\theta_0+1)^2(\theta_0+2)}t^3 + O(t^5), \\ z &= \frac{\theta_0 + \theta_\infty}{2\theta_0}t - \frac{\theta_0^2 - \theta_\infty^2}{2\theta_0^2(\theta_0^2 - 1)}t^3 + O(t^5). \end{aligned} \tag{4}$$

This is a symmetric solution with respect to the action of $(y,z,t) \mapsto (-y,-z,-t)$.

3 The Linear Monodromy

In this paper, we use Okamoto's SL-type linearization [10]:

$$\frac{\partial^2 \Psi}{\partial x^2} = p(x,\tau)\Psi, \tag{5}$$

$$\frac{\partial \Psi}{\partial \tau} = -\frac{1}{2}\frac{\partial A(x,\tau)}{\partial x}\Psi + A(x,\tau)\frac{\partial \Psi}{\partial x}, \tag{6}$$

where

$$p(x,\tau) = \frac{a_0\tau^2}{x^4} + \frac{a_0'\tau}{x^3} + \frac{\tau K_{III}'}{x^2} + \frac{a_\infty'}{x} + a_\infty + \frac{3}{4(x-\lambda)^2} - \frac{\lambda\nu}{x(x-\lambda)},$$

$$A(x,\tau) = \frac{\lambda x}{\tau(x-\lambda)},$$

$$a_0 = \frac{1}{4}, \qquad a_0' = -\frac{1+\theta_0}{2}, \qquad a_\infty = \frac{1}{4}, \qquad a_\infty' = -\frac{\theta_\infty}{2},$$

$$\tau = t^2, \qquad \lambda = ty, \qquad \nu = \frac{(2z-t)y^2 + (1-\theta_0)y + t}{2ty^2},$$

$$K_{III}' = \frac{1}{\tau}\left\{\lambda^2\nu^2 - \lambda\nu - \frac{a_0\tau^2}{\lambda^2} - \frac{a_0'\tau}{\lambda} - a_\infty'\lambda - a_\infty\lambda^2\right\}.$$

The compatibility condition of (5) and (6) becomes the Hamiltonian system

$$\frac{d\lambda}{d\tau} = \frac{\partial K_{III}'}{\partial \nu}, \qquad \frac{d\nu}{d\tau} = -\frac{\partial K_{III}'}{\partial \lambda},$$

which is equivalent to (3).

The monodromy data of (5) are expressed in $\{S_1, S_2, M_0, \Gamma, G_1, G_2, M_\infty\}$, where M_0 is a formal monodromy matrix around $x=0$, M_∞ is a formal monodromy matrix around $x=\infty$, Γ is a connection matrix, and S_1, S_2, G_1, G_2 are the Stokes matrices around $x=0$ and $x=\infty$ defined as follows:

Around $x=\infty$, there exists a fundamental system $Y^{(j)} = (Y_1^{(j)}, Y_2^{(j)})$ of solutions of (5) satisfying

$$Y_1^{(j)} \sim (1+O(x^{-1}))x^{-\theta_\infty/2}e^{x/2}, \qquad Y_2^{(j)} \sim (1+O(x^{-1}))x^{\theta_\infty/2}e^{-x/2}$$

as $x \to \infty$ in

$$\mathscr{S}_j = \{x \in \mathbb{C} | -\tfrac{\pi}{2}(2j-1) < \arg x < -\tfrac{\pi}{2}(2j-5)\}.$$

Then the Stokes matrices are defined by

$$Y^{(2)}(x) = Y^{(1)}(x)G_1, \qquad Y^{(3)}(x) = Y^{(2)}(x)G_2 = Y^{(1)}(xe^{2\sqrt{-1}\pi})M_\infty^{-1}.$$

Around $x=0$, there exists a fundamental system $\bar{Y}^{(j)} = (\bar{Y}_1^{(j)}, \bar{Y}_2^{(j)})$ of solutions of (5) satisfying

$$\bar{Y}_1^{(j)} \sim g(\tau)(1+O(x^{-1}))x^{(3+\theta_0)/2}e^{\tau/(2x)}, \quad \bar{Y}_2^{(j)} \sim g(\tau)(1+O(x^{-1}))x^{(1-\theta_0)/2}e^{-\tau/(2x)}$$

as $x \to 0$ in

$$\bar{\mathscr{S}}_j = \{x \in \mathbb{C} | -\tfrac{\pi}{2}(2j-1) < \arg(\tfrac{\tau}{x}) < -\tfrac{\pi}{2}(2j-5)\}.$$

Then the Stokes matrices are defined by

$$\bar{Y}^{(2)}(x) = \bar{Y}^{(1)}(x)S_1^{-1}, \qquad \bar{Y}^{(3)}(x) = \bar{Y}^{(2)}(x)S_2^{-1} = \bar{Y}^{(1)}(xe^{-2\sqrt{-1}\pi})M_0.$$

