# Toward the exact WKB analysis for instanton-type solutions of Painlevé hierarchies

Yoshitsugu Takei*
Research Institute for Mathematical Sciences
Kyoto University, Kyoto 606-8502, Japan
E-mail: takei@kurims.kyoto-u.ac.jp

## 1 Introduction

Together with T. Aoki, T. Kawai, T. Koike and partly with Y. Nishikawa, as a generalization of the exact WKB analysis for traditional (i.e., second order) Painlevé equations, we have now been trying to develop a program to analyze $(P_J)$ ($J =$ I, II or IV) hierarchies of higher order Painlevé equations. After the venue of the conference held at Toulouse in 2003, where Kawai first proposed the program, we named this program "the Toulouse Project". The purpose of this paper is to discuss to what extent the Toulouse Project is carried out and what kind of open problems there are in conjunction with this Project.

Recently the so-called instanton-type formal solutions of higher order Painlevé equations are constructed first for the $(P_{\mathrm{I}})$ hierarchy ([T5]) and later for the $(P_{\mathrm{II}})$ and $(P_{\mathrm{IV}})$ hierarchies as well (cf. [Ko]). The construction of instanton-type solutions is one of the most important steps in the Toulouse Project; the instanton-type solutions are expected to be suitable formal solutions for the description of Stokes phenomena for the $(P_J)$ hierarchies, as is suggested by the explicit connection formula for the traditional $(P_{\mathrm{I}})$ equation given in [T1]. The final goal of the Toulouse Project is to give the connection formula for the $(P_J)$ hierarchies explicitly in terms of instanton-type solutions.

Roughly speaking, instanton-type solutions of higher order Painlevé equations play the role of WKB solutions of linear ordinary differential equations with a large parameter. Our exact WKB analysis for instanton-type solutions of Painlevé hierarchies is, however, NOT a straightforward generalization of the exact WKB analysis for linear differential equations. Rather the exact WKB analysis for Painlevé hierarchies is a generalization of the asymptotic analysis for integral representations of solutions of linear equations; we make full use of the underlying Lax pair (i.e., the

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associated isomonodromic deformation) as a substitute of integral representations. Note that the existence of the Lax pair is an expression of the "integrability" of (higher order) Painlevé equations. In this paper we discuss the exact WKB analysis for instanton-type solutions of the $(P_J)$ hierarchies from this viewpoint.

The concrete plan of the paper is as follows: In Section 2 we first recall the definition of the $(P_{\mathrm{I}})$ hierarchy and review the construction of its instanton-type solutions. (In this paper we mainly discuss the $(P_{\mathrm{I}})$ hierarchy for the sake of simplicity and definiteness.) Then, after making a very brief review of the exact WKB analysis for linear differential equations in Section 3, we explain the relevance of the underlying Lax pair (i.e., the associated isomonodromic deformation) in the definition of the Stokes geometry of higher order Painlevé equations and the mechanism how the Stokes phenomena for instanton-type solutions occur in Section 4. Finally in Section 5 we discuss conjectures, results obtained so far, and some important open problems toward the determination of the connection formula for instanton-type solutions.

## 2 $(P_{\mathrm{I}})$ hierarchy and its instanton-type solutions

The main object of the discussion in this paper is the $(P_{\mathrm{I}})$ hierarchy studied by Kudryashov ([Ku, KuSo]), Gordoa and Pickering ([GP]), Shimomura ([S1, S2]) and so on. In what follows we use the following expression $(P_{\mathrm{I}})_m$ $(m = 1, 2, \ldots)$ of the hierarchy, which is obtained through a slight modification of that of Shimomura ([S2]) and appropriate introduction of a large parameter $\eta$ $(> 0)$.

$$(P_{\mathrm{I}})_m \qquad \begin{cases} \dfrac{du_j}{dt} = 2\eta v_j \\ \dfrac{dv_j}{dt} = 2\eta(u_{j+1} + u_1 u_j + w_j) \end{cases} \qquad (j = 1, \ldots, m).$$

Here $u_j$ and $v_j$ are unknown functions (we conventionally assume $u_{m+1} \equiv 0$) and $w_j$ denotes a polynomial of $\{u_k, v_l\}_{1 \le k,l \le j}$ recursively defined by the following relations:

$$w_j = \frac{1}{2}\sum_{k+l=j+1} u_k u_l + \sum_{k+l=j} u_k w_l - \frac{1}{2}\sum_{k+l=j} v_k v_l + c_j + \delta_{jm} t, \tag{1}$$

where $c_j$ is a constant and $\delta_{jm}$ stands for Kronecker's delta. For example, the first member of the hierarchy

$$\begin{cases} \dfrac{du_1}{dt} = 2\eta v_1, \\ \dfrac{dv_1}{dt} = \eta(3u_1^2 + 2c_1 + 2t), \end{cases} \tag{2}$$

that is,

$$\frac{d^2 u_1}{dt^2} = \eta^2(6u_1^2 + 4c_1 + 4t), \tag{3}$$

is equivalent to the traditional first Painlevé equation $(P_{\rm I})$ by appropriate scaling and translation of the independent variable $t$. (This is the reason why we call $(P_{\rm I})_m$ "the $(P_{\rm I})$ hierarchy".) Similarly, the second member $(P_{\rm I})_2$ is equivalent to the following fourth order nonlinear equation for $u = u_1$:

$$\frac{d^4u}{dt^4} = \eta^2 \left( 20u\frac{d^2u}{dt^2} + 10\left(\frac{du}{dt}\right)^2 \right) - \eta^4 \left(40u^3 + 16cu - 16t\right). \tag{4}$$

