The Lax pair for the sixth Painlevé equation arising from Drinfeld-Sokolov hierarchy

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Introduction

In a recent work [FS], we showed that the sixth Painlevé equation arises from a Drinfeld-Sokolov hierarchy of type $D_4^{(1)}$ by a similarity reduction. We actually discuss a derivation of the symmetric representation of P_{VI} given in [Kaw].

On the other hand, P_{VI} can be expressed as the Hamiltonian system; see [IKSY, O]. Also it is known that this Hamiltonian system is equivalent to the compatibility condition of the Lax pair associated with $\widehat{\mathfrak{so}}(8)$; see [NY].

In this article, we discuss the derivation of this Lax pair from the Drinfeld-Sokolov hierarchy.

1 Lax pair for P_{VI} associated with $\widehat{\mathfrak{so}}(8)$

The sixth Painlevé equation can be expressed as the following Hamiltonian system:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \tag{1.1}$$

with the Hamiltonian

$$\begin{aligned} t(t-1)H = q(q-1)(q-t)p^2 - \{&(\alpha_0 - 1)q(q-1) \\ &+ \alpha_3 q(q-t) + \alpha_4(q-1)(q-t)\}p + \alpha_2(\alpha_1+\alpha_2)q, \end{aligned} \tag{1.2}$$

satisfying the relation

$$\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1.$$

Let $\varepsilon_1, \ldots, \varepsilon_4$ be complex constants defined by

$$\begin{aligned}
&\alpha_0 = 1 - \varepsilon_1 - \varepsilon_2, \quad \alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \\
&\alpha_3 = \varepsilon_3 - \varepsilon_4, \qquad\quad \alpha_4 = \varepsilon_3 + \varepsilon_4.
\end{aligned}$$

Consider the system of linear differential equations

$$(z\partial_z + M)\boldsymbol{\psi} = 0, \quad \partial_t \boldsymbol{\psi} = B\boldsymbol{\psi}, \tag{1.3}$$

for a vector of unknown functions $\boldsymbol{\psi} = {}^t(\psi_1, \ldots, \psi_8)$. Here we assume that the matrix M is defined as

$$M = \begin{bmatrix}
\varepsilon_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \varepsilon_2 & -p & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & \varepsilon_3 & q-1 & q & 0 & 0 & 0 \\
0 & 0 & 0 & \varepsilon_4 & 0 & -q & 1 & 0 \\
0 & 0 & 0 & 0 & -\varepsilon_4 & 1-q & 1 & 0 \\
-z & 0 & 0 & 0 & 0 & -\varepsilon_3 & p & 0 \\
(t-q)z & 0 & 0 & 0 & 0 & 0 & -\varepsilon_2 & -1 \\
0 & (q-t)z & z & 0 & 0 & 0 & 0 & -\varepsilon_1
\end{bmatrix},$$

and the matrix B is defined as

$$B = \begin{bmatrix}
u_1 & x_1 & y_1 & 0 & 0 & 0 & 0 & 0 \\
0 & u_2 & x_2 & -y_3 & -y_4 & 0 & 0 & 0 \\
0 & 0 & u_3 & x_3 & x_4 & 0 & 0 & 0 \\
0 & 0 & 0 & u_4 & 0 & -x_4 & y_4 & 0 \\
0 & 0 & 0 & 0 & -u_4 & -x_3 & y_3 & 0 \\
0 & 0 & 0 & 0 & 0 & -u_3 & -x_2 & -y_1 \\
-z & 0 & 0 & 0 & 0 & 0 & -u_2 & -x_1 \\
0 & z & 0 & 0 & 0 & 0 & 0 & -u_1
\end{bmatrix}.$$

Theorem 1.1 ([NY])**.** *Under the compatibility condition for* (1.3)*, the variables x_i, y_i and u_i are determined as elements of $\mathbb{C}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, q, p, t)$. The compatibility condition is then equivalent to the Hamiltonian system* (1.1) *with* (1.2)*.*

Here we do not describe the explicit forms of u_i, x_i and y_i.