We fix a path γ joining ∞ and 0. Along γ the connection matrix is defined as

$$Y^{(1)}(x) = \bar{Y}^{(2)}(x)\Gamma.$$

Then we have the relation

$$\Gamma^{-1}S_1M_0S_2\Gamma G_1G_2M_\infty = I_2.$$

By the isomonodromy condition, the linear monodromy of (5) is invariant for any τ. We substitute the symmetric solution (4) into $p(x,\tau)$, $A(x,\tau)$ and $\partial A/\partial x$, and take the limit $\tau \to 0$. Then we have

$$\begin{aligned}
p &\to \frac{1}{4} - \frac{\theta_\infty}{2x} + \frac{2\theta_0 + {\theta_0}^2}{4x^2}, \\
A &\to \frac{1}{1+\theta_0}, \\
\frac{\partial A}{\partial x} &\to 0.
\end{aligned}$$

The linearization (5) becomes the Whittaker equation:

$$\frac{d^2\Psi}{dx^2} = \left\{ \frac{1}{4} - \frac{\frac{\theta_\infty}{2}}{x} - \frac{\frac{1}{4} - \left(\frac{\theta_0+1}{2}\right)^2}{x^2} \right\} \Psi. \tag{7}$$

For our symmetric solution of the third Painlevé equation, we can calculate the monodromy data, because (5) reduces to the Whittaker equation when $t = 0$ ($\tau = 0$).

Let us take a fundamental system of solutions of (7) as follows:
around $x = 0$,

$$M_{\frac{\theta_\infty}{2}, -\frac{\theta_0+1}{2}}(x), \qquad M_{\frac{\theta_\infty}{2}, \frac{\theta_0+1}{2}}(x),$$

and around $x = \infty$,

$$W_{\frac{\theta_\infty}{2}, -\frac{\theta_0+1}{2}}(x), \qquad W_{-\frac{\theta_\infty}{2}, -\frac{\theta_0+1}{2}}(x e^{-\sqrt{-1}\pi}),$$

where

$$\begin{aligned}
M_{\kappa,\mu}(x) &:= x^{\frac{1}{2}+\mu} e^{-\frac{x}{2}} {}_1F_1\left(\frac{1}{2}+\mu-\kappa, 2\mu+1, x\right), \\
W_{\kappa,\mu}(x) &= \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\kappa)} M_{\kappa,\mu}(x) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\kappa)} M_{\kappa,-\mu}(x) \\
&\sim e^{\frac{-x}{2}} x^{\kappa} \left[1 + \sum_{n=1}^{\infty} \frac{1}{n! x^n} \prod_{m=0}^{n} \left(\mu^2 - \left(\kappa + m - \frac{1}{2} \right)^2 \right) \right]
\end{aligned}$$

$$(-\tfrac{3}{2}\pi < \arg x < \tfrac{3}{2}\pi).$$

Theorem 3.1 *The linear monodromy of our symmetric solution* (4) *is explicitly given as follows:*

$$\Gamma = \begin{pmatrix} \frac{\Gamma(\theta_\infty+1)}{\Gamma(1+\frac{\theta_0-\theta_\infty}{2})} & \frac{\Gamma(\theta_\infty+1) e^{\sqrt{-1}\pi \frac{\theta_0-\theta_\infty}{2}}}{\Gamma(1+\frac{\theta_0+\theta_\infty}{2})} \\ \frac{\Gamma(-\theta_\infty-1)}{\Gamma(-\frac{\theta_0+\theta_\infty}{2})} & \frac{\Gamma(-\theta_\infty-1) e^{\sqrt{-1}\pi(1-\frac{\theta_0+\theta_\infty}{2})}}{\Gamma(-\frac{\theta_0-\theta_\infty}{2})} \end{pmatrix},$$

$$M_0 = \begin{pmatrix} e^{-\pi\sqrt{-1}\theta_0} & 0 \\ 0 & e^{\pi\sqrt{-1}\theta_0} \end{pmatrix}, \qquad M_\infty = \begin{pmatrix} e^{\pi\sqrt{-1}\theta_\infty} & 0 \\ 0 & e^{-\pi\sqrt{-1}\theta_\infty} \end{pmatrix},$$

$$S_1 = S_2 = I_2,$$

$$G_1 = \begin{pmatrix} 1 & 0 \\ \frac{-2\pi\sqrt{-1} e^{\pi\sqrt{-1}\theta_\infty}}{\Gamma(-\frac{\theta_0+\theta_\infty}{2})\Gamma(1+\frac{\theta_0-\theta_\infty}{2})} & 1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 1 & \frac{-2\pi\sqrt{-1} e^{-2\pi\sqrt{-1}\theta_\infty}}{\Gamma(-\frac{\theta_0-\theta_\infty}{2})\Gamma(1+\frac{\theta_0+\theta_\infty}{2})} \\ 0 & 1 \end{pmatrix}.$$

Remark 3.2 *The third Painlevé equation admits classical solutions for some special parameters. However, our solution exists for general parameters, and thus our solution is not classical in most cases. If $\theta_0 + \theta_\infty = 0$, every element of the linear monodromy of our solution becomes an upper triangular matrix, and our solution becomes a Riccati solution which is represented by the Bessel function. Our solution is not an algebraic solution for any parameters.*

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