As $(P_{\rm I})_m$ contains a large parameter $\eta$ in a singular-perturbative manner, we can easily construct a formal power series (in $\eta^{-1}$) solution of $(P_{\rm I})_m$ of the form

$$\hat{u}_j(t,\eta) = u_{j,0}(t) + \eta^{-1}u_{j,1}(t) + \cdots, \quad \hat{v}_j(t,\eta) = v_{j,0}(t) + \eta^{-1}v_{j,1}(t) + \cdots. \tag{5}$$

Note that the top order part $(u_{j,0}(t), v_{j,0}(t))$ of (5) satisfies a system of algebraic equations and the higher order part $(u_{j,l}(t), v_{j,l}(t))$ $(l \geq 1)$ is uniquely determined in a recursive manner once $(u_{j,0}(t), v_{j,0}(t))$ is fixed (cf. [KKoNT]). The solution (5) is called "a 0-parameter solution" of $(P_{\rm I})_m$. The 0-parameter solutions are, however, not sufficient to discuss the Stokes phenomena for $(P_{\rm I})_m$ since they contain no free parameters. A wider class of formal solutions of $(P_{\rm I})_m$ recently constructed in [T5] are expected to play the fundamental role in the description of the Stokes phenomena: They contain $2m$ free parameters and have the following form:

$$\left\{ \begin{array}{rcl} u_j(t,\eta;\alpha) & = & u_{j,0}(t) + \eta^{-1/2} \displaystyle\sum_{1\leq k\leq 2m} \alpha_k \exp\left(\eta \int^t \nu_k dt\right) u_{jk,1/2}(t) + \cdots, \\ v_j(t,\eta;\alpha) & = & v_{j,0}(t) + \eta^{-1/2} \displaystyle\sum_{1\leq k\leq 2m} \alpha_k \exp\left(\eta \int^t \nu_k dt\right) v_{jk,1/2}(t) + \cdots, \end{array} \right. \tag{6}$$

where $\alpha_k \in \mathbb{C}$ $(k = 1, \ldots, 2m)$ are free parameters and $\nu_k = \nu_k(t)$ denote the eigenvalues of the leading coefficient $C_0(t)$ of the following linearized equation $(\Delta P_{\rm I})_m$ of $(P_{\rm I})_m$ at a 0-parameter solution $(\hat{u}_j, \hat{v}_j)$:

$$(\Delta P_{\rm I})_m \qquad \frac{d}{dt}\begin{pmatrix} \Delta u_1 \\ \vdots \\ \Delta v_m \end{pmatrix} = \eta\left(C_0(t) + \eta^{-1}C_1(t) + \cdots\right)\begin{pmatrix} \Delta u_1 \\ \vdots \\ \Delta v_m \end{pmatrix}.$$

(Note that the eigenvalues $\{\nu_k\}_{1\leq k\leq 2m}$ can be numbered so that $\nu_l + \nu_{l+m} = 0$ holds for $l = 1, \ldots, m$.) The solution (6) is called "an instanton-type solution" of $(P_{\rm I})_m$.

**Outline of the construction of instanton-type solutions**

In [T5] instanton-type solutions of $(P_{\rm I})_m$ are constructed by using reduction to Birkhoff normal form. To be more specific, we first express $(P_{\rm I})_m$ in the form of a Hamiltonian system with an appropriately chosen canonical variable $(q_j, p_j)$ and next consider the "localization" at a 0-parameter solution $(\hat{q}_j, \hat{p}_j)$ corresponding to $(\hat{u}_j, \hat{v}_j)$:

$$q_j = \hat{q}_j + \eta^{-1/2}\psi_j, \quad p_j = \hat{p}_j + \eta^{-1/2}\varphi_j. \tag{7}$$

It is readily confirmed that $(\psi_j, \varphi_j)$ also satisfies a Hamiltonian system of the form

$$\frac{d\psi_j}{dt} = \eta\frac{\partial K}{\partial \varphi_j}, \quad \frac{d\varphi_j}{dt} = -\eta\frac{\partial K}{\partial \psi_j}. \tag{8}$$

We then consider the reduction of (8) to its Birkhoff normal form, that is, we construct a (formal) canonical transform

$$\psi_j = \sum_{k=0}^{\infty} \eta^{-k/2}\psi_j^{(k)}(t, \tilde{\psi}, \tilde{\varphi}, \eta^{-1/2}), \quad \varphi_j = \sum_{k=0}^{\infty} \eta^{-k/2}\varphi_j^{(k)}(t, \tilde{\psi}, \tilde{\varphi}, \eta^{-1/2}) \tag{9}$$

in such a way that (8) is transformed into

$$\frac{d\tilde{\psi}_j}{dt} = \eta\frac{\partial \tilde{K}}{\partial \tilde{\varphi}_j}, \quad \frac{d\tilde{\varphi}_j}{dt} = -\eta\frac{\partial \tilde{K}}{\partial \tilde{\psi}_j} \quad \text{with } \tilde{K} = \tilde{K}(t, \rho_1, \ldots, \rho_m, \eta^{-1/2})\Big|_{\rho_j = \tilde{\psi}_j\tilde{\varphi}_j}. \tag{10}$$

Since the Birkhoff normal form (10) can be easily solved, by substituting its solution into (9) we obtain a formal solution of (8) and hence that of $(P_{\mathrm{I}})_m$. This is an outline of the construction of instanton-type solutions.

