RIMS Kôkyûroku Bessatsu
B20 (2010), 129–162

Heckman-Opdam hypergeometric functions and their specializations

By

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Abstract

We discuss three topics, confluence, restrictions, and real forms for the Heckman-Opdam hypergeometric functions.

Introduction

In this article, we discuss three topics on the Heckman-Opdam hypergeometric functions. The Heckman-Opdam hypergeometric functions were introduced by Heckman and Opdam [HO, Hec1, Op1, Op2, Op3]. They are joint eigenfunction of a Weyl group invariant commuting family of differential operators on an Euclidean space associated to a root system and a parameter, which is a function on the roots. By a gauge transformation, the commuting family of differential operators gives an quantum integrable system, which is called the Sutherland model. Among joint eigenfunctions, the Heckman-Opdam hypergeometric function is characterized by the properties that it is real analytic and its value at the origin is 1. Opdam [Op3] proved the Gauss summation formula for the Heckman-Opdam hypergeometric function, which asserts that the value of the hypergeometric function at the origin is 1. Thus the Heckman-Opdam hypergeometric function is determined by a root system, a parameter attached to the roots (that is generic) and a parameter of the eigenvalue. For some special values of the

Received December 20, 2008. Accepted May 18, 2010.
2000 Mathematics Subject Classification(s): Primary 33C67; Secondary 22E45, 33D67, 33D80, 33F10, 34M35, 43A90
Key Words: *hypergeometric functions, root systems*
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parameter for the roots, the hypergeometric functions are radial parts of zonal spherical functions on Riemannian symmetric spaces. Thus the hypergeometric functions are generalization of the zonal spherical functions. If the root system is of rank one, then the hypergeometric function turns out to be the Gauss hypergeometric function by a change of variable. Though there is no underlying Lie group and their integral representations are not known in general, the theory of the Heckman-Opdam hypergeometric functions are well developed by using techniques used by Harish-Chandra in group case and also those of representations of Hecke algebras, and serve a good class of multivariable hypergeometric functions. A brief review of the Heckman-Opdam hypergeometric functions is given in §1.

The first topic, which is discussed in §2, is "confluences". Among hypergeometric differential equations of one variable, the Gauss equation has three regular singular points, and the Whittaker equation has two singular points one is regular and the other is irregular. There exists a limit transition between differential equations from Gauss to Whittaker, which we call a confluence, because two of the three regular singular points confluent to one irregular singular point. For the solution of those equations, corresponding limit of the Gauss hypergeometric function is the Whittaker function, which is rapidly decreasing at infinity. We consider higher rank counterpart of this phenomenon. A typical example of confluences on the level of commuting families of differential operators, or integrable systems is a confluence from the Sutherland model to the Toda model, by translating the origin to infinity along with the parameter for the roots going to infinity. We show that the corresponding limit of the Heckman-Opdam hypergeometric function is up to constant multiples a unique joint eigenfunction of the Toda model with moderate growth. We also discuss intermediate confluences between Sutherland and Toda models, and give some estimates for the limit functions.

The second topic, which is discussed in §3, is "restrictions". The system of Heckman-Opdam hypergeometric differential equations has singularities on the walls of the Weyl group for the root system. We studied ordinary differential equations that are satisfied by the restriction of the Heckman-Opdam hypergeometric function to a generic point of singular sets of dimension one. The monodromies of the resulting ordinary differential equations can be analyzed by using representation theory of Iwahori-Hecke algebras. In certain restrictions in the case of root systems of type A_{n-1} and BC_n, resulting ordinary differential equations are rigid, that is, free from accessory parameters, which mean that the differential equations are determined uniquely by their Riemann schemes. For A_{n-1} it is the generalized hypergeometric equation of rank n, and for BC_n it corresponds to the even family of rank $2n$ in Simpson's list of rigid local systems [Si]. For these two cases, we give another proof of the Gauss summation formula for the Heckman-Opdam hypergeometric function together with its asymptotic behavior at infinity by using connection formulae for Fuchsian differential equations.

The third topic, which is discussed in §4, is "real forms" of the Heckman-Opdam hypergeometric system. The Heckman-Opdam hypergeometric system and hypergeometric function are considered mainly on a real Euclidean space, though sometimes techniques of complex analysis are employed. We consider the hypergeometric system on other real sections in the complexification of the original Euclidean space by changes of variables and study the spaces of real analytic solutions. For some special values of parameter for the roots, the hypergeometric systems are radial parts of invariant differential equations on Riemannian symmetric spaces G/K. In these cases, another real form means radial parts of invariant differential operators on a pseudo-Riemannian symmetric space G/K_ϵ with respect to the generalized Cartan decomposition $G = KAK_\epsilon$. In those group cases, Oshima and Sekiguchi [OS2] studied the space of real analytic joint eigenfunctions. Namely they determined the dimension of the solution space, gave an explicit formula for a basis and proved a functional equation. We generalize their results for generic parameters for the roots. The results are compatible with those of [OS2] with minor changes, though the method is different because of the absence of underling Lie groups.

We do not give proofs of the statements in this article. Sometimes the statements are not given in full details. They will be given in our forthcoming papers. Instead, we include detailed examples of rank one or two cases, because they might give ideas or feelings and be useful for readers.

§ 1. Heckman-Opdam hypergeometric functions

In this section, we review on the Heckman-Opdam hypergeometric functions. We refer to [HS, Part I], [Hec4], and [Op5] for details.

§ 1.1. Commuting family of differential operators

Let $\mathfrak{a}$ be an n-dimensional Euclidean space and $\mathfrak{a}^*$ denote its dual space. The inner products on $\mathfrak{a}$ and $\mathfrak{a}^*$ are denoted by $\langle\, ,\, \rangle$. We often identify $\mathfrak{a}$ and $\mathfrak{a}^*$ by using the inner products. Let $\Sigma \subset \mathfrak{a}^*$ be a root system of rank n and W denote its Weyl group. Let R denote one of types of irreducible root systems. If Σ is the irreducible root system of type R, then we write $\Sigma = \Sigma_R$. For example, the root system of type A_{n-1} is denoted by $\Sigma_{A_{n-1}}$. Let k be a complex valued function on Σ such that $k_\alpha = k_{w\alpha}$ for all $\alpha \in \Sigma$, $w \in W$. We call k a multiplicity function. We choose a positive system $\Sigma^+ \subset \Sigma$ and let Ψ denote the set of simple roots.

For $\alpha \in \mathfrak{a}^*$, let ∂_α denote the differential operator on $\mathfrak{a}$ defined by

$$(\partial_\alpha \phi)(x) = \frac{d}{dt}\phi(x + t\,\alpha)\big|_{t=0}\,.$$

Choose an orthonormal basis $\{e_1,\dots,e_n\}$ of $\mathfrak{a}^*$ and define

$$L(k)=\sum_{i=1}^n \partial_{e_i}^2+\sum_{\alpha\in\Sigma^+}2k_\alpha\coth\alpha\,\partial_\alpha.$$

Example 1.1. We give examples for Σ of type BC_n and A_{n-1}.

$$\Sigma^+_{BC_n}=\{e_i\pm e_j,\,(1\le i<j\le n),\,e_p,\ 2e_p\,(1\le p\le n)\},$$
$$\Psi_{BC_n}=\{e_i-e_{i+1}\,(1\le i<n),\,e_n\},$$
$$\begin{aligned}L(k)_{BC_n}&=\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}+\sum_{k=1}^n\bigl(2k_3\coth x_k+4k_2\coth 2x_k\bigr)\frac{\partial}{\partial x_k}\\&+\sum_{1\le i<j\le n}2k_1\Bigl(\coth(x_i-x_j)\Bigl(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\Bigr)+\coth(x_i+x_j)\Bigl(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial x_j}\Bigr)\Bigr).\end{aligned}\tag{1.1}$$

Here we put $k_1=k_{e_i\pm e_j}\ (i\ne j)$, $k_2=k_{2e_i}$, $k_3=k_{e_i}$.

For Σ with type A_{n-1}, we embed $\mathfrak{a}$ in $\mathbb{R}^n$ with orthonormal basis $\{e_1,e_2,\dots,e_n\}$.

$$\mathfrak{a}=\{x\in\mathbb{R}^n\ :\ x_1+x_2+\cdots+x_n=0\},$$
$$\Sigma^+_{A_{n-1}}=\{e_i-e_j\ (1\le i<j\le n)\},$$
$$\Psi_{A_{n-1}}=\{e_i-e_{i+1}\ (1\le i<n)\},$$
$$L(k)_{A_{n-1}}=\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}+\sum_{1\le i<j\le n}2k\coth(x_i-x_j)\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right).\tag{1.2}$$

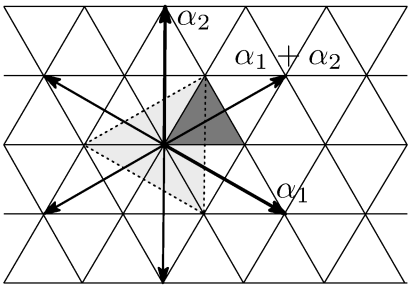
Here we put $k=k_\alpha\,(\alpha\in\Sigma_{A_{n-1}})$.

