The Lax pair for the sixth Painlevé equation arising from Drinfeld-Sokolov hierarchy

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Introduction

In a recent work [FS], we showed that the sixth Painlevé equation arises from a Drinfeld-Sokolov hierarchy of type $D_4^{(1)}$ by a similarity reduction. We actually discuss a derivation of the symmetric representation of $P_{VI}$ given in [Kaw].

On the other hand, $P_{VI}$ can be expressed as the Hamiltonian system; see [IKSY, O]. Also it is known that this Hamiltonian system is equivalent to the compatibility condition of the Lax pair associated with $\hat{\mathfrak{s}0}(8)$; see [NY].

In this article, we discuss the derivation of this Lax pair from the Drinfeld-Sokolov hierarchy.

1 Lax pair for $P_{VI}$ associated with $\hat{\mathfrak{s}0}(8)$

The sixth Painlevé equation can be expressed as the following Hamiltonian system:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

(1.1)

with the Hamiltonian

$$t(t-1)H = q(q-1)(q-t)p^2 - \{(\alpha_0 - 1)q(q-1)$$

$$+ \alpha_3q(q-t) + \alpha_4(q-1)(q-t)\}p + \alpha_2(\alpha_1 + \alpha_2)q,$$

(1.2)

satisfying the relation

$$\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1.$$
Let $\epsilon_1, \ldots, \epsilon_4$ be complex constants defined by

\[ \alpha_0 = 1 - \epsilon_1 - \epsilon_2, \quad \alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = \epsilon_2 - \epsilon_3, \]
\[ \alpha_3 = \epsilon_3 - \epsilon_4, \quad \alpha_4 = \epsilon_3 + \epsilon_4. \]

Consider the system of linear differential equations

\[ (z\partial_z + M)\psi = 0, \quad \partial_t\psi = B\psi, \quad (1.3) \]

for a vector of unknown functions $\psi = (\psi_1, \ldots, \psi_8)$. Here we assume that the matrix $M$ is defined as

\[
M = \begin{bmatrix}
\epsilon_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \epsilon_2 & -p & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & \epsilon_3 & q - 1 & q & 0 & 0 & 0 \\
0 & 0 & 0 & \epsilon_4 & 0 & -q & 1 & 0 \\
0 & 0 & 0 & 0 & -\epsilon_4 & 1 - q & 1 & 0 \\
-t & 0 & 0 & 0 & 0 & -\epsilon_3 & p & 0 \\
(q-t)z & 0 & 0 & 0 & 0 & 0 & -\epsilon_2 & -1 \\
0 & (q-t)z & z & 0 & 0 & 0 & 0 & -\epsilon_1 \\
\end{bmatrix},
\]

and the matrix $B$ is defined as

\[
B = \begin{bmatrix}
u_1 & x_1 & y_1 & 0 & 0 & 0 & 0 & 0 \\
u_2 & x_2 & -y_3 & -y_4 & 0 & 0 & 0 & 0 \\
0 & u_3 & x_3 & x_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & u_4 & 0 & -x_4 & y_4 & 0 \\
0 & 0 & 0 & 0 & -u_4 & -x_3 & y_3 & 0 \\
0 & 0 & 0 & 0 & 0 & -u_3 & -x_2 & -y_1 \\
-z & 0 & 0 & 0 & 0 & 0 & -u_2 & -x_1 \\
0 & z & 0 & 0 & 0 & 0 & 0 & -u_1 \\
\end{bmatrix}.
\]

**Theorem 1.1** ([NY]). Under the compatibility condition for (1.3), the variables $x_i, y_i$ and $u_i$ are determined as elements of $\mathbb{C}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, q, p, t)$. The compatibility condition is then equivalent to the Hamiltonian system (1.1) with (1.2).

Here we do not describe the explicit forms of $u_i, x_i$ and $y_i$.