For example, an instanton-type solution of (4) (i.e., the second member $(P_{\mathrm{I}})_2$ of the hierarchy) is given as follows:

$$\begin{aligned} u(t,\eta;\alpha,\beta) &= u_0(t) + \\ &\eta^{-1/2}\Bigg[\frac{\alpha_1}{(\nu_1^2\Delta)^{1/4}}\,\theta_{11}^{\alpha_1\beta_1}\theta_{12}^{\alpha_2\beta_2}\,e^{\eta\int^t \nu_1 dt} + \frac{\alpha_2}{(\nu_2^2\Delta)^{1/4}}\,\theta_{21}^{\alpha_1\beta_1}\theta_{22}^{\alpha_2\beta_2}\,e^{\eta\int^t \nu_2 dt} + \\ &\frac{\beta_1}{(\nu_1^2\Delta)^{1/4}}\,\theta_{11}^{-\alpha_1\beta_1}\theta_{12}^{-\alpha_2\beta_2}\,e^{-\eta\int^t \nu_1 dt} + \frac{\beta_2}{(\nu_2^2\Delta)^{1/4}}\,\theta_{21}^{-\alpha_1\beta_1}\theta_{22}^{-\alpha_2\beta_2}\,e^{-\eta\int^t \nu_2 dt}\Bigg] + \cdots, \end{aligned} \tag{11}$$

where $u_0$ is the top order part of the 0-parameter solution satisfying $40u_0^3 + 16cu_0 - 16t = 0$, $\pm\nu_1$ and $\pm\nu_2$ denote the eigenvalues of the coefficient $C_0(t)$ of $(\Delta P_{\mathrm{I}})_2$ which are explicitly given by

$$\nu_1^2 = 10u_0 + 2\sqrt{\Delta}, \quad \nu_2^2 = 10u_0 - 2\sqrt{\Delta} \quad \text{with} \quad \Delta = -(5u_0^2 + 4c), \tag{12}$$

and $\theta_{jk}(t)$ are functions of $t$ defined by the following formulas:

$$\begin{cases} \theta_{11} = \exp\left(\displaystyle\int^t \frac{8\nu_1^4 - 27\nu_1^2\nu_2^2 + 15\nu_2^4}{\nu_1^4\nu_2^2\Delta}dt\right), \\ \theta_{12} = \exp\left(-\displaystyle\int^t \frac{12\nu_1^4 - 16\nu_1^2\nu_2^2 + 12\nu_2^4}{\nu_1^3\nu_2^3\Delta}dt\right) = \theta_{21}^{-1}, \\ \theta_{22} = \exp\left(-\displaystyle\int^t \frac{15\nu_1^4 - 27\nu_1^2\nu_2^2 + 8\nu_2^4}{\nu_1^2\nu_2^4\Delta}dt\right). \end{cases} \tag{13}$$

Generally speaking, some exponentially small terms should be added to an original asymptotic solution in the Stokes phenomenon. In the expression (6) of instanton-type solutions exponential terms appearing in the coefficients of $\eta^{-1/2}$ are expected to correspond to (the principal part of) such exponentially small terms. In view of the concrete form of (6) we are thus led to the following definition of turning points and Stokes curves of $(P_{\rm I})_m$.

**Definition 1.** (i) A turning point of $(P_{\rm I})_m$ is, by definition, a point where $\nu_j$ and $\nu_{j'}$ merge for some $j \neq j'$. In particular, a point where $\nu_j$ and $\nu_{j+m}$ vanish for some $1 \leq j \leq m$ is called a turning point of the first kind and a point where $\nu_j = \nu_{j'}$ or $\nu_j = \nu_{j'+m}$ holds for some $1 \leq j, j' \leq m$ is called a turning point of the second kind.
(ii) A Stokes curve of $(P_{\rm I})_m$ is a curve defined by the following relation:

$$\mathrm{Im} \int_{\tau}^{t} (\nu_j - \nu_{j'}) dt = 0. \tag{14}$$

Note that this definition of the Stokes geometry of $(P_{\rm I})_m$ coincide with that given in [KKoNT] thanks to the fact that $\nu_j$ are nothing but the eigenvalues of $C_0(t)$ of $(\Delta P_{\rm I})_m$.

The goal of our Project is to analyze the Stokes phenomenon observed on a Stokes curve defined by (14) and to give the connection formula which describes the Stokes phenomenon explicitly.

# 3 Brief review of the exact WKB analysis for linear ordinary differential equations

Before discussing the Stokes phenomenon for the hierarchy $(P_{\rm I})_m$ of higher order Painlevé equations, let us briefly review, as its prototype, the exact WKB analysis for a linear ordinary differential equation with a large parameter $\eta$ of the form

$$\left( \frac{d^m}{dx^m} + a_1(x)\eta \frac{d^{m-1}}{dx^{m-1}} + \cdots + a_m(x)\eta^m \right) \psi = 0. \tag{15}$$