If Σ is the restricted root system for a Riemannian symmetric space G/K of the noncompact type and $2k_\alpha$ is the dimension of the root space for every $\alpha\in\Sigma$, then $L(k)$ is the radial part of the Laplace-Beltrami operator on $A=\exp\mathfrak{a}$ with respect to the Cartan decomposition $G=KAK$. We call it the group case. For example, $\Sigma=\Sigma_{BC_n}$, $k_1=k_2=1/2$, $k_3=0$ for $G/K=Sp(n,\mathbb{R})/U(n)$, and $\Sigma=\Sigma_{A_{n-1}}$, $k=1/2$ for $G/K=SL(n,\mathbb{R})/SO(n)$.

In the group case, there exists a commuting family of differential operators containing $L(k)$, which consists of radial parts of invariant differential operators, and there is a theory of joint eigenfunction, that is the zonal spherical function on a symmetric space. Heckman and Opdam generalized them for arbitrary k.

There exists a commuting family of differential operators $\mathbb{D}(k)$ containing $L(k)$ for arbitrary k. There exists an algebra isomorphism γ of $\mathbb{D}(k)$ onto $S(\mathfrak{a})^W$, the Weyl group invariants in the symmetric algebra on $\mathfrak{a}$. For $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$, we consider simultaneous eigenvalue problem for $\mathbb{D}(k)$:

$$D\,u=\gamma(D)(\lambda)\,u\quad({}^\forall D\in\mathbb{D}(k)).\tag{1.3}$$



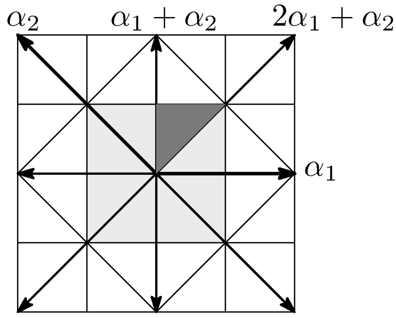


Figure 1. root diagrams for A_2 (left) and B_2 (right)

We call (1.3) the Heckman-Opdam hypergeometric system or (HO) system, shortly. In particular, (HO) system contains the following equation:

$$L(k)u = (\langle\lambda,\lambda\rangle - \langle\rho(k),\rho(k)\rangle)u. \tag{1.4}$$

Here $\rho(k) = \sum_{\alpha\in\Sigma^+} k_\alpha\alpha$.

We mention on a relation between $\mathbb{D}(k)$ and a quantum integrable system. Let

$$\delta(k)^{\frac{1}{2}} = \prod_{\alpha\in\Sigma^+} (2\sinh\alpha)^{k_\alpha}.$$

Then

$$\delta(k)^{\frac{1}{2}} \circ (L(k) + \langle\rho(k),\rho(k)\rangle) \circ \delta(k)^{-\frac{1}{2}} = \sum_{i=1}^n \partial_{e_i}^2 - \sum_{\alpha\in\Sigma^+} \frac{k_\alpha(k_\alpha + 2k_{2\alpha} - 1)\langle\alpha,\alpha\rangle}{\sinh^2\alpha}. \tag{1.5}$$

The right hand side of (1.5) is of the form the Euclidean Laplacian plus a potential function, which is (-2 times) the Sutherland Hamiltonian. $\delta(k)^{\frac{1}{2}} \circ \mathbb{D}(k) \circ \delta(k)^{-\frac{1}{2}}$ gives a commuting family of differential operators containing the Hamiltonian. This proves the complete integrability of the model.

Example 1.2. We give generators of $\mathbb{D}(k)$ for rank 2 cases. Put $\partial_i = \frac{\partial}{\partial x_i}$. For $R = A_2$,

$$\mathfrak{a} = \{x \in \mathbb{R}^3 \,:\, x_1 + x_2 + x_3 = 0\},$$
$$\Sigma_{A_2}^+ = \{e_1 - e_2, e_2 - e_3, e_1 - e_3\}$$

and the simple roots are $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2 - e_3$. The Weyl group is generated by the simple reflection s_i corresponding to α_i for $i = 1, 2$. s_i is the permutation of i and $i+1$. $\mathbb{D}(k)$ (considered as operators on $\mathbb{R}^3$) is generated by algebraically independent differential operators

$$L_1 = \partial_1 + \partial_2 + \partial_3,$$

$$
\begin{aligned}
L_2 =&\partial_1\partial_2+\partial_2\partial_3+\partial_3\partial_1-k\coth(x_1-x_2)(\partial_1-\partial_2)\\
&-k\coth(x_1-x_3)(\partial_1-\partial_3)-k\coth(x_2-x_3)(\partial_2-\partial_3)-4k^2,\\
L_3 =&\partial_1\partial_2\partial_3-k\coth(x_1-x_2)(\partial_1\partial_3-\partial_2\partial_3)\\
&-k\coth(x_1-x_3)(\partial_1\partial_2-\partial_3\partial_2)-k\coth(x_2-x_3)(\partial_2\partial_1-\partial_3\partial_1)\\
&+4k^2\frac{e^{2(x_1-x_3)}+e^{2(x_1-x_2)}}{(e^{2(x_1-x_3)}-1)(e^{2(x_1-x_2)}-1)}\partial_1+4k^2\frac{e^{2(x_2-x_3)}+e^{2(x_2-x_1)}}{(e^{2(x_2-x_3)}-1)(e^{2(x_2-x_1)}-1)}\partial_2\\
&+4k^2\frac{e^{2(x_3-x_1)}+e^{2(x_3-x_2)}}{(e^{2(x_3-x_1)}-1)(e^{2(x_3-x_2)}-1)}\partial_3,
\end{aligned}
$$

with $L(k)=L_1^2-2L_2-8k^2$ (cf. [Sekj1]). Let $(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{C}^3$ with $\lambda_1+\lambda_2+\lambda_3=0$. (HO) system for A_2 is equivalent to the following system of differential equations on $\mathbb{R}^3$:

$$
\begin{aligned}
&L_1u=0,\\
&L_2u=(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)u,\\
&L_3u=\lambda_1\lambda_2\lambda_3u.
\end{aligned}
$$

For $R=BC_2$, we include here the construction of a fourth order operator in $\mathbb{D}(k)$ due to Koornwinder [Koo], instead of the construction by means of trigonometric Dunkl operators. We put $a=k_{2e_1}+k_{e_1}=k_2+k_1$, $b=k_{2e_1}=k_2$, $c=k_{e_1+e_2}=k_3$. ($a=\alpha+1/2$, $b=\beta+1/2$, $c=\gamma+1/2$ in the notation of [Koo].) By the change of variables $t_i=\cosh 2x_i\,(i=1,2)$,

$$
\begin{aligned}
-\frac{L(k)}{4}=&(1-t_1^2)\frac{\partial^2}{\partial t_1^2}+(1-t_2^2)\frac{\partial^2}{\partial t_2^2}\\
&+\left\{b-a-(a+b+1)t_1+\frac{2c(1-t_1^2)}{t_1-t_2}\right\}\frac{\partial}{\partial t_1}\\
&+\left\{b-a-(a+b+1)t_2+\frac{2c(1-t_2^2)}{t_2-t_1}\right\}\frac{\partial}{\partial t_2}.
\end{aligned}
$$

Define differential operators D_- and D_+ by

$$
\begin{aligned}
D_-=&\frac{\partial^2}{\partial t_1\partial t_2}-\frac{c}{t_1-t_2}\left(\frac{\partial}{\partial t_1}-\frac{\partial}{\partial t_2}\right),\\
D_+=&((1-t_1)(1-t_2))^{-a+1/2}((1+t_1)(1+t_2))^{-b+1/2}D_-\\
&\circ((1-t_1)(1-t_2))^{a+1/2}((1+t_1)(1+t_2))^{b+1/2}.
\end{aligned}
$$

Then the operator $L_4=D_+D_-$ satisfies $[L(k),L_4]=0$ and is algebraically independent with $L(k)$, and $\mathbb{D}(k)$ is generated by $L(k)$ and L_4. $D_\pm$ is the hypergeometric shift operator with the shift $k_1=\mp1$, that is

$$
(L_2(k_1\mp1,k_2,k_3)+\rho(k_1\mp1,k_2,k_3))\circ D_\pm
$$

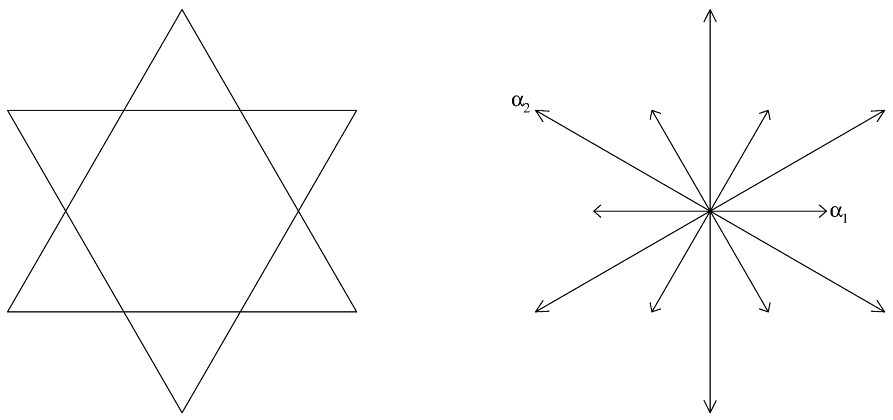


Figure 2. root diagram for G_2

$$= D_{\pm} \circ (L_2(k_1, k_2, k_3) + \rho(k_1, k_2, k_3)).$$

For $R = G_2$,

$$\begin{aligned}
&\mathfrak{a} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_1 + x_2 + x_3 = 0\},\\
&\Sigma_{G_2} = \{\alpha = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \,:\, \lambda_i \in \mathbb{Z} \ (i = 1,\, 2,\, 3),\, \langle \alpha, \alpha \rangle = 2 \text{ or } 6\},\\
&\Sigma^+_{G_2} = \{e_1 - e_2, -2e_1 + e_2 + e_3, -e_1 + e_3, -e_2 + e_3, -2e_2 + e_1 + e_3, 2e_3 - e_1 - e_2\}
\end{aligned}$$

and the simple roots are

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = -2e_1 + e_2 + e_3.$$

Fundamental weights ϖ_1, ϖ_2 are defined by

$$\frac{2\langle \varpi_i,\, \alpha_j \rangle}{\langle \alpha_j,\, \alpha_j \rangle} = \delta_{ij} \quad (1 \le i,\, j \le 2),$$

which turn out to be

$$\begin{aligned}
\varpi_1 &= 2\alpha_1 + \alpha_2 = -e_2 + e_3,\\
\varpi_2 &= 3\alpha_1 + 2\alpha_2 = -e_1 - e_2 + 2e_3.
\end{aligned}$$