## 2 Affine Lie algebra

In the notation of [Kac], $\mathfrak{g} = \mathfrak{g}(D_4^{(1)})$ is the affine Lie algebra generated by the Chevalley generators $e_i, f_i, \alpha_i^\vee (i = 0, \ldots, 4)$ and the scaling element $d$.
with the generalized Cartan matrix defined as

\[
A = (a_{ij})_{i,j=0}^{4} = \begin{bmatrix}
2 & 0 & -1 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 \\
-1 & -1 & 2 & -1 & -1 \\
0 & 0 & -1 & 2 & 0 \\
0 & 0 & -1 & 0 & 2
\end{bmatrix}.
\]

We denote the Cartan subalgebra of \( \mathfrak{g} \) by \( \mathfrak{h} \). The canonical central element of \( \mathfrak{g} \) is given by

\[
K = \alpha_0^\vee + \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee.
\]

We consider the \( \mathbb{Z} \)-gradation \( \mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k(s) \) of type \( s = (1,1,0,1,1) \) by setting

\[
\deg \mathfrak{h} = \deg e_2 = \deg f_2 = 0, \quad \deg e_i = 1, \quad \deg f_i = -1 \quad (i = 0, 1, 3, 4).
\]

This gradation is defined by

\[
\mathfrak{g}_k(s) = \{ x \in \mathfrak{g} \mid [d_s, x] = kx \} \quad (k \in \mathbb{Z}),
\]

where

\[
d_s = 4d + 2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + 2\alpha_4^\vee \in \mathfrak{h}.
\]

Denoting by \( e_{2i} = [e_2, e_i] \), we choose the graded Heisenberg subalgebra of \( \mathfrak{g} \)

\[
\mathfrak{s} = \{ x \in \mathfrak{g} \mid [x, \Lambda] = \mathbb{C}K \},
\]

of type \( s = (1,1,0,1,1) \) with

\[
\Lambda = e_0 - e_1 + e_3 - e_{20} + e_{23} + e_{24}.
\]

The positive part of \( \mathfrak{s} \) has a graded basis \( \{ \Lambda_{2k-1,1}, \Lambda_{2k-1,2} \}_{k=1}^{\infty} \) such that

\[
\Lambda_{1,1} = \Lambda, \quad \Lambda_{1,2} = e_0 - e_3 + e_4 + e_{20} + e_{21} + e_{23},
\]

\[
[d_s, \Lambda_{2k-1,i}] = (2k-1)\Lambda_{2k-1,i}, \quad [\Lambda_{2k-1,i}, \Lambda_{2l-1,j}] = 0.
\]

Let \( \mathfrak{n}_+ \) be the subalgebra of \( \mathfrak{g} \) generated by \( e_j \) \((j = 0, \ldots, 4)\), and let \( \mathfrak{b}_+ \) be the borel subalgebra of \( \mathfrak{g} \) defined by \( \mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+ \). Then the compatibility condition for (1.3) is equivalent to the system on \( \mathfrak{b}_+ \)

\[
\partial_t(M) = [B, d_s + M],
\]

with

\[
M = h(\varepsilon) + (q - t)e_0 + e_1 - pe_2 + (q - 1)e_3 + ge_4 - e_{20} - e_{23} - e_{24},
\]

\[
B = h(u) + e_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + y_1e_{21} + y_3e_{23} + y_4e_{24},
\]

3
where $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ and $u = (u_1, u_2, u_3, u_4)$. Here we set

$$h(\varepsilon) = (1 - \varepsilon_1 - \varepsilon_2)\alpha_0^\vee + (\varepsilon_1 - \varepsilon_2)\alpha_1^\vee + (\varepsilon_2 - \varepsilon_3)\alpha_2^\vee + (\varepsilon_3 - \varepsilon_4)\alpha_3^\vee + (\varepsilon_3 + \varepsilon_4)\alpha_4^\vee.$$  

We derive the system (2.1) from the Drinfeld-Sokolov hierarchy associated with the Heisenberg subalgebra $\mathfrak{h}$ by a similarity reduction.

### 3 Drinfeld-Sokolov hierarchy

In the following, we use the notation of infinite dimensional groups

$$G_{<0} = \exp(\hat{\mathfrak{g}}_{<0}), \quad G_{\geq 0} = \exp(\hat{\mathfrak{g}}_{\geq 0}),$$

where $\hat{\mathfrak{g}}_{<0}$ and $\hat{\mathfrak{g}}_{\geq 0}$ are completions of $\mathfrak{g}_{<0} = \bigoplus_{k<0} g(s)$ and $\mathfrak{g}_{\geq 0} = \bigoplus_{k \geq 0} \mathfrak{g}_k(s)$ respectively.