For Eq. (15) there exists a WKB solution

$$\psi_j = \exp\left( \eta \int^x \lambda_j(x) dx \right) \sum_{l=0}^{\infty} \psi_{j,l}(x) \eta^{-(l+1/2)} \quad (j = 1, \ldots, m), \tag{16}$$

where $\lambda_j(x)$ is a root of the characteristic equation

$$\lambda^m + a_1(x)\lambda^{m-1} + \cdots + a_m(x) = 0 \tag{17}$$

of (15). In the exact WKB analysis a WKB solution (16), although being divergent, is given an analytic meaning through the Borel resummation technique, that is, instead of (16) we consider its Borel sum

$$\Psi_j = \int_{-y_j(x)}^{\infty} e^{-\eta y}\psi_{j,B}(x,y)dy \quad (\text{where } y_j(x) = \int^x \lambda_j(x)dx). \tag{18}$$

Here $\psi_{j,B}(x,y)$ denotes the Borel transform

$$\psi_{j,B}(x,y) = \sum_{l=0}^{\infty} \frac{\psi_{j,l}(x)}{\Gamma(l+1/2)}(y+y_j(x))^{l-1/2} \tag{19}$$

of $\psi_j$ and the path of integration of (18) is conventionally taken to be parallel to the positive real axis (cf. Fig. 1).

In the case of a linear equation (15) a Stokes curve is defined by a relation $\mathrm{Im}(y_j(x)-y_{j'}(x)) = 0$, whose true meaning is as follows: For example, let us consider a simple turning point $x = x_0$ where two characteristic roots $\lambda_j(x)$ and $\lambda_{j'}(x)$ merge. Then in a neighborhood of $x_0$ the Borel transform $\psi_{j,B}(x,y)$ has singularity both at $y = -y_j(x)$ and at $y = -y_{j'}(x)$ (cf. Fig. 1). (In general $\psi_{j,B}(x,y)$ is expected to have singularity at $y = -y_k(x)$, $k = 1,\ldots,m$.) Hence at each point of a Stokes

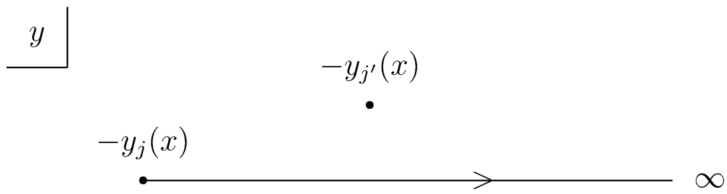


Figure 1 : The path of integration for $\Psi_j$ and singular points of $\psi_{j,B}(x,y)$.

curve the singular point $y = -y_{j'}(x)$ crosses the path of integration of (18) and consequently the Borel sum $\Psi_j$ picks up the contour integral around the singular point $y = -y_{j'}(x)$. This is the Stokes phenomenon for (the Borel sum of) a WKB solution of (15) and such a phenomenon occurs on a Stokes curve. (For the details see, e.g., [T3] and references cited therein.)

Our approach to the analysis of the Stokes phenomenon for a (higher order) Painlevé equation is, however, quite different from that for a general linear equation (15); rather our approach for a Painlevé equation resembles more to the exact WKB analysis for a linear equation whose solutions admit an integral representation of the form

$$\psi = \int_\Gamma e^{\eta f(x,\zeta)}g(x,\zeta)d\zeta. \tag{20}$$

When such an integral representation exists, a WKB solution $\psi_j$ "lives" at a saddle point of (20), i.e., a point $\zeta_j = \zeta_j(x)$ satisfying $(\partial f/\partial\zeta)(x,\zeta_j) = 0$. To be more precise, the Borel sum $\Psi_j$ of a WKB solution corresponds to a solution given by

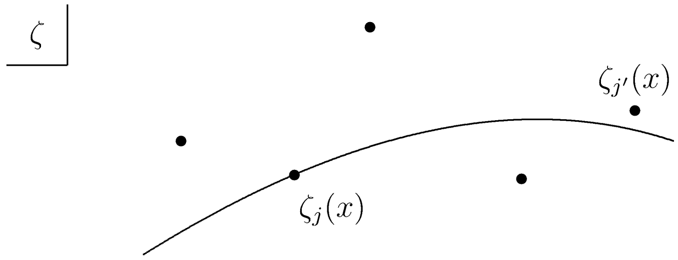


Figure 2 : Steepest descent path of $\mathrm{Re}f(x,\zeta)$ through a saddle point $\zeta_j$.

(20) with $\Gamma$ being a steepest descent path of $\mathrm{Re}f(x,\zeta)$ passing through a saddle point $\zeta_j$. Thus, in this case, a Stokes curve is characterized in the following way:

(21) A point $x$ lies in a Stokes curve if and only if two saddle points $\zeta_j(x)$ and $\zeta_{j'}(x)$ are connected by a steepest descent path of $\mathrm{Re}f(x,\zeta)$.

In view of (21) and the fact that each WKB solution $\psi_j$ lives at a saddle point (in the above sense), we can readily find that the Stokes phenomenon occurs on a Stokes curve as a consequence of the topological change of configuration of steepest descent paths. See [T2] for more detailed discussions.

# 4 Underlying Lax pair of $(P_{\mathrm{I}})_m$ and Stokes phenomena for instanton-type solutions

As is mentioned in the preceding section, the exact WKB analysis for a linear equation whose solutions admit an integral representation can be regarded as a prototype of our WKB analysis for a (higher order) Painlevé equation. Then, *what is the integral representation for a (higher order) Painlevé equation?* The answer is the isomonodromic deformations (or the so-called "Lax pair") of linear equations that underlie a Painlevé equation in question.