Let $z_i = \sum_{w \in (W/\mathrm{stab}(\varpi_i))} \exp(-w\varpi_i(x))$ and write ∂_i for $\partial/\partial z_i$. In this coordinates $L(k)$ and $\rho(k)$ are given by the following expressions (cf. [Op1, Table 2.6]):

$$\begin{aligned}
L(k) =& 2\left(z_1^2 - 3\,z_1 - z_2 - 12\right)\partial_1^2 + 6\left(-2\,z_1^2 + z_1 z_2 + 4\,z_2 + 12\right)\partial_1\partial_2\\
&+ 6\left(-z_1^3 + 3\,z_1 z_2 + z_2^2 + 9\,z_1 + 3\,z_2\right)\partial_2^2
\end{aligned}$$

$$
\begin{aligned}
&+ 2\left((2k_1 + 3k_2 + 1) z_1 + 6k_1\right) \partial_1 \\
&+ 6\left((k_1 + 2k_2 + 1) z_2 + 2k_1 z_1 + 6k_2\right) \partial_2, \\
\rho(k) =& k_1 \varpi_1 + k_2 \varpi_2, \\
\langle \rho(k), \rho(k) \rangle =& 2k_1^2 + 6k_2^2 + 6k_1 k_2.
\end{aligned}
$$

Define

$$
\begin{aligned}
f_1 &= z_1^2 - 4z_2 - 12, \\
f_2 &= -4z_1^3 + z_2^2 + 12z_1 z_2 + 24z_2 + 36z_1 + 36.
\end{aligned}
$$

The weight function becomes

$$\delta(k) = f_1^{k_1} f_2^{k_2}.$$

Let $e_1 = (1,0)$, $e_2 = (0,1)$, and $e = e_1 + e_2 = (1,1)$. The following shift operators $G(l,k)$ for $l = e_1$ and e_2 are given by Opdam ([Op1, Section 2]).

$$
\begin{aligned}
G(e_1,k) =& (2z_1^2 + 18z_1 + z_2 + 30)\partial_1^3 \\
&+ 9\left(z_1 z_2 + 4z_1^2 + z_2 + 6z_1 - 6\right)\partial_1^2\partial_2 \\
&+ 9\left(2z_1^3 + z_2^2 + 3z_1 z_2 + 9z_2 + 18\right)\partial_1\partial_2^2 \\
&+ 27\left(z_1^2 z_2 - {z_2}^2 + 2z_1^2 - 5z_2 - 6\right)\partial_2^3 + (k_1 + 9k_2 + 6)(z_1 + 3)\partial_1^2 \\
&+ 3\left((k_1 + 6k_2 + 6) z_2 + 2(k_1 + 9k_2 + 12) z_1 + 18k_2 + 36\right)\partial_1\partial_2 \\
&+ 9\left((k_1 + 3k_2 + 6) z_1^2 - (2k_1 + 3k_2 + 6) z_2 - 2k_1 z_1 - 3k_1 - 9k_2 - 18\right)\partial_2^2 \\
&+ \left(3k_1 k_2 + 9k_2^2 + k_1 + 9k_2 + 2\right)\partial_1 - 6k_1(3k_2 + 1)\partial_2, \\
G(e_2,k) =& (z_1 + 2)\partial_1^3 + \left(z_1^2 + 3z_1 + 2z_2 + 6\right)\partial_1^2\partial_2 \\
&+ \left(3z_1^2 + 3z_1 z_2 + 9z_1 - 6z_2 - 18\right)\partial_1\partial_2^2 \\
&+ \left(z_1^3 - 3z_1 z_2 - 9z_1 + 2z_2^2 + 12z_2 + 18\right)\partial_2^3 \\
&+ (k_1 + k_2 + 2)\partial_1^2 + \left((2k_1 + k_2 + 4) z_1 - 6k_1\right)\partial_1\partial_2 \\
&+ \left(9k_1 + 3k_2 + 18 + (3k_1 + k_2 + 6) z_2 - 3k_1 z_1\right)\partial_2^2 \\
&+ \left(k_1^2 + 3k_1 + k_1 k_2 + k_2 + 2\right)\partial_2.
\end{aligned}
$$

These two operators are hypergeometric shift operator with shift l in the sense that they satisfy

$$G(l,k) \circ ML(k) = ML(k+l) \circ G(l,k). \tag{1.6}$$

Here we put

$$ML(k) = L(k) + (\rho(k), \rho(k)).$$

By (1.5), we have

$$\delta(k - e/2) \circ ML(k) \circ \delta(k - e/2)^{-1} = ML(e - k) \tag{1.7}$$

Define $G(-e_i,k)$ $(i=1,\,2)$ by

$$G(-e_i,k)=\delta(-k+e_i+e/2)\circ G(e_i,e-k)\circ\delta(k-e/2). \tag{1.8}$$

By (1.6) and (1.7), it is easy to see that $G(-e_i,k)$ is a shift operator with shift $-e_i$. We can see by calculating the image under the Harish-Chandra homomorphism that $G(-e_2,k)$ coincides with the shift operator

$$\tilde{G}(e_2,k)=\delta(e_2-k)\circ G(e_2,k)^*\circ\delta(k)$$

given by Opdam [Op1, Definition 3.3]). Here $G(e_2,k)^*$ means formal transpose as a differential operator on $A=\exp\mathfrak{a}$ with respect to the Haar measure da.

Define L_6 by

$$L_6=G(-e_2,k+e_2)\circ G(e_2,k). \tag{1.9}$$

By (1.6) and (1.8), $[L(k),\,L_6]=0$. Moreover the proof of [Op1, Theorem 3.6] shows that $L(k)$ and L_6 are algebraically independent. Commutativity of $L(k)$ and L_6 also can be proved by the following commutation relations.

$$\begin{aligned}
&[L(k),f_2^{1/2}G(e_2,k)]\\
&\qquad\qquad=-12\,k_2\,(z_1+3)\left({z_1}^2-3\,z_1-3\,z_2\right)f_2^{-1/2}\,G(e_2,k),\\
&[L(k),\,G(-e_2,k+e_2)\circ f_2^{-1/2}]\\
&\qquad\qquad=G(-e_2,k)\circ 12\,k_2(z_1+3)(z_1^2-3\,z_1-3\,z_2)\,f_2^{-3/2}.
\end{aligned}$$

§ 1.2. Heckman-Opdam hypergeometric functions

(HO) system has singularities on the walls of the Weyl group. For generic λ, there exists a unique local solution for (1.4) of the form

$$\Phi(\lambda,k;x)=e^{\langle\lambda-\rho(k),x\rangle}+\cdots\quad(x\to\infty).$$

Here $x\to\infty$ means $\langle\alpha,x\rangle\to\infty$ for all $\alpha\in\Psi$. It turns out that Φ satisfies all equations in (HO) system. In the group case, Φ is the series solution given by Harish-Chandra.

For generic λ, $\{\Phi(w\lambda,k\,;\,\cdot)\,:\,w\in W\}$ forms a basis of the space of local solutions of (HO) system. The connection coefficients are given by $c(\lambda,k)$, which is a generalization of the Harish-Chandra c-function. It is defined by

$$\tilde{c}(\lambda,k)=\prod_{\alpha\in\Sigma^+}\frac{\Gamma\left(\frac{\langle\lambda,\alpha^\vee\rangle+k_{\alpha/2}}{2}\right)}{\Gamma\left(\frac{\langle\lambda,\alpha^\vee\rangle+k_{\alpha/2}+2k_\alpha}{2}\right)},\quad c(\lambda,k)=\frac{\tilde{c}(\lambda,k)}{\tilde{c}(\rho(k),k)}.$$

Here $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$ and $k_{\alpha/2} = 0$ if $\alpha/2 \notin \Sigma$. Define the Heckman-Opdam hypergeometric function by

$$F(\lambda, k; x) = \sum_{w\in W} c(w\lambda, k)\Phi(w\lambda, k; x). \tag{1.10}$$

In a series of papers [HO, Hec1, Op1, Op2, Op3] Heckman and Opdam proved the following theorem:

Theorem 1.3 (Heckman-Opdam). *If k is generic ($\Leftarrow k_\alpha > 0$ for all $\alpha \in \Sigma$), then $F(\lambda, k; x)$ is the unique real analytic solution on $\mathfrak{a}$ of (HO) system with value 1 at $x = 0$. Moreover, it satisfies*

$$F(w\lambda, k; x) = F(\lambda, k; x) \quad (w \in W),$$
$$F(\lambda, k; wx) = F(\lambda, k; x) \quad (w \in W).$$

If the rank of the root system Σ is one, then the Heckman-Opdam hypergeometric function can be written by the Gauss hypergeometric function. Let α denote the unique positive simple root. λ and $\rho(k)$ in $\mathfrak{a}^* \simeq \mathbb{C}$ are given by $\lambda \in \mathbb{C}$, $\rho(k) = k_\alpha + 2k_{2\alpha}$. Then

$$F(\lambda, k; x) = {}_2F_1(\tfrac{1}{2}(\rho(k) - \lambda), \tfrac{1}{2}(\rho(k) + \lambda)\,;\, k_\alpha + k_{2\alpha} + \frac{1}{2}; -\sinh^2 x). \tag{1.11}$$

In the group case, $F(\lambda, k; \cdot)$ is the radial parts of the zonal spherical function on a Riemannian symmetric space.