Introducing the time variables $t_{k,i}$ ($i = 1, 2; k = 1, 3, 5, \ldots$), we consider the Sato equation for a $G_{<0}$-valued function $W = W(t_{1,1}, t_{1,2}, \ldots)$

$$\partial_{k,i}(W) = B_{k,i}W - W\Lambda_{k,i} \quad (i = 1, 2; k = 1, 3, 5, \ldots), \quad (3.1)$$

where $\partial_{k,i} = \partial/\partial t_{k,i}$ and $B_{k,i}$ stand for the $\mathfrak{g}_{\geq 0}$-component of $W\Lambda_{k,i}W^{-1} \in \hat{\mathfrak{g}}_{<0} \oplus \mathfrak{g}_{\geq 0}$. The Zakharov-Shabat equation

$$[\partial_{k,i} - B_{k,i}, \partial_{l,j} - B_{l,j}] = 0 \quad (i, j = 1, 2; k, l = 1, 3, 5, \ldots), \quad (3.2)$$

follows from the Sato equation (3.1). Let

$$\Psi = W \exp(\xi), \quad \xi = \sum_{i=1,2} \sum_{k=1,3,\ldots} t_{k,i}\Lambda_{k,i}.$$  

Then the Zakharov-Shabat equation (3.2) can be regarded as the compatibility condition of the Lax form

$$\partial_{k,i}(\Psi) = B_{k,i}\Psi \quad (i = 1, 2; k = 1, 3, 5, \ldots). \quad (3.3)$$

Assuming that $t_{k,1} = t_{k,2} = 0$ for $k \geq 3$, we require that the following similarity condition is satisfied:

$$d_s(\Psi) = (t_{1,1}B_{1,1} + t_{1,2}B_{1,2})\Psi. \quad (3.4)$$

The compatibility condition for (3.3) and (3.4) is expressed as

$$[d_s - t_{1,1}B_{1,1} - t_{1,2}B_{1,2}, \partial_{1,i} - B_{1,i}] = 0 \quad (i = 1, 2). \quad (3.5)$$
We regard the systems (3.2) and (3.5) as a similarity reduction of the Drinfeld-Sokolov hierarchy of type $D_4^{(1)}$.

Let $S \subset \mathbb{C}^2$ be an open subset with coordinates $t = (t_{1,1}, t_{1,2})$. Also let
\[
\mathcal{M} = d_s - t_{1,1}B_{1,1} - t_{1,2}B_{1,2} \in \mathcal{O}(S; \mathfrak{g} \geq 0),
\]
\[
\mathcal{B} = B_{1,1}dt_{1,1} + B_{1,2}dt_{1,2} \in \Omega^1(S; \mathfrak{g} \geq 0).
\]

Then the similarity reduction is expressed as
\[
d_t\mathcal{M} = [\mathcal{B}, \mathcal{M}], \quad d_t\mathcal{B} = \mathcal{B} \wedge \mathcal{B}. \tag{3.6}
\]

4 Derivation of $P_{VI}$

The operator $\mathcal{M} \in \mathfrak{g} \geq 0$ is expressed as
\[
\mathcal{M} = \text{(terms of degree 0)} - t_{1,1}\Lambda_{1,1} - t_{1,2}\Lambda_{1,2}.
\]

We consider the gauge transformation for the Lax form (3.4) such that $\mathcal{M} \rightarrow \mathcal{\tilde{M}} \in \mathcal{O}(S; \mathfrak{b}_+)$.

We first consider a gauge transformation $\hat{\Psi} = \exp(\zeta)\exp(\xi e_2)\Psi$, where
\[
\zeta = \sum_{j=0,1,3,4}\zeta_j\alpha_j^\vee.
\]
This is lifted to the transformation on $\mathfrak{g} \geq 0$:
\[
\mathcal{\tilde{M}} = \exp(\text{ad}(\zeta))\exp(\text{ad}(\xi e_2))\mathcal{M},
\]
\[
d_t - \mathcal{\tilde{B}} = \exp(\text{ad}(\zeta))\exp(\text{ad}(\xi e_2))(d_t - \mathcal{B}).
\]