It is well-known that each member $(P_{\mathrm{I}})_m$ of the $(P_{\mathrm{I}})$ hierarchy describes the compatibility condition of the following system of first order $2\times 2$ linear differential equations

$$(L_{\mathrm{I}})_m \qquad \frac{\partial}{\partial x}\vec{\varphi} = \eta A\vec{\varphi}, \quad \frac{\partial}{\partial t}\vec{\varphi} = \eta B\vec{\varphi},$$

where

$$(22) \qquad A = \begin{pmatrix} V(x)/2 & U(x) \\ (2x^{m+1} - xU(x) + 2W(x))/4 & -V(x)/2 \end{pmatrix},$$

$$(23) \qquad B = \begin{pmatrix} 0 & 2 \\ u_1 + x/2 & 0 \end{pmatrix},$$

or, equivalently, the compatibility condition of the following two differential equations with one unknown function

$$(SL_{\mathrm{I}})_m \qquad \left(\frac{\partial^2}{\partial x^2} - \eta^2 Q_{(\mathrm{I},m)}\right)\psi = 0,$$

$$(D_{\mathrm{I}})_m \qquad \frac{\partial \psi}{\partial t} = A_{(\mathrm{I},m)}\frac{\partial \psi}{\partial x} - \frac{1}{2}\frac{\partial A_{(\mathrm{I},m)}}{\partial x}\psi,$$

where

$$\begin{aligned} Q_{(\mathrm{I},m)} &= \frac{1}{4}(2x^{m+1} - xU + 2W)U + \frac{1}{4}V^2 \\ &\quad -\eta^{-1}\frac{U_x V}{2U} + \eta^{-1}\frac{V_x}{2} + \eta^{-2}\frac{3U_x^2}{4U^2} - \eta^{-2}\frac{U_{xx}}{2U}, \end{aligned} \tag{24}$$

$$A_{(\mathrm{I},m)} = \frac{2}{U}. \tag{25}$$

Here $U = U(x)$, $V = V(x)$ and $W = W(x)$ respectively denote the following polynomials in $x$ and $U_x$ etc. designate their derivatives with respect to $x$:

$$U(x) = x^m - u_1 x^{m-1} - \cdots - u_m, \tag{26}$$

$$V(x) = v_1 x^{m-1} + \cdots + v_m, \tag{27}$$

$$W(x) = w_1 x^{m-1} + \cdots + w_m. \tag{28}$$

In our WKB analysis of $(P_{\mathrm{I}})_m$ the underlying Lax pair $(L_{\mathrm{I}})_m$ (or $(SL_{\mathrm{I}})_m$ and $(D_{\mathrm{I}})_m$; in what follows we mainly use $(SL_{\mathrm{I}})_m$ and $(D_{\mathrm{I}})_m$ for the sake of convenience of explanation) plays the same role as an integral representation of solutions in the following sense.

We first substitute an instanton-type solution $(u_j(t,\eta;\alpha), v_j(t,\eta;\alpha))$ into the coefficients of $(L_{\mathrm{I}})_m$ (or $(SL_{\mathrm{I}})_m$ and $(D_{\mathrm{I}})_m$). Then we can verify

**Proposition 1.** *(i) Let $Q_{(I,m),0}$ denote the top order part of the potential $Q_{(I,m)}$ of $(SL_{\mathrm{I}})_m$ (i.e., the top order part of the discriminant of the characteristic equation of $A$, the coefficient of the first equation of $(L_{\mathrm{I}})_m$). Then it is factorized as*

$$Q_{(I,m),0} = \frac{1}{4}(x + 2u_{1,0})U_0(x)^2, \tag{29}$$

*where $u_{1,0}$ and $U_0(x)$ denote the top order part of $u_1(t,\eta;\alpha)$ and $U(x)$, respectively. Hence $(SL_{\mathrm{I}})_m$ (or the first equation of $(L_{\mathrm{I}})_m$) has one simple turning point at $x = -2u_{1,0}$, which will be denoted by $a(t)$ in what follows, and $m$ double turning points at zeros of $U_0(x)$, which will be denoted by $b_j(t)$ $(j = 1,\ldots,m)$ as well.*

*(ii) At each point in a Stokes curve of $(P_{\mathrm{I}})_m$ two turning points of $(SL_{\mathrm{I}})_m$ are connected by a Stokes curve of $(SL_{\mathrm{I}})_m$ (cf. Fig. 3). More specifically, a simple turning point $a(t)$ and a double turning point $b_j(t)$ (resp., two double turning points $b_j(t)$ and $b_{j'}(t)$) are connected by a Stokes curve of $(SL_{\mathrm{I}})_m$ at a point in a Stokes curve of $(P_{\mathrm{I}})_m$ emanating from a turning point of the first kind (resp., the second kind).*

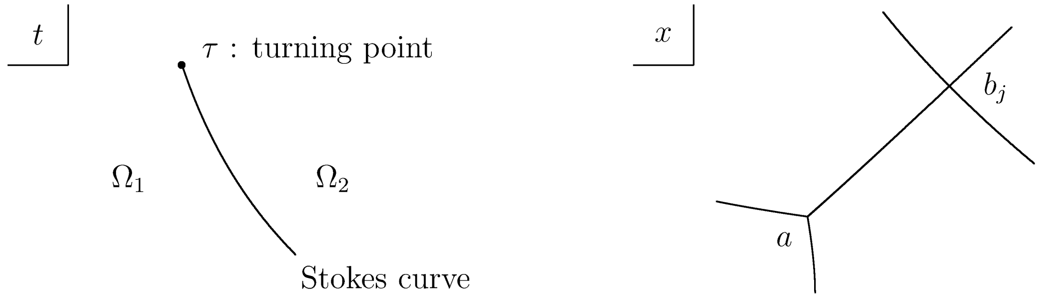


Figure 3 : Degenerate configuration of the Stokes geometry of $(SL_{\mathrm{I}})_m$ observed on a Stokes curve of $(P_{\mathrm{I}})_m$.