Remark 1. Differences between notation of symmetric spaces, Heckman-Opdam, and ours may cause confusions. In the group case, Heckman and Opdam use root system 2Σ, where Σ is the restricted root system for a symmetric space, and $2k_\alpha$ is the multiplicity of the restricted root α for the symmetric space. In this article, we use the root system Σ and the multiplicity $2k_\alpha$.

§ 2. Confluences of the Heckman-Opdam system to the Toda systems

The right hand side of (1.5) times $-1/2$ is of the form

$$P = -\frac{1}{2}\sum_{i=1}^{n} \partial_{e_i}^2 + \sum_{\alpha\in\Sigma^+} \frac{C_\alpha}{\sinh^2\langle\alpha, x\rangle} \tag{2.1}$$

where C_α ($\alpha \in \Sigma$) are constants such that $C_{w\alpha} = C_\alpha$ for all $w \in W$. P is the Hamiltonian (Schrödinger operator) for the quantum Sutherland model associated with Σ.

The Sutherland model is completely integrable, that is there exist n algebraically independent mutually commuting differential operators containing (2.1). More generally, consider the Hamiltonian

$$P = -\frac{1}{2}\sum_{i=1}^{n}\partial_{e_i}^2 + \sum_{\alpha\in\Sigma^+} C_\alpha u_\alpha(\langle\alpha, x\rangle), \tag{2.2}$$

where u_α ($\alpha \in \Sigma^+$) are functions with $u_{w\alpha} = u_\alpha$ for all $w \in W$. The quantum model with the Hamiltonian (2.2) is known to be completely integrable for $u(t) = \wp(t, 2\omega_1, 2\omega_2)$ (Weierstrass $\wp$-function), $u(t) = \sinh^{-2}\lambda t$, and $u(t) = t^{-2}$. There is a hierarchy among completely integrable systems. By hierarchy we mean that a complete integrable system is a limit of another one, and so on. See [O2] for the complete list of integrable potentials and the hierarchy among them for classical root systems.

In particular, we are interested in a limit transition from the Sutherland Hamiltonian (2.1) to the Toda Hamiltonian. For $R = A_1$,

$$P = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{C_1}{\sinh^2 x}.$$

Put $C_1 = Ce^{2K}/4$ and let $x \mapsto x + K$. Then we have

$$P \to -\frac{1}{2}\frac{d^2}{dx^2} + Ce^{-2x} \quad (K\to\infty),$$

because

$$\lim_{K\to\infty}\frac{e^{2K}}{4\sinh^2(x+K)} = \lim_{K\to\infty}\frac{e^{-2x}}{(1-e^{-2x-2K})^2} = e^{-2x}.$$

For $R = A_{n-1}$, let $x_i \mapsto x_i - iK$ and ($K\to\infty$). Then $e^{x_j-x_i} \mapsto e^{-(j-i)K}e^{x_j-x_i}$ and

$$\begin{aligned}\sum_{1\le i<j\le n}\frac{Ce^{2K}}{4\sinh^2(x_i-x_j)} &\mapsto \sum_{1\le i<j\le n}\frac{Ce^{2(1-j+i)K}e^{2(x_j-x_i)}}{\left(1-e^{2(x_j-x_i)}e^{-2(j-i)K}\right)^2}\\ &\to \sum_{i=1}^{n-1} Ce^{-2(x_i-x_{i+1})} \qquad (K\to+\infty).\end{aligned}$$

The right hand side is the potential function for the non-periodic Toda model for A_{n-1}.

Non-trivial limits of the Sutherland Hamiltonian (2.1) associated with a root system Σ when $x \mapsto x + Kv$ and $K\to\infty$ with a suitable vector $v \in \mathfrak{a}$ is a direct sum of Hamiltonians for lower rank root systems with the potential functions in the following list. (Here "trivial" means the potential function is a constant.)

(HO-R) *Heckman-Opdam potential of type R:*

$$\sum_{\alpha\in\Sigma_R^+} C_\alpha \sinh^{-2}\langle\alpha, x\rangle \qquad (C_\alpha = C_{\alpha'} \text{ if } |\alpha| = |\alpha'|)$$

(Toda-R) *Toda potential of type R (R: reduced)*:

$$\sum_{\alpha\in\Psi_R} C_\alpha e^{-2\langle\alpha,x\rangle} \qquad (C_\alpha = C_{\alpha'} \ \text{ if } \ |\alpha| = |\alpha'|)$$

(Toda-BC_n) *Toda potential of type BC_n* :

$$C_0 \sum_{i=1}^{n-1} e^{x_i - x_{i+1}} + C_3\, e^{-2x_n} + C_4\, e^{-x_n}$$

(Trig-A_{n-1}-bry-reg) *Trigonometric potential of type A_{n-1} with RS boundary conditions*:

$$\sum_{1\le i<j\le n} C_3 \sinh^{-2}(x_i - x_j) + \sum_{k=1}^{n} \bigl(C_1 e^{-2x_k} + C_2 e^{-4x_k}\bigr)$$

(Toda-B_n-bry-reg) *Toda potential of type B_n with RS boundary conditions*:

$$\sum_{i=1}^{n-1} C_3 e^{-2(x_i - x_{i+1})} + C_3 e^{-2(x_{n-1}+x_n)} + C_1 \sinh^{-2} x_n + C_2 \sinh^{-2} 2x_n$$

(Trig-A_2-Toda-D_4) *Trigonometric and Toda potential of type D_4*:

$$\begin{aligned} &C_1\bigl(\sinh^{-2}(x_2 - x_3) + \sinh^{-2}(x_3 - x_4) + \sinh^{-2}(x_2 - x_4)\bigr) \\ &\qquad + C_2\bigl(e^{-2x_2} + e^{-2x_3} + e^{-2x_4} + e^{x_2+x_3+x_4-x_1}\bigr) \end{aligned}$$

(Trig-A_2-Toda-$D_4^{(d)}$)

$$\begin{aligned} &C_1\bigl(\sinh^{-2}(x_2 - x_3) + \sinh^{-2}(x_3 - x_4) + \sinh^{-2}(x_2 - x_4)\bigr) \\ &\qquad + C_2\bigl(e^{-4x_2} + e^{-4x_3} + e^{-4x_4} + e^{2(x_2+x_3+x_4-x_1)}\bigr) \end{aligned}$$

(HOP-G_2) *Partial confluent Heckman-Opdam potential of type G_2*:

$$C_1 \sinh^{-2}(x_2 - x_3) + C_2\bigl(e^{x_1-2x_2+x_3} + e^{x_1+x_2-2x_3}\bigr)$$

(HOP-$G_2^{(d)}$)

$$C_1 \sinh^{-2}(x_2 - x_3) + C_2\bigl(e^{2(x_1-2x_2+x_3)} + e^{2(x_1+x_2-2x_3)}\bigr)$$

We call these limit transitions "confluences". The confluence to the Hamiltonian with the potential (Toda-R) was first proved by Inozemtsev [In] (see also [E]).

Not only the Hamiltonian but the commuting family of differential operators has a limit and it turns out that the quantum models with the above potential functions are completely integrable. The proof for classical R and references to related studies are given in [O2]. Among integrable systems, they form a class that their joint eigenfunctions are easy to analyze, because they have regular singularities at an infinite point (cf. [O1, O4]) and have no accessory parameter.

Remark 2. Some of the Hamiltonians with potential functions in the above list appear in group cases. If Σ is the restricted root system of a Riemannian symmetric space G/K and Σ_0 is of type R, then Hashizume [Has] showed that (Toda-R) appears as the radial part of the Casimir operator with respect to the Iwasawa decomposition $G = NAK$, a non-degenerate one-dimensional representation of N and the trivial representation of K from the left and right respectively. (Toda-BC_n) appears as the radial part of the Casimir operator for a Hermitian symmetric space G/K of tube type with respect to the Iwasawa decomposition $G = NAK$, a non-degenerate one-dimensional representation of N and a one-dimensional representation of K from the left and right respectively. (Trig-A_{n-1}-bry-reg) with a special value of C_3 appear as the radial part of the Casimir operator for a Hermitian symmetric space of tube type with respect to $G = ((L_s \cap K) \ltimes N_s)AK$ where $P_s = L_s \ltimes N_s$ is the Siegel parabolic subgroup of G, the trivial representation of $L_s \cap K$, a non-degenerate one-dimensional representation of N_s from the left, and a one-dimensional representation of K from the right. It was observed by Ishii [Is] for $G = SO_0(2, n)$ and by the second author in general. In these group cases, the commutativity of the algebras of invariant differential operators on G/K prove the complete integrability of the quantum models.