2 Affine Lie algebra

In the notation of [Kac], $\mathfrak{g} = \mathfrak{g}(D_4^{(1)})$ is the affine Lie algebra generated by the Chevalley generators e_i, f_i, $\alpha_i^\vee$ $(i = 0, \ldots, 4)$ and the scaling element d

with the generalized Cartan matrix defined as

$$A = (a_{ij})_{i,j=0}^{4} = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}.$$

We denote the Cartan subalgebra of $\mathfrak{g}$ by $\mathfrak{h}$. The canonical central element of $\mathfrak{g}$ is given by

$$K = \alpha_0^\vee + \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee.$$

We consider the $\mathbb{Z}$-gradation $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k(s)$ of type $s = (1,1,0,1,1)$ by setting

$$\deg \mathfrak{h} = \deg e_2 = \deg f_2 = 0, \quad \deg e_i = 1, \quad \deg f_i = -1 \quad (i = 0,1,3,4).$$

This gradation is defined by

$$\mathfrak{g}_k(s) = \{x \in \mathfrak{g} \mid [d_s, x] = kx\} \quad (k \in \mathbb{Z}),$$

where

$$d_s = 4d + 2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + 2\alpha_4^\vee \in \mathfrak{h}.$$

Denoting by $e_{2i} = [e_2, e_i]$, we choose the graded Heisenberg subalgebra of $\mathfrak{g}$

$$\mathfrak{s} = \{x \in \mathfrak{g} \mid [x, \Lambda] = \mathbb{C}K\},$$

of type $s = (1,1,0,1,1)$ with

$$\Lambda = e_0 - e_1 + e_3 - e_{20} + e_{23} + e_{24}.$$

The positive part of $\mathfrak{s}$ has a graded basis $\{\Lambda_{2k-1,1}, \Lambda_{2k-1,2}\}_{k=1}^{\infty}$ such that

$$\Lambda_{1,1} = \Lambda, \quad \Lambda_{1,2} = e_0 - e_3 + e_4 + e_{20} + e_{21} + e_{23},$$
$$[d_s, \Lambda_{2k-1,i}] = (2k-1)\Lambda_{2k-1,i}, \quad [\Lambda_{2k-1,i}, \Lambda_{2l-1,j}] = 0.$$

Let $\mathfrak{n}_+$ be the subalgebra of $\mathfrak{g}$ generated by e_j $(j = 0, \ldots, 4)$, and let $\mathfrak{b}_+$ be the borel subalgebra of $\mathfrak{g}$ defined by $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$. Then the compatibility condition for (1.3) is equivalent to the system on $\mathfrak{b}_+$

$$\partial_t(M) = [B, d_s + M], \tag{2.1}$$

with

$$M = h(\boldsymbol{\varepsilon}) + (q-t)e_0 + e_1 - pe_2 + (q-1)e_3 + qe_4 - e_{20} - e_{23} - e_{24},$$
$$B = h(\boldsymbol{u}) + e_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + y_1e_{21} + y_3e_{23} + y_4e_{24},$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ and $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$. Here we set

$$\begin{aligned} h(\boldsymbol{\varepsilon}) = &(1 - \varepsilon_1 - \varepsilon_2)\alpha_0^\vee + (\varepsilon_1 - \varepsilon_2)\alpha_1^\vee \\ &+ (\varepsilon_2 - \varepsilon_3)\alpha_2^\vee + (\varepsilon_3 - \varepsilon_4)\alpha_3^\vee + (\varepsilon_3 + \varepsilon_4)\alpha_4^\vee. \end{aligned}$$

We derive the system (2.1) from the Drinfeld-Sokolov hierarchy associated with the Heisenberg subalgebra $\mathfrak{s}$ by a similarity reduction.

3 Drinfeld-Sokolov hierarchy

In the following, we use the notation of infinite dimensional groups

$$G_{<0} = \exp(\widehat{\mathfrak{g}}_{<0}), \quad G_{\geq 0} = \exp(\widehat{\mathfrak{g}}_{\geq 0}),$$

where $\widehat{\mathfrak{g}}_{<0}$ and $\widehat{\mathfrak{g}}_{\geq 0}$ are completions of $\mathfrak{g}_{<0} = \bigoplus_{k<0} \mathfrak{g}_k(s)$ and $\mathfrak{g}_{\geq 0} = \bigoplus_{k\geq 0} \mathfrak{g}_k(s)$ respectively.