For the proof see [KKoNT, Section 2]. Note that the claim (ii) of Prop. 1 is a counterpart of (21).

In the case of a linear equation with an integral representation of solutions the Stokes phenomenon on a Stokes curve is a consequence of (21), that is, it is caused by the topological change of configuration of steepest descent paths of (20). For $(P_{\mathrm{I}})_m$ such a straightforward understanding of the Stokes phenomenon is not possible. However, combining the claim (ii) of Prop. 1 with the isomonodromic property of $(SL_{\mathrm{I}})_m$, we can explain the mechanism how the Stokes phenomenon occurs on a Stokes curve also for $(P_{\mathrm{I}})_m$ as follows.

**Mechanism how the Stokes phenomenon for $(P_{\mathrm{I}})_m$ occurs**

As is schematically shown in Fig. 3, two Stokes regions $\Omega_1$ and $\Omega_2$ in $t$-plane are sharing a Stokes curve of $(P_{\mathrm{I}})_m$ in question as a common boundary. Let $(u_j(t,\eta;\alpha), v_j(t,\eta;\alpha))$ be an instanton-type solution in $\Omega_1$ and let us denote its analytic continuation in $\Omega_2$ across the Stokes curve by $(u_j(t,\eta;\tilde{\alpha}), v_j(t,\eta;\tilde{\alpha}))$. It is expected that a Stokes phenomenon occurs on the Stokes curve and some exponentially small terms are added to $(u_j(t,\eta;\alpha), v_j(t,\eta;\alpha))$; consequently the free parameter $\tilde{\alpha}$ of the corresponding solution in $\Omega_2$ may become different from the original parameter $\alpha$. We want to explain why such a change of free parameters occurs on a Stokes curve and how an explicit formula describing it can be obtained.

Prop. 1, (ii) claims that on a Stokes curve of $(P_{\mathrm{I}})_m$ (i.e., curve in $t$-plane) two turning points are connected by a Stokes curve of $(SL_{\mathrm{I}})_m$ (i.e., curve in $x$-plane). Hence the configuration of Stokes curves of $(SL_{\mathrm{I}})_m$ when $t$ belongs to $\Omega_1$ is different from the configuration when $t$ belongs to $\Omega_2$ (cf. Fig. 4). Now, applying the exact WKB analysis for linear ordinary differential equations (cf. [KT1] and references cited therein), we compute the monodromy data of $(SL_{\mathrm{I}})_m$ when $t \in \Omega_1$ and $t \in \Omega_2$, respectively. Since the computation of monodromy data through the exact WKB analysis heavily depends on the configuration of Stokes curves, the concrete expressions of monodromy data thus obtained, which become functions of the parameters $\alpha$ and $\tilde{\alpha}$, should be different (as functions of the parameters) according as $t$ belongs

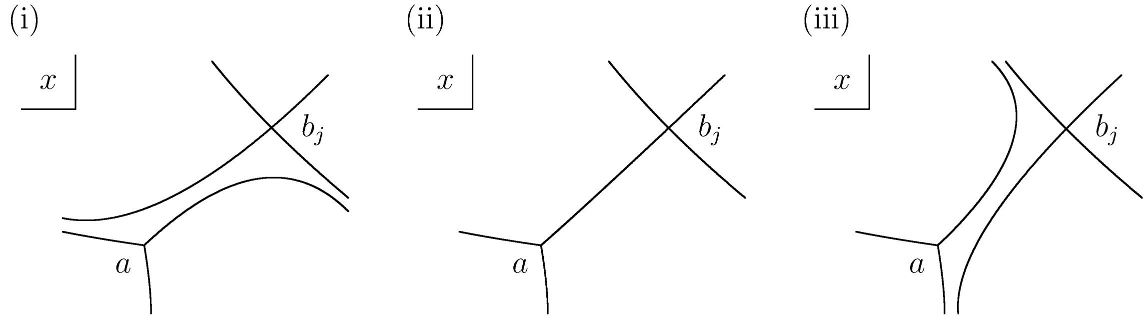


Figure 4 : Configuration of Stokes curves of $(SL_{\mathrm{I}})_m$ when (i) $t$ belongs to $\Omega_1$, (ii) $t$ lies in a Stokes curve and (iii) $t$ belongs to $\Omega_2$.

to $\Omega_1$ or $\Omega_2$.

$$\begin{aligned} &\text{If } t \in \Omega_1 \quad \Rightarrow \quad \text{monodromy data of } (SL_{\mathrm{I}})_m : M_k(\alpha). \\ &\text{If } t \in \Omega_2 \quad \Rightarrow \quad \text{monodromy data of } (SL_{\mathrm{I}})_m : \widetilde{M}_k(\tilde{\alpha}). \end{aligned} \tag{30}$$

Although the expressions $M_k(\alpha)$ and $\widetilde{M}_k(\tilde{\alpha})$ are different, their values, i.e., the monodromy data themselves should be unchanged thanks to the isomonodromic property if $(u_j(t,\eta;\tilde{\alpha}), v_j(t,\eta;\tilde{\alpha}))$ is the analytic continuation of $(u_j(t,\eta;\alpha), v_j(t,\eta;\alpha))$. We thus obtain

$$M_k(\alpha) = \widetilde{M}_k(\tilde{\alpha}). \tag{31}$$

The relation (31) immediately implies $\alpha$ and $\tilde{\alpha}$ are different in general. Moreover (31) describes an explicit formula ("general connection formula" for instanton-type solutions) which relates $\tilde{\alpha}$ to $\alpha$. The relation (31) thus explains the mechanism how the Stokes phenomenon for instanton-type solutions occurs on a Stokes curve of $(P_{\mathrm{I}})_m$.