We give an answer to the following problem:

Problem. What is the limit of the Heckman-Opdam hypergeometric function $F(\lambda, k; x)$ corresponding to confluences of the Sutherland model to the Toda models described above?

The answer is that the limit is a joint eigenfunction of the confluent system that is of moderate growth.

For $R = BC_1$, the Heckman-Opdam hypergeometric function can be written by the Gauss hypergeometric function as in (1.11) and $\delta(k)^{1/2} = (\sinh x)^{k_\alpha}(\sinh 2x)^{k_{2\alpha}}$. Put $4\,k_{2\alpha}\,(k_{2\alpha} - 1) = e^{2K}$ and $x \mapsto x + K$. Then we have

$$\lim_{K\to\infty} k_{2\alpha}^{-(k_\alpha+1)/2}\, 2^{-\rho(k)}\, \delta(k, x + K)^{1/2}\, F(\lambda, k, x + K) = e^x W_{-k_\alpha/2,\lambda/2}(e^{-2x}). \tag{2.3}$$

Here $W_{\kappa,\mu}(z)$ is the classical Whittaker function. The Whittaker function is up to constant multiples unique analytic solution of the Whittaker differential equation that is of moderate growth.

We prove (2.3). Recall that for generic λ,

$$F(\lambda,k;x)=c(\lambda,k)\Phi(\lambda,k,x)+c(-\lambda,k)\Phi(-\lambda,k,x),$$

where

$$\Phi(\lambda,k;x)\sim e^{(\lambda-\rho(k))x}\quad (x\to\infty)$$

is the series solution of (1.4) and

$$c(\lambda,k)=\frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2}(\lambda+k_\alpha+1))}\frac{2^{-\lambda+\rho(k)}\,\Gamma(k_{2\alpha}+k_\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2}\lambda+\frac{1}{2}k_\alpha+k_{2\alpha})}.$$

$F(\lambda,k;x)$ is the unique real analytic eigenfunction of

$$L(k)=\frac{d^2}{dx^2}+(2k_\alpha\coth x+4k_{2\alpha}\coth 2x)\frac{d}{dx}$$

with eigenvalue $\lambda^2-\rho(k)^2$ such that $F(\lambda,k;0)=1$. We have

$$\delta(k)\circ(L(k)+\rho(k)^2)\circ\delta(k)^{-1}=\frac{d^2}{dx^2}+\frac{k_\alpha(1-k_\alpha-2k_{2\alpha})}{\sinh^2 x}+\frac{4k_{2\alpha}(1-k_{2\alpha})}{\sinh^2 2x}. \tag{2.4}$$

Let $4\,k_{2\alpha}\,(k_{2\alpha}-1)=e^{2K}$ and $x\mapsto x+K$. Then as $K\to\infty$, the right hand side of (2.4) goes to

$$\frac{d^2}{dx^2}-2k_\alpha e^{-2x}-e^{-4x}. \tag{2.5}$$

As $K\to\infty$, we have

$$\delta(k,x+K)^{1/2}\,\Phi(\lambda,k,x+K)\sim 2^\lambda\,k_{2\alpha}^{\lambda/2}\,\Phi_{\mathrm{T}}(\lambda,k_\alpha,x),$$

where $\Phi_{\mathrm{T}}(\lambda,k_\alpha,x)$ is the eigenfunction of (2.5) with eigenvalue λ^2 such that

$$\Phi_{\mathrm{T}}(\lambda,k_\alpha,x)\sim e^{\lambda x}\quad (x\to\infty)$$

and

$$c(\lambda,k)\sim 2^{-\lambda+\rho(k)}\,k_{2\alpha}^{(-\lambda+k_\alpha+1)/2}\frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2}(\lambda+k_\alpha+1))}.$$

Hence the limit in the left hand side of (2.3) turns out to be

$$\frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2}(\lambda+k_\alpha+1))}\Phi_{\mathrm{T}}(\lambda,k_\alpha,x)+\frac{\Gamma(-\lambda)}{\Gamma(\frac{1}{2}(-\lambda+k_\alpha+1))}\Phi_{\mathrm{T}}(-\lambda,k_\alpha,x),$$

which is equal to $e^xW_{-k_\alpha/2,\lambda/2}(e^{-2x})$ by using a connection formula for the Whittaker function (cf. [WW]).

For the confluence of (HO) to each of eight cases (Toda-R) to (HOP-$G_2^{(d)}$) listed above, a scaled limit of the Heckman-Opdam hypergeometric function is a joint eigenfunction of the confluent system with moderate growth. Let $\mathcal{C}\subset\mathfrak{a}\simeq\mathfrak{a}^*$ denote the positive Weyl chamber. We give main results of this section:

Theorem 2.1. (i) *For a suitable vector $v \in \mathbb{R}^n$ and k_α, (HO) system is holomorphically continued to the confluent commuting system by $x \mapsto x + vK$ with $K \to \infty$.*

(ii) *For* $\operatorname{Re}\lambda \in \bar{\mathcal{C}}$, *the normalized Heckman-Opdam hypergeometric function*

$$\bar{F}(\lambda, k; x) = \delta(k)^{-\frac{1}{2}} c(\lambda, k)^{-1} e^{-\langle \lambda, v\rangle K} F(\lambda, k, x + vK)\cdot\pi(\lambda)$$

and its expansion (at the infinity of $\mathcal{C}$) converge to the solution $\bar{W}(x)$ of the confluent system with moderate growth, that is, there exist $C > 0$ and $m > 0$ such that

$$|\bar{W}(x)| \leq C e^{m|x|}.$$

Here $\pi(\lambda)$ is a certain normalizing factor, which satisfies $\pi(\lambda) = 1$ for $\operatorname{Re}\lambda \in \mathcal{C}$.

(iii) *The global solutions of the confluent system with moderate growth are unique up to constant multiples.*

(iv) *We have an explicit estimate of $\bar{W}(x)$ such as*

$$|\bar{W}(x)| \leq e^{\operatorname{Re}\langle \lambda, x\rangle}, \; |\bar{W}(x)| \leq C \exp(-e^{K \operatorname{dist}(x, \mathcal{C})}) \quad \textit{(Toda)}_{R:reduced}.$$

The existence and uniqueness of joint eigenfunction with moderate growth and estimates for the eigenfunction are proved by rank 1 reduction. We use an estimate of the Heckman-Opdam hypergeometric function (cf. [Sc]) to prove part (iv) of the theorem.

Remark 3. For (Toda-R), it is known that there exists a unique joint eigenfunction with moderate growth up to constant multiples. The eigenfunction with moderate growth is given by the Jacquet integral on a semisimple Lie group. The eigenfunction is the radial part of the class one Whittaker function on a semisimple Lie group, and Hashizume [Has] gave connection formula for the Whittaker function that is similar to (1.10). Shimeno [Sh3] proved that a scaled limit of the Heckman-Opdam hypergeometric function is the radial part of the Jacquet integral, by using an argument similar to the above proof of (2.3).

Hirano-Ishii-Oda [HIO] studied a problem closely related to ours for Whittaker functions on $Sp(2, \mathbb{R})$.

§ 3. Restrictions of Heckman-Opdam systems

In this section, we discuss restrictions of (HO) system to singular sets, that is, intersections of walls of the Weyl group. In particular we restrict (HO) system to a generic point of a singular set of dimension one to get an ordinary differential equation. Monodromies at the origin for the restriction equation are related to representation theory of Hecke algebras. In some cases, we have ordinary differential equations that are free from accessory parameters (cf. [Har, G, Koh, Si]). As an application to the Heckman-Opdam hypergeometric function for A_{n-1} and BC_n, we give another new proof of Theorem 1.3, in particular, the Gauss summation formula, which was proved by Opdam [Op3] for general root systems.

§ 3.1. Rank 2 cases

First we consider rank 2 cases, where we can explicitly compute ordinary differential equation by restricting (HO) system to singular lines. In this subsection, we use the notations given in Example 1.1 and Example 1.2. We employed the Computer Algebra System Maple to do very complicated computations.

3.1.1. A_2 **case** Let $R = A_2$. Then the walls of the Weyl group $\mathfrak{S}_3$ are $x_i = x_j$ $(1 \le i < j \le 3)$ in $\mathfrak{a} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$. $\mathbb{D}(k)$ is generated by $L(k)$ and a third order differential operator, as we gave in Example 1.2.