We look for gauge parameters $\zeta$ and $\xi$ such that
\[
\mathcal{\tilde{M}} = \text{(terms of degree 0)} - c_0e_0 - e_1 - c_3e_3 - c_4e_4 - e_{20} - e_{23} - e_{24}.
\]

where $c_j \in \mathbb{C}(t) \ (j = 0, 3, 4)$. Such gauge parameters are determined uniquely as $\xi = t_{1,2}/t_{1,1}$ and
\[
\zeta_0 = \frac{1}{2}\log\{(t_{1,1}^2 + 2t_{1,1}t_{1,2} - t_{1,2}^2)(t_{1,1}^2 + t_{1,2}^2)t_{1,1}^{-2}\},
\]
\[
\zeta_1 = -\frac{1}{2}\log(-t_{1,1}),
\]
\[
\zeta_3 = \frac{1}{2}\log\{(-t_{1,1}^2 + 2t_{1,1}t_{1,2} + t_{1,2}^2)(t_{1,1}^2 + t_{1,2}^2)t_{1,1}^{-2}\},
\]
\[
\zeta_4 = \frac{1}{2}\log\{(t_{1,1}^2 + 2t_{1,1}t_{1,2} - t_{1,2}^2)(-t_{1,1}^2 + 2t_{1,1}t_{1,2} + t_{1,2}^2)t_{1,1}^{-2}\}.\]
Here each \( c_j \) is described explicitly as
\[
\begin{align*}
  c_0 &= -\frac{1}{t_{1,1}}(t_{1,1} + t_{1,2})(t_{1,1}^2 + 2t_{1,1}t_{1,2} - t_{1,2}^2)(t_{1,1}^2 + t_{1,2}^2), \\
  c_3 &= -\frac{1}{t_{1,1}}(t_{1,1} - t_{1,2})(-t_{1,1}^2 + 2t_{1,1}t_{1,2} + t_{1,2}^2)(t_{1,1}^2 + t_{1,2}^2), \\
  c_4 &= -\frac{1}{t_{1,1}}t_{1,2}(t_{1,1}^2 + 2t_{1,1}t_{1,2} - t_{1,2}^2)(-t_{1,1}^2 + 2t_{1,1}t_{1,2} + t_{1,2}^2).
\end{align*}
\]

We next consider a gauge transformation \( \widetilde{\Psi} = \exp(-\lambda f_2) \hat{\Psi} \). This is lifted to the transformation on \( \mathfrak{g}_{\geq 0} \):
\[
\widetilde{\mathcal{M}} = \exp(\text{ad}(-\lambda f_2)) \mathcal{M}, \quad d_t \widetilde{B} = \exp(\text{ad}(-\lambda f_2))(d_t - \hat{B}).
\]

Denoting by \( \eta + \phi e_2 + \psi f_2 \) and \( u + x e_2 + y f_2 \) the terms of degree 0 of \( \mathcal{M} \) and \( \hat{B} \) respectively, we look for a gauge parameter \( \lambda \) such that \( \mathcal{M} \in \mathcal{O}(S; \mathfrak{b}_+) \) and \( B \in \Omega^1(S; \mathfrak{b}_+) \), namely
\[
\varphi \lambda^2 + (\eta|\alpha_j^\vee) \lambda - \psi = 0, \quad d_t \lambda = x \lambda^2 + (u|\alpha_j^\vee) \lambda - y,
\]
where \(( | )\) stands for the normalized invariant form. We can verify that the second equation of (4.1) follows from the first equation. Hence the gauge parameter \( \lambda = \lambda(t) \) can be determined and we obtain
\[
\mathcal{M} = \kappa + \mu e_2 + (\lambda - c_0)e_0 - e_1 + (\lambda - c_3)e_3 + (\lambda - c_4)e_4 - e_{20} - e_{23} - e_{24},
\]
where \( \kappa \in \mathfrak{h} \) and \( \mu = \mu(t) \). Note that \( d_t \kappa = 0 \). By definition, it is clear that the operators \( \mathcal{M} \) and \( \hat{B} \) satisfy
\[
d_t \mathcal{M} = [\hat{B}, \mathcal{M}].
\]

Finally, we consider a transformation of time variables \((t_{1,1}, t_{1,2}) \to (t_1, t_2)\) such that
\[
\partial_1(c_0 - c_4) = -4, \quad \partial_1(c_3 - c_4) = 0.
\]
Then by setting
\[
q = \frac{\lambda - c_4}{c_3 - c_4}, \quad p = \frac{1}{4}(c_3 - c_4) \mu, \quad \alpha_j = \frac{1}{4}(\kappa|\alpha_j^\vee), \quad t = \frac{c_0 - c_4}{c_3 - c_4},
\]
we arrive at

**Theorem 4.1.** Under the specialization \( t_2 = 1 \), the system (4.2) is equivalent to the compatibility condition of (1.3) that gives the sixth Painlevé equation.
References