In conclusion, Prop. 1, (ii) together with the explicit computation of monodromy data of $(SL_{\mathrm{I}})_m$ enables us to explicitly analyze the Stokes phenomena for $(P_{\mathrm{I}})_m$. However it is a quite complicated and troublesome task to compute the monodromy data of $(SL_{\mathrm{I}})_m$ in general. In the subsequent section, to write down the connection formula in a neat way, we will discuss the generalization of "the normal form theory at a turning point" established for traditional (i.e., second order) Painlevé equations in [KT2] to hierarchies of higher order Painlevé equations.

# 5 Toward the connection formula for instanton-type solutions — Discussion and open problems

In [KT2] we showed that every 2-parameter instanton-type solution of a traditional Painlevé equation $(P_J)$ $(J = \mathrm{I}, \ldots, \mathrm{VI})$ can be transformed (in the formal sense) to that of the first Painlevé equation ($P_{\mathrm{I}}$) near a simple turning point. (Note that all turning points of ($P_J$) are of the first kind in the sense of Def. 1.) Otherwise stated, ($P_{\mathrm{I}}$) is the normal form (or, canonical equation) at a simple turning point for ($P_J$). This reduction (more precisely, local equivalence) theorem would imply that the connection formula for instanton-type solutions of ($P_J$) should be the same as that of ($P_{\mathrm{I}}$). In this section we discuss its generalization to the hierarchy $(P_{\mathrm{I}})_m$ of higher order first Painlevé equations.

Let us here recall the fact that in the case of a linear equation with an integral representation of solutions each WKB solution lives at a saddle point of the integral representation. Toward the verification of the reduction (local equivalence) theorem near a turning point for $(P_{\mathrm{I}})_m$, we first consider a counterpart of this fact, that is, taking the claim (ii) of Prop. 1 into account, we ask the following question: *Does an instanton-type solution of $(P_{\mathrm{I}})_m$ "live" at a double turning point $x = b_j(t)$ of the underlying linear equation $(SL_{\mathrm{I}})_m$?* The precise formulation of our expectation is the following

**Conjecture 1.** *Assume that an instanton-type solution $(u_j(t,\eta;\alpha), v_j(t,\eta;\alpha))$ is substituted into the coefficients of $(SL_{\mathrm{I}})_m$ and $(D_{\mathrm{I}})_m$. Then at a double turning point $x = b_j(t)$ of $(SL_{\mathrm{I}})_m$ the simultaneous equations $(SL_{\mathrm{I}})_m$ and $(D_{\mathrm{I}})_m$ can be transformed into*

$$(Can) \qquad \left(\frac{\partial^2}{\partial z^2} - \eta^2 Q_{\mathrm{can}}(z,s,\eta)\right)\varphi = 0,$$

$$(D_{\mathrm{can}}) \qquad \frac{\partial \varphi}{\partial s} = A_{\mathrm{can}}\frac{\partial \varphi}{\partial z} - \frac{1}{2}\frac{\partial A_{\mathrm{can}}}{\partial z}\varphi,$$

*where*

$$Q_{\mathrm{can}} = 4z^2 + \eta^{-1}(\rho^2 - 4\sigma^2) + \frac{\eta^{-3/2}\rho}{z - \eta^{-1/2}\sigma} + \frac{3\eta^{-2}}{4(z-\eta^{-1/2}\sigma)^2}, \tag{32}$$

$$A_{\mathrm{can}} = \frac{1}{2(z-\eta^{-1/2}\sigma)}. \tag{33}$$

*(Here $\rho$ and $\sigma$ are considered to be functions of $t$.)*

We can readily verify that ($Can$) and ($D_{\mathrm{can}}$) are compatible if $\rho$ and $\sigma$ satisfy the following Hamiltonian system:

$$\begin{cases} \dfrac{d\rho}{dt} = -4\eta\sigma, \\ \dfrac{d\sigma}{dt} = -\eta\rho. \end{cases} \tag{34}$$

As the compatibility condition of ($Can$) and ($D_{\mathrm{can}}$) is described by a second order equation (34), Conjecture 1 implies that a 2-parameter family of (instanton-type) solutions of $(P_{\mathrm{I}})_m$ lives at a double turning point $x = b_j(t)$ of $(SL_{\mathrm{I}})_m$.