We consider the restriction of (HO) system to the wall $x_1 = x_2$. Let u be a solution of (HO) system given by the series expansion

$$u(x_1, x_2, x_3) = \sum_{j=0}^{\infty} u_j \left(\frac{x_1 + x_2}{2} - x_3 \right) (x_1 - x_2)^j$$

near $x_1 = x_2$. Substitute u into (HO) system

$$L_1 u = 0, \tag{3.1}$$

$$L_2 u = (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)u, \tag{3.2}$$

$$L_3 u = \lambda_1\lambda_2\lambda_3 u \tag{3.3}$$

and put $x_1 = x_2$. From (3.2) we have

$$-\tfrac{3}{4}u_0''(t) - (4k+2)u_1(t) - 3k\coth t\, u_0'(t) = (\mu_2 + 4k^2)u_0(t).$$

Here $t = x_2 - x_3$ and set $\mu_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$, $\mu_3 = \lambda_1\lambda_2\lambda_3$. Thus

$$(4k+2)u_1(t) = -\tfrac{3}{4}u_0''(t) - 3k\coth t\, u_0'(t) - (\mu_2 + 4k^2)u_0(t), \tag{3.4}$$

$$\begin{aligned}(4k+2)u_1'(t) = -\tfrac{3}{4}u_0'''(t) - 3k(1 - \coth^2 t)u_0'(t)\\ - 3k\coth t\, u_0''(t) - (\mu_2 + 4k^2)u_0'(t).\end{aligned} \tag{3.5}$$

From (3.3) we have

$$\begin{aligned}-\tfrac{1}{4}u_0'''(t) - \tfrac{3}{2}k\coth t\, u_0''(t) + 2k^2(1 - \coth^2 t)u_0'(t)\\ + (4k+2)u_1'(t) + 2k(4k+2)\coth t\, u_1(t) = \mu_3 u_0(t).\end{aligned} \tag{3.6}$$

Eliminating u_1 from (3.4), (3.5), and (3.6), we obtain a differential equation for u_0.

$$\begin{aligned}(3.7)\qquad & u_0'''(t) + 6k\coth t\, u_0''(t) \\ & +\{(10k^2-3k)\coth^2 t - k^2\sinh^{-2} t + (2k^2+3k+\mu_2)\}u_0'(t) \\ & + \{2k(\mu_2+4k^2)\coth t + \mu_3\}u_0(t) = 0\end{aligned}$$

By the change of variable $z = e^{-2t}$, $w(z) = z^{-k-\lambda_3/2}(1-z)^{-1+3k}u_0(t)$ satisfies the generalized hypergeometric equation of rank 3

$$(3.8)\qquad \left\{\left(z\tfrac{d}{dz}+\beta_1\right)\left(z\tfrac{d}{dz}+\beta_2\right)\tfrac{d}{dz} - \left(z\tfrac{d}{dz}+\alpha_1\right)\left(z\tfrac{d}{dz}+\alpha_2\right)\left(z\tfrac{d}{dz}+\alpha_3\right)\right\} w = 0,$$

with

$$\begin{aligned}&\alpha_1 = 1-k+\tfrac{1}{2}(\lambda_3-\lambda_2), \quad \alpha_2 = 1-k+\tfrac{1}{2}(\lambda_3-\lambda_1), \quad \alpha_3 = 1-k, \\ &\beta_1 = 1+\tfrac{1}{2}(\lambda_3-\lambda_2), \quad \beta_2 = 1+\tfrac{1}{2}(\lambda_3-\lambda_1).\end{aligned}$$

Characteristic exponents at the regular singular points 0, 1, ∞ are given by the following Riemann scheme

$$(3.9)\qquad P\begin{Bmatrix} z=0 & z=1 & z=\infty \\ 0 & 3k-1 & 1-k+(\lambda_3-\lambda_2)/2 \\ (\lambda_2-\lambda_3)/2 & 0 & 1-k+(\lambda_3-\lambda_1)/2 \\ (\lambda_1-\lambda_3)/2 & 1 & 1-k \end{Bmatrix}.$$

The point $z=1$ corresponds to the origin $x_1 = x_2 = x_3$ and the solution with the exponent $3k-1$ corresponds to the restriction of the Heckman-Opdam hypergeometric function. The generalized hypergeometric equation is free from accessory parameters, that is, the Riemann scheme determines the differential equation uniquely. We call such Fuchsian equation rigid.

3.1.2. BC_2 **case** Let $R = BC_2$. Then the walls of the Weyl group are $x_1 \pm x_2 = 0$, $x_1 = 0$, $x_2 = 0$, which corresponds to roots $e_1 \pm e_2$, e_1, e_2, respectively.

First we consider the restriction of (HO) system to the wall $x_2 = 0$. Let u be a solution of (HO) system given by series expansion

$$(3.10)\qquad u(x_1,x_2) = \sum_{j=0}^{\infty} u_j(x_1)x_2^j$$

near $x_2 = 0$ with $x_1 \neq 0$. Substituting u into (HO) system and putting $x_2 = 0$, we get a fourth order differential equation for $u_0(x_1)$. By the change of variable $z = -\sinh^2 x_1$, $w(z) = z^{a+c-1/2}u_0(x_1)$ satisfies the differential equation

$$(3.11)\qquad p_4(z)\frac{d^4w}{dz^4} = p_3(z)\frac{d^3w}{dz^3} + p_2(z)\frac{d^2w}{dz^2} + p_1(z)\frac{dw}{dz} + p_0(z)w$$

with

$$\begin{aligned}
p_0(z) &= -\tfrac{1}{16}(a-b-1+\lambda_1)(a-b-1-\lambda_1)(a-b-1+\lambda_2)(a-b-1-\lambda_2),\\
p_1(z) &= \tfrac{1}{4}(a-b-2)(2a^2-4ab-8a+2b^2+8b+10-{\lambda_1}^2-{\lambda_2}^2)z\\
&\qquad + \tfrac{15}{4}+\tfrac{3}{4}b^2-\tfrac{1}{2}c^2+\tfrac{11}{4}a^2-\tfrac{11}{2}a+3b+bc-\tfrac{7}{2}ab+\tfrac{1}{2}c\\
&\qquad - \tfrac{1}{2}ab^2-bc^2-\tfrac{1}{2}a^3+a^2b+\tfrac{1}{8}(2a-3)(\lambda_1^2+\lambda_2^2),\\
p_2(z) &= \left(-\tfrac{3}{2}(a^2+b^2)+3ab+9a-9b-\tfrac{29}{2}+\tfrac{1}{4}(\lambda_1^2+\lambda_2^2)\right)z^2\\
&\qquad + \left(\tfrac{5}{2}a^2-13a-3ab+\tfrac{1}{2}b^2+7b-c^2+c+\tfrac{35}{2}-\tfrac{1}{4}(\lambda_1^2+\lambda_2^2)\right)z\\
&\qquad - \tfrac{1}{4}(2a+2c-5)(2a-2c-3),\\
p_3(z) &= z(z-1)((2a-2b-8)z-2a+5),\\
p_4(z) &= z^2(z-1)^2.
\end{aligned}$$

The Riemann scheme is

$$P\begin{Bmatrix} z=0 & z=1 & z=\infty \\ 0 & 0 & (-a+b+1+\lambda_1)/2 \\ 1 & 1 & (-a+b+1-\lambda_1)/2 \\ a-c+1/2 & -b+1/2 & (-a+b+1+\lambda_2)/2 \\ a+c-1/2 & -b+3/2 & (-a+b+1-\lambda_2)/2 \end{Bmatrix}. \tag{3.12}$$

For generic k and λ, the local monodromy representation for each singular point is semisimple. Local monodromy type at a singular point is a partition of 4 consisting of multiplicities of the eigenvalues of the local monodromy matrix at the singular point. The local monodromy types for $z = 0$, 1 and ∞ are $(2,1,1)$, $(2,2)$, and $(1,1,1,1)$ respectively. By consulting Simpson's list (3.14), the resulting differential equation turns out to be free from accessory parameters, which is the even family of rank 4 (cf. [G, Si]). In other words, it is rigid in the sense that local monodromies uniquely determine the global monodromy. The point $z = 0$ corresponds to the origin $x_1 = 0$ and the exponent $a + c - 1/2$ is multiplicity one, which corresponds to the restriction of the Heckman-Opdam hypergeometric function.

Remark 4. Let $P(x, \frac{d}{dx})u = 0$ be an ordinary differential equation on the Riemann sphere with $p+1$ isolated singular points. Suppose that the equation is Fuchsian, mamely, all the singular points are regular. Moreover suppose the local monodromies are semisimple and the global monodromy group is irreducible. Let $(m_{0,1}, \dots, m_{0,n_0})$,..., $(m_{p,1}, \dots, m_{p,n_p})$ be the local monodromy types. Here $\sum_{\nu=1}^{n_j} m_{j,\nu}$, denoted by n, is the order of the differential equation.

(i) Katz [Katz] proved that the corresponding local system is rigid if and only if

$$\sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}^2 - (p-1)n^2 = 2. \tag{3.13}$$

(ii) Simpson [Si] classified the monodromy types of irreducible rigid local systems with $m_{0,1} = \cdots = m_{0,n_0} = 1$. In this case they satisfy $p = 2$ and they are as follows

$$
\begin{array}{lll}
 & (1,\dots,1),\ (1,\dots,1),\ (1,n-1) & \text{(hypergeometric family of rank } n\text{)}\\
(3.14) & (1,\dots,1),\ (1,m-1,m),\ (m,m) & \text{(even family of rank } 2m\text{)}\\
 & (1,\dots,1),\ (1,m,m),\ (m,m+1) & \text{(odd family of rank } 2m+1\text{)}\\
 & (1,1,1,1,1,1),\ (2,4),\ (2,2,2) & \text{(extra case)}
\end{array}
$$