Introducing the time variables $t_{k,i}$ $(i = 1, 2; k = 1, 3, 5, \ldots)$, we consider the *Sato equation* for a $G_{<0}$-valued function $W = W(t_{1,1}, t_{1,2}, \ldots)$

$$\partial_{k,i}(W) = B_{k,i}W - W\Lambda_{k,i} \quad (i = 1, 2; k = 1, 3, 5, \ldots), \tag{3.1}$$

where $\partial_{k,i} = \partial/\partial t_{k,i}$ and $B_{k,i}$ stand for the $\mathfrak{g}_{\geq 0}$-component of $W\Lambda_{k,i}W^{-1} \in \widehat{\mathfrak{g}}_{<0} \oplus \mathfrak{g}_{\geq 0}$. The Zakharov-Shabat equation

$$[\partial_{k,i} - B_{k,i}, \partial_{l,j} - B_{l,j}] = 0 \quad (i, j = 1, 2; k, l = 1, 3, 5, \ldots), \tag{3.2}$$

follows from the Sato equation (3.1). Let

$$\Psi = W \exp(\xi), \quad \xi = \sum_{i=1,2} \sum_{k=1,3,\ldots} t_{k,i}\Lambda_{k,i}.$$

Then the Zakharov-Shabat equation (3.2) can be regarded as the compatibility condition of the Lax form

$$\partial_{k,i}(\Psi) = B_{k,i}\Psi \quad (i = 1, 2; k = 1, 3, 5, \ldots). \tag{3.3}$$

Assuming that $t_{k,1} = t_{k,2} = 0$ for $k \geq 3$, we require that the following similarity condition is satisfied:

$$d_s(\Psi) = (t_{1,1}B_{1,1} + t_{1,2}B_{1,2})\Psi. \tag{3.4}$$

The compatibility condition for (3.3) and (3.4) is expressed as

$$[d_s - t_{1,1}B_{1,1} - t_{1,2}B_{1,2}, \partial_{1,i} - B_{1,i}] = 0 \quad (i = 1, 2). \tag{3.5}$$

We regard the systems (3.2) and (3.5) as a similarity reduction of the Drinfeld-Sokolov hierarchy of type $D_4^{(1)}$.

Let $S \subset \mathbb{C}^2$ be an open subset with coordinates $\boldsymbol{t} = (t_{1,1}, t_{1,2})$. Also let

$$\mathcal{M} = d_s - t_{1,1}B_{1,1} - t_{1,2}B_{1,2} \in \mathcal{O}(S; \mathfrak{g}_{\geq 0}),$$
$$\mathcal{B} = B_{1,1}dt_{1,1} + B_{1,2}dt_{1,2} \in \Omega^1(S; \mathfrak{g}_{\geq 0}).$$

Then the similarity reduction is expressed as

$$d_{\boldsymbol{t}}\mathcal{M} = [\mathcal{B}, \mathcal{M}], \quad d_{\boldsymbol{t}}\mathcal{B} = \mathcal{B} \wedge \mathcal{B}. \tag{3.6}$$

4 Derivation of P_{VI}

The operator $\mathcal{M} \in \mathfrak{g}_{\geq 0}$ is expressed as

$$\mathcal{M} = (\text{terms of degree } 0) - t_{1,1}\Lambda_{1,1} - t_{1,2}\Lambda_{1,2}.$$

We consider the gauge transformation for the Lax form (3.4) such that $\mathcal{M} \to \widetilde{\mathcal{M}} \in \mathcal{O}(S; \mathfrak{b}_+)$.

We first consider a gauge transformation $\widehat{\Psi} = \exp(\zeta)\exp(\xi e_2)\Psi$, where $\zeta = \sum_{j=0,1,3,4} \zeta_j \alpha_j^\vee$. This is lifted to the transformation on $\mathfrak{g}_{\geq 0}$:

$$\widehat{\mathcal{M}} = \exp(\mathrm{ad}(\zeta))\exp(\mathrm{ad}(\xi e_2))\mathcal{M},$$
$$d_{\boldsymbol{t}} - \widehat{\mathcal{B}} = \exp(\mathrm{ad}(\zeta))\exp(\mathrm{ad}(\xi e_2))(d_{\boldsymbol{t}} - \mathcal{B}).$$