As a matter of fact, there are some supporting evidences of Conjecture 1. For example, let us consider the second member $(P_{\mathrm{I}})_2$ (see (4) for the concrete expression) of the hierarchy and its underlying linear equations $(SL_{\mathrm{I}})_2$ and $(D_{\mathrm{I}})_2$ with an instanton-type solution $(u_j(t,\eta;\alpha,\beta), v_j(t,\eta;\alpha,\beta))$ (cf. (11)) of $(P_{\mathrm{I}})_2$ being substituted into their coefficients. There exist two double turning points $x = b_1(t)$ and $x = b_2(t)$ of $(SL_{\mathrm{I}})_2$. We can then verify that at $x = b_j(t)$ $(j = 1, 2)$, if we temporarily ignore the deformation equation $(D_{\mathrm{I}})_2$, $(SL_{\mathrm{I}})_2$ is transformed into $(Can)$. To be more specific, we can find

$$\sigma^{(j)}(t,\eta) = \sum_l \eta^{-l/2}\sigma^{(j)}_{l/2}, \qquad \rho^{(j)}(t,\eta) = \sum_l \eta^{-l/2}\rho^{(j)}_{l/2}, \tag{35}$$

so that $(SL_{\mathrm{I}})_2$ can be transformed into the following equation:

$$\left(\frac{\partial^2}{\partial z^2} - \eta^2 Q^{(j)}_{\mathrm{can}}\right)\varphi = 0, \tag{36}$$

with

$$Q^{(j)}_{\mathrm{can}} = 4z^2 + \eta^{-1}((\rho^{(j)})^2 - 4(\sigma^{(j)})^2) + \frac{\eta^{-3/2}\rho^{(j)}}{z - \eta^{-1/2}\sigma^{(j)}} + \frac{3\eta^{-2}}{4(z - \eta^{-1/2}\sigma^{(j)})^2}. \tag{37}$$

Furthermore the top order part of $\sigma^{(j)}$ and that of $\rho^{(j)}$ are explicitly given by

$$\sigma^{(j)}_0 = \frac{1}{2\sqrt{2}}\left[\alpha_j\theta_{j1}^{\alpha_1\beta_1}\theta_{j2}^{\alpha_2\beta_2}e^{\eta\int^t \nu_j dt} + \beta_j\theta_{j1}^{-\alpha_1\beta_1}\theta_{j2}^{-\alpha_2\beta_2}e^{-\eta\int^t \nu_j dt}\right], \tag{38}$$

$$\rho^{(j)}_0 = \frac{1}{\sqrt{2}}\left[-\alpha_j\theta_{j1}^{\alpha_1\beta_1}\theta_{j2}^{\alpha_2\beta_2}e^{\eta\int^t \nu_j dt} + \beta_j\theta_{j1}^{-\alpha_1\beta_1}\theta_{j2}^{-\alpha_2\beta_2}e^{-\eta\int^t \nu_j dt}\right]. \tag{39}$$

One important point of the formulas (38) and (39) is the following: $\sigma^{(1)}$ and $\rho^{(1)}$ (resp., $\sigma^{(2)}$ and $\rho^{(2)}$) contain the free parameters $\alpha_1$ and $\beta_1$ (resp., $\alpha_2$ and $\beta_2$) only, that is, "separation of free parameters" is occurring with the top order part of $\sigma^{(j)}$ and $\rho^{(j)}$.

More generally, for $(P_{\mathrm{I}})_m$ and its underlying linear equation $(SL_{\mathrm{I}})_m$ we can confirm the following:

**Proposition 2.** *(i) If we substitute an instanton-type solution $(u_j(t,\eta;\alpha), v_j(t,\eta;\alpha))$ of $(P_{\mathrm{I}})_m$ into the coefficients of $(SL_{\mathrm{I}})_m$, we can find $\sigma^{(j)}(t,\eta)$ and $\rho^{(j)}(t,\eta)$ of the form* (35) *so that $(SL_{\mathrm{I}})_m$ is transformed into* (36) *with* (37) *at a double turning point $x = b_j(t)$ $(j = 1, \ldots, m)$.*

*(ii) Furthermore "separation of free parameters" (in the above sense) is observed with the top order part of $\sigma^{(j)}$ and $\rho^{(j)}$.*

In view of the discussion employed in [AKT] and [KT2], we believe Prop. 2 should be the first step toward the verification of Conjecture 1 and the reduction (local equivalence) to $(P_{\mathrm{I}})$ near a turning point of the first kind. In particular, Prop. 2, (ii), i.e., "separation of free parameters", which is a phenomenon peculiar to higher order equations, would play an important role in the construction of a transformation of the simultaneous equations $(SL_{\mathrm{I}})_m$ and $(D_{\mathrm{I}})_m$ to $(Can)$ and $(D_{\mathrm{can}})$. Once such a transformation is constructed (i.e., Conjecture 1 is verified), it is expected that the reduction (local equivalence) to $(P_{\mathrm{I}})$ may be proved in a similar manner to the case of traditional Painlevé equations. We hope we can prove Conjecture 1 and discuss the reduction (local equivalence) to $(P_{\mathrm{I}})$ somewhere in the near future.

Finally, in ending this report, we list up some relevant open problems in the exact WKB analysis for instanton-type solutions of $(P_{\mathrm{I}})_m$.

**Some open problems in the exact WKB analysis of** $(P_{\mathrm{I}})_m$

(A) To prove Conjecture 1.

(B) To establish the reduction (local equivalence) of $(P_{\mathrm{I}})_m$ to $(P_{\mathrm{I}})$ near a turning point of the first kind. Note that the local equivalence theorem for 0-parameter solutions has already been proved in [KT4]. (See [KT3] for its announcement; cf. [T4] also.)

(C) To establish the reduction (local equivalence) theorem near a turning point of the second kind. In particular, which equation is the normal form at a turning point of the second kind?

(D) To study the Stokes phenomenon on a "new Stokes curve" of $(P_{\mathrm{I}})_m$ discovered by Nishikawa (cf. [N], [KKoNT]).

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