Let u be a solution of (HO) system given by series expansion

$$u(x_1,x_2) = \sum_{j=0}^{\infty} u_j(x_1+x_2)(x_1-x_2)^j \tag{3.15}$$

near $x_1 = x_2 \neq 0$. Substituting u into (HO) system and putting $x_1 = x_2$, we get a differential equation for $u_0(2x_1)$. Then $w(z) = z^{a+c-1/2}(z-1)^{b+c-1/2}u_0(2x_1)$ with the variable $z = -\sinh^2 x_1$ satisfies the differential equation (3.11) with

$$
\begin{aligned}
p_0(z) &= -\tfrac{1}{16}\left(2-2c+\lambda_1+\lambda_2\right)\left(2-2c-\lambda_1-\lambda_2\right)\\
&\qquad\times\left(2-2c+\lambda_1-\lambda_2\right)\left(2-2c-\lambda_1+\lambda_2\right),\\
p_1(z) &= (2\,c-3)\left(2\,c^2-6\,c+5-\tfrac{1}{2}(\lambda_1^2+\lambda_2^2)\right)z\\
&\qquad+(2\,c-3)\left(-c^2+3\,c+\tfrac{1}{2}\,a^2-\tfrac{1}{2}\,b^2+\tfrac{1}{2}\,b-\tfrac{1}{2}\,a-\tfrac{5}{2}+\tfrac{1}{4}(\lambda_1^2+\lambda_2^2)\right),\\
p_2(z) &= \left(-6\,c^2+24\,c-25+\tfrac{1}{2}(\lambda_1^2+\lambda_2^2)\right)z^2\\
&\qquad+\left(6\,c^2-24\,c+25-\tfrac{1}{2}(\lambda_1^2+\lambda_2^2)-a^2+b^2+a-b\right)z\\
&\qquad\quad+a^2-c^2-a+4\,c-\tfrac{15}{4},\\
p_3(z) &= z\,(z-1)\,(2\,z-1)\,(2\,c-5)\,,\\
p_4(z) &= (z-1)^2\,z^2.
\end{aligned}
$$

The Riemann scheme is

$$
P\left\{\begin{matrix}
z=0 & z=1 & z=\infty\\
0 & 0 & 1-c+(\lambda_1+\lambda_2)/2\\
1 & 1 & 1-c-(\lambda_1+\lambda_2)/2\\
c-a+1/2 & -b+c+1/2 & 1-c+(\lambda_1-\lambda_2)/2\\
a+c-1/2 & b+c-1/2 & 1-c-(\lambda_1-\lambda_2)/2
\end{matrix}\right\}. \tag{3.16}
$$

For generic k and λ, the local monodromy representation for each singular point is semisimple. The local monodromy types are $(2,1,1)$, $(2,1,1)$, $(1,1,1,1)$, hence this equation is not rigid by the classification (3.14) of Simpson.

3.1.3. G_2 **case** Let $R = G_2$. Choose simple roots α_1 and α_2 with $|\alpha_2| > |\alpha_1|$ and let ϖ_1 and ϖ_2 denote the fundamental weights corresponding to α_1 and α_2. Put $k_i = k_{\alpha_i}$ $(i = 1, 2)$ and let $\lambda = \lambda_1\varpi_1 + \lambda_2\varpi_2$ with $(\lambda_1, \lambda_2) \in \mathbb{C}^2$.

First we consider the restriction of (HO) system to the wall $\alpha_1 = 0$. The resulting equation is of sixth order and has regular singularities at 0, 1, ∞ in the coordinate $z = \tanh^2(\varpi_2/2)$. The Riemann scheme is

$$P\begin{Bmatrix} z=0 & z=1 & z=\infty \\ 0 & k_1+2\,k_2+\lambda_2 & 0 \\ \frac{3}{2}-3\,k_2 & k_1+2\,k_2-\lambda_2 & 1 \\ 1-\frac{3}{2}(k_1+k_2) & k_1+2\,k_2+\lambda_1+\lambda_2 & 2 \\ 2-\frac{3}{2}(k_1+k_2) & k_1+2\,k_2-\lambda_1-\lambda_2 & 1/2-k_2 \\ 1/2-\frac{3}{2}(k_1+k_2) & k_1+2\,k_2+\lambda_1+2\lambda_2 & 3/2-k_2 \\ 5/2-\frac{3}{2}(k_1+k_2) & k_1+2\,k_2-\lambda_1-2\lambda_2 & 5/2-k_2 \end{Bmatrix}. \tag{3.17}$$

For generic k and λ, the local monodromy representation for each singular point is semisimple. The local monodromy types are $(1,1,2,2)$, $(1,1,1,1,1,1)$, $(3,3)$, which shows that the resulting differential equation is not rigid by the list (3.14).

Next we consider the restriction of (HO) system to the wall $\alpha_2 = 0$. The resulting equation is of sixth order and has regular singularities at -1, $-1/2$, 1, ∞ in the coordinate $z = \cosh(\varpi_1/2)$. The Riemann scheme is

$$P\begin{Bmatrix} z=-1 & z=-1/2 & z=1 & z=\infty \\ 0 & 0 & 0 & 2k_1+3k_2+\lambda_1 \\ 1 & 3 & 3/2-3k_1 & 2k_1+3k_2-\lambda_1 \\ 2 & 1-3k_2 & (1-3k_1-3k_2)/2 & 2k_1+3k_2+2\lambda_1+3\lambda_2 \\ 1/2-k_1 & 2-3k_2 & (5-3k_1-3k_2)/2 & 2k_1+3k_2-2\lambda_1-3\lambda_2 \\ 3/2-k_1 & 4-3k_2 & (2-3k_1-3k_2)/2 & 2k_1+3k_2+\lambda_1+3\lambda_2 \\ 5/2-k_1 & 5-3k_2 & (4-3k_1-3k_2)/2 & 2k_1+3k_2-\lambda_1-3\lambda_2 \end{Bmatrix}. \tag{3.18}$$

For generic k and λ, the local monodromy representation for each singular point is semisimple. The local monodromy types are $(3,3)$, $(2,4)$, $(1,1,2,2)$, $(1,1,1,1,1,1)$, which shows that the resulting differential equation is not rigid by the list (3.14).

Remark 5. There are several algorithms to compute restrictions of partial differential equations to singular sets. For example, Oaku [Oa] gave an algorithm for computing the restriction of a Fuchsian system of partial differential equations to a singular set. He call the resulting system "tangent system". Though we do not use his algorithm, it may work well for our cases.

§ 3.2. General cases

The ordinary differential equation obtained by restricting (HO) system to a singular line is not necessarily rigid, as in the cases of $x_1 = x_2$ for BC_2 and both walls for G_2. But its monodromy can be calculated by using representations of the Hecke algebra.

In this subsection, we assume that Σ is an irreducible root system. First we review on the monodromy representation for (HO) system around the origin. See [Op4, Section 5, 7], [Hec2], [HS, Part I, Section 4.3], and [KO, Appendix] for details. (HO) system is W-invariant and it can be viewed as a system on $W\backslash\mathfrak{a}_{\mathbb{C}}$. Let

$$\mathfrak{a}_{\mathbb{C}}^{\mathrm{reg}} = \{x \in \mathfrak{a}_{\mathbb{C}} \,:\, \alpha(x) \neq 0 \,(\alpha \in \Sigma)\}.$$

Let $\Psi = \{\alpha_1, \ldots, \alpha_n\}$ and let $s_i \in W$ denote the simple reflection corresponding to α_i. W is generated by $s_1, \ldots, s_n$ with the relation of the form

$$s_i^2 = 1 \,(1 \leq i \leq n), \quad (s_i s_j)^{m_{ij}} = 1 \,(1 \leq i \neq j \leq n).$$

Let A_W denote the associated Artin group generated by elements $\delta_1, \ldots, \delta_n$ satisfying the relations

$$\delta_i \delta_j \delta_i \cdots = \delta_j \delta_j \delta_i \cdots \quad (1 \leq i \neq j \leq n,\, m_{ij} \text{ factors on both sides}).$$

Fix a point $x_0 \in \mathfrak{a}_+$. For $1 \leq i \leq n$ let $g_i \in \pi_1(W\backslash\mathfrak{a}_{\mathbb{C}}, x_0)$ be defined by the loop

$$g_i(t) = (1-t)x_0 + t\, r_i x_0 + \sqrt{-1}\epsilon(t)\alpha_i,$$

where $\epsilon \,:\, [0,1] \to [0,1]$ is continuous function with $\epsilon(0) = \epsilon(1) = 0$ and $\epsilon(1/2) > 0$. The fundamental group $\pi_1(W\backslash\mathfrak{a}_{\mathbb{C}}, x_0)$ is isomorphic to A_W by $\delta_i \mapsto g_i$.

Let $q_j = e^{-2\pi\sqrt{-1}(k_{\alpha_j} + k_{2\alpha_j})}\,(1 \leq i \leq n)$. Define

$$\Sigma_0 = \{\alpha \in \Sigma \,:\, \frac{\alpha}{2} \notin \Sigma\}$$

and $\Sigma_0^+ = \Sigma_0 \cap \Sigma^+$. Let q denote the function on Σ_0 defined by $q_\alpha = q_{\alpha_i}$ if $\alpha = w\alpha_i$ for $w \in W$. The Iwahori-Hecke algebra of the Weyl group W is the complex algebra $\mathcal{H}_W(q)$ generated by elements T_i with the relations

$$\begin{aligned}
&(T_i - 1)(T_i + q_i) = 0 \quad (1 \leq i \leq n),\\
&T_i T_j T_i \cdots = T_j T_i T_j \cdots \quad (1 \leq i \neq j \leq n,\, m_{ij} \text{ factors on both sides}).
\end{aligned}$$

Let $V(\lambda, k)$ denote the local solution space of (HO) system around x_0 and

$$\mu(\lambda, k) \,:\, \pi_1(W\backslash\mathfrak{a}_{\mathbb{C}}) \to GL(V(\lambda, k))$$

the monodromy representation. The monodromy representation for (HO) system on $W\backslash\mathfrak{a}_{\mathbb{C}}$ factors through $\tau : \mathbb{C}A_W \to \mathcal{H}_W(q)$ defined by $\delta_i \mapsto T_i$, that is

$$(\mu(\lambda,k)(g_i)-1)(\mu(\lambda,k)(g_i)+q_i)=0 \quad (1\le i\le n).$$

Let $\nu(\lambda,k) : \mathcal{H}_W(q)\to GL(V(\lambda,k))$ be the representation such that $\mu(\lambda,k)=\nu(\lambda,k)\circ\tau$. If k is generic, $\nu(\lambda,k)$ is equivalent with the regular representation of $\mathcal{H}_W(q)$.