We look for gauge parameters ζ and ξ such that

$$\widehat{\mathcal{M}} = (\text{terms of degree } 0) - c_0e_0 - e_1 - c_3e_3 - c_4e_4 - e_{20} - e_{23} - e_{24}.$$

where $c_j \in \mathbb{C}(\boldsymbol{t})$ $(j = 0, 3, 4)$. Such gauge parameters are determined uniquely as $\xi = t_{1,2}/t_{1,1}$ and

$$\begin{aligned}
\zeta_0 &= \frac{1}{2}\log\{(t_{1,1}^2 + 2t_{1,1}t_{1,2} - t_{1,2}^2)(t_{1,1}^2 + t_{1,2}^2)t_{1,1}^{-2}\},\\
\zeta_1 &= -\frac{1}{2}\log(-t_{1,1}),\\
\zeta_3 &= \frac{1}{2}\log\{(-t_{1,1}^2 + 2t_{1,1}t_{1,2} + t_{1,2}^2)(t_{1,1}^2 + t_{1,2}^2)t_{1,1}^{-2}\},\\
\zeta_4 &= \frac{1}{2}\log\{(t_{1,1}^2 + 2t_{1,1}t_{1,2} - t_{1,2}^2)(-t_{1,1}^2 + 2t_{1,1}t_{1,2} + t_{1,2}^2)t_{1,1}^{-2}\}.
\end{aligned}$$

Here each c_j is described explicitly as

$$\begin{aligned}
c_0 &= -\frac{1}{t_{1,1}}(t_{1,1}+t_{1,2})(t_{1,1}^2+2t_{1,1}t_{1,2}-t_{1,2}^2)(t_{1,1}^2+t_{1,2}^2),\\
c_3 &= -\frac{1}{t_{1,1}}(t_{1,1}-t_{1,2})(-t_{1,1}^2+2t_{1,1}t_{1,2}+t_{1,2}^2)(t_{1,1}^2+t_{1,2}^2),\\
c_4 &= -\frac{1}{t_{1,1}}t_{1,2}(t_{1,1}^2+2t_{1,1}t_{1,2}-t_{1,2}^2)(-t_{1,1}^2+2t_{1,1}t_{1,2}+t_{1,2}^2).
\end{aligned}$$

We next consider a gauge transformation $\widetilde{\Psi}=\exp(-\lambda f_2)\widehat{\Psi}$. This is lifted to the transformation on $\mathfrak{g}_{\geq 0}$:

$$\widetilde{\mathcal{M}}=\exp(\mathrm{ad}(-\lambda f_2))\widehat{\mathcal{M}},\quad d_{\boldsymbol{t}}-\widetilde{\mathcal{B}}=\exp(\mathrm{ad}(-\lambda f_2))(d_{\boldsymbol{t}}-\widehat{\mathcal{B}}).$$

Denoting by $\eta+\varphi e_2+\psi f_2$ and $u+xe_2+yf_2$ the terms of degree 0 of $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{B}}$ respectively, we look for a gauge parameter λ such that $\widetilde{\mathcal{M}}\in\mathcal{O}(S;\mathfrak{b}_+)$ and $\widetilde{\mathcal{B}}\in\Omega^1(S;\mathfrak{b}_+)$, namely

$$\varphi\lambda^2+(\eta|\alpha_2^\vee)\lambda-\psi=0,\quad d_{\boldsymbol{t}}\lambda=x\lambda^2+(u|\alpha_2^\vee)\lambda-y, \tag{4.1}$$

where $(\,|\,)$ stands for the normalized invariant form. We can verify that the second equation of (4.1) follows from the first equation. Hence the gauge parameter $\lambda=\lambda(\boldsymbol{t})$ can be determined and we obtain

$$\widetilde{\mathcal{M}}=\kappa+\mu e_2+(\lambda-c_0)e_0-e_1+(\lambda-c_3)e_3+(\lambda-c_4)e_4-e_{20}-e_{23}-e_{24},$$

where $\kappa\in\mathfrak{h}$ and $\mu=\mu(\boldsymbol{t})$. Note that $d_{\boldsymbol{t}}\kappa=0$. By definition, it is clear that the operators $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{B}}$ satisfy

$$d_{\boldsymbol{t}}\widetilde{\mathcal{M}}=[\widetilde{\mathcal{B}},\widetilde{\mathcal{M}}]. \tag{4.2}$$

Finally, we consider a transformation of time variables $(t_{1,1},t_{1,2})\to(t_1,t_2)$ such that

$$\partial_1(c_0-c_4)=-4,\quad \partial_1(c_3-c_4)=0.$$

Then by setting

$$q=\frac{\lambda-c_4}{c_3-c_4},\quad p=\frac{1}{4}(c_3-c_4)\mu,\quad \alpha_j=\frac{1}{4}(\kappa|\alpha_j^\vee),\quad t=\frac{c_0-c_4}{c_3-c_4},$$

we arrive at

Theorem 4.1. *Under the specialization $t_2=1$, the system (4.2) is equivalent to the compatibility condition of (1.3) that gives the sixth Painlevé equation.*

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