For $w\in W$ we define

$$T_w = T_{s_{j_1}}T_{s_{j_2}}\cdots T_{s_{j_m}}$$

where $w=s_{j_1}s_{j_2}\cdots s_{j_m}$ is a minimal expression. It does not depend on the choice of minimal expressions of w. Let w^* denote the longest element of W. The monodromy for the loop $g(t)=e^{2\pi\sqrt{-1}t\cdot x_0}$ $(0\le t\le 1)$ is given by the action of $T_{w^*}^2$ on the regular representation of $\mathcal{H}_W(q)$. We can compute the monodromy by using the following theorem of Springer ([GP, Theorem 9.2.2]). (In the context of fundamental groups, this theorem was given by [D] and [Op4].)

Theorem 3.1. *$T_{w^*}^2\in Z(\mathcal{H}_W(q))$ the center of the $\mathcal{H}_W(q)$. For any irreducible representation V of $\mathcal{H}_W(q)$*

$$T_{w^*}^2|_V=\prod_{\alpha\in\Sigma_0^+} q_\alpha^{1-\frac{\chi_V(s_\alpha)}{\chi_V(1)}},$$

where χ_V denote the character of V.

For $1\le j\le n$, let H_j denote the singular line of (HO) system defined by

$$H_j=\{t\in\mathfrak{a} : \langle t,\alpha_i\rangle=0\ (\forall i\in\{1,\dots,n\}\setminus\{j\})\}.$$

Set $k_j=k_{\alpha_j}$ for $1\le j\le n$. They are complex parameters such that $k_i=k_j$ if $|\alpha_i|=|\alpha_j|$. (HO) system is W-invariant and has regular singularity along the line H_j with the characteristic exponents 0 and $1-2k_j$. Put

$$\begin{aligned}\mathcal{H}_j&=\{v\in\mathcal{H}_W(q) : (T_i-1)v=0 \quad (\forall i\in\{1,\dots,n\}\setminus\{j\})\},\\ W_j&=\langle s_i : (\forall i\in\{1,\dots,n\}\setminus\{j\})\rangle.\end{aligned}$$

We have the following results for the restriction of (HO) system to H_j.

Theorem 3.2. *Fix j with $1\le j\le n$. If q is generic, then the restrictions of local holomorphic solutions of the Heckman-Opdam system at a generic point of H_j form $\#\mathcal{H}_j$-dimensional vector space and they satisfy an ordinary differential equation of Fuchsian type with order $\#\mathcal{H}_j$. The local monodromy matrix of the equation at the origin is semisimple and isomorphic to $T_{w^*}^2|_{\mathcal{H}_j}$.*

Example 3.3 (A_{n-1}). For $R = A_{n-1}$, $W_{A_{n-1}} \simeq \mathfrak{S}_n$ and

$$T_{w^*}^2\big|_V = q^{n(n-1)\dim\{v\in V|_{q=1}\,:\,s_1 v=-v\}/\dim V}$$

by Theorem 3.1. We have $W_j \simeq \mathfrak{S}_j \times \mathfrak{S}_{n-j}$ for $1 \le j \le \frac{n}{2}$ and

$$\mathrm{Ind}_{W_j}^W(1) \simeq \bigoplus_{i=0}^{j} V_{n-i,i} \qquad \dim \mathrm{Ind}_{W_j}^W(1) = \binom{n}{j},$$

where $V_{n-i,i}$ is the representation of W that corresponds to the partition $(n-i,i)$ and

$$T_{w^*}^2|_{V_{n-i,i}} = q^{i(n+1-i)}.$$

For the restriction (A_{n-1}, A_{n-2}), we have

$$T_{w^*}^2\big|_{\mathrm{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(1)} \simeq I_1 \oplus q^n I_{n-1}$$

and the characteristic exponents at the origin are 0, $j - nk$ ($j = 1, \ldots, n-1$) and the local monodromy type is $(1, n-1)$.

Remark 6. (i) In the group case, the system of differential equation for the Heckman-Opdam hypergeometric function that is mentioned in (iii) of the above theorem is the radial part of the "Hua system". The Hua system characterizes the image of the Poisson transform from the boundary G/P_Θ of a Riemannian symmetric space G/K with the restricted root system Σ (cf. [Sh1, Sh2]).

(ii) Beerends [B] studied restrictions to some singular lines of the BC type Heckman-Opdam hypergeometric function with some degenerate eigenvalues and special values of the hypergeometric functions.

§ 3.3. Cases of (A_{n-1}, A_{n-2}) and (B_n, B_{n-1})

By using general results of §3.2, we have Fuchsian differential equations that are free from accessory parameters for certain restrictions in A_{n-1} and B_n cases, which are generalizations of those given in §3.1.

3.3.1. (A_{n-1}, A_{n-2})

First we consider the case of $R = A_{n-1}$. Then

$$\mathfrak{a} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, x_1 + \cdots + x_n = 0\}$$

and (HO) system has singularities along the walls $x_i = x_j$ of the Weyl group $W \simeq \mathfrak{S}_n$. We consider the restriction of (HO) system to the singular line

$$H_1 = \{x \in \mathfrak{a} \,:\, x_2 = x_3 = \cdots = x_n\}.$$

For this restriction of type (A_{n-1}, A_{n-2}), (HO) system gives an ordinary differential equation of order n with the following Riemann scheme:

$$P\left\{\begin{matrix} z=0 & z=1\ (\text{origin}) & z=\infty \\ \frac{n-1}{2}k+\frac{\lambda_1}{2} & 0 & \frac{n-1}{2}k-\frac{\lambda_1}{2} \\ \frac{n-1}{2}k+\frac{\lambda_2}{2} & 1-nk & \frac{n-1}{2}k-\frac{\lambda_2}{2} \\ \frac{n-1}{2}k+\frac{\lambda_3}{2} & 2-nk & \frac{n-1}{2}k-\frac{\lambda_3}{2} \\ \cdots & \cdots & \cdots \\ \frac{n-1}{2}k+\frac{\lambda_n}{2} & n-1-nk & \frac{n-1}{2}k-\frac{\lambda_n}{2} \end{matrix}\right\}. \tag{3.19}$$

The characteristic exponents at $z = 0$ and $z = \infty$ come from the characteristic exponents of (HO) system at infinity and the characteristic exponents at 1 are given in Example 3.3. If $n = 3$, then adding and subtracting $k + \lambda_3/2$ from exponents at $x = \infty$ and $x = 0$ respectively, (3.19) becomes (3.9).

For generic k and λ, the local monodromy representation for each singular point is semisimple. Local monodromy type at a singular point is a partition of n consisting of multiplicities of the eigenvalues of the local monodromy matrix at the singular point. Though characteristic exponents $1-nk, 2-nk, \ldots, n-1-nk$ have integral differences, they contribute by the same eigenvalue $\exp(2\pi\sqrt{-1}(1-nk))$ in the local monodromy matrix. Hence the local monodromy types at the singular points 0, 1, ∞ are

$$(1,\ldots,1),\ (1,n-1),\ (1,\ldots,1),$$

respectively. The ordinary differential equation is essentially determined by these local monodromy types. It turns out to be the differential equation for the generalized hypergeometric function ${}_nF_{n-1}$ up to some multiples of powers of z and $z-1$.

The characteristic exponent 0 at $z = 1$ is of multiplicity one and it corresponds to the restriction of the Heckman-Opdam hypergeometric function. By using this fact, we can prove that the dimension of the space of the real analytic solutions of A_{n-1}-type (HO) system is at most one for generic k. Hence it is really one with the Heckman-Opdam hypergeometric function as a basis for generic k.

Levelt [L] and Okubo et al. [OTY] gave the connection coefficient $c((n-1)k/2 + \lambda_i/2 \rightsquigarrow 0)$ from the local solution at $z = 0$ with the characteristic exponent $(n-1)k/2 + \lambda_i/2$ $(1 \le i \le n)$ to the local solution at $z = 1$ with the characteristic exponent 0. There is also an explicit formula for the connection coefficient $c(0 \rightsquigarrow (n-1)k/2 + \lambda_i/2)$ for the opposite direction (cf. [O5]). By using these connection formulae, the following identity for trigonometric functions follows:

$$\sum_{j=1}^{n} \frac{\prod_{\substack{1\le\nu\le n \\ \nu\ne j}} \sin(\mu_\nu - \mu_j + t)}{\prod_{\substack{1\le\nu\le n \\ \nu\ne j}} \sin(\mu_\nu - \mu_j)} = \frac{\sin nt}{\sin t} \quad ({}^\forall(\mu_1,\ldots,\mu_n) \in \mathfrak{a}_{\mathbb{C}}^*). \tag{3.20}$$

By using these results, we can prove that the value of the A_{n-1}-type Heckman-Opdam hypergeometric function at the origin is 1. We will explain about the proof in the next subsection.

3.3.2. (B_n, B_{n-1})

For $R = BC_n$, $\Sigma_0 = \Sigma_{B_n}$ and (HO) system has singularities along the walls

$$H^{\pm}_{(i,j)} : x_i = \pm x_j \, (1 \le i \ne j \le n), \quad H_{(p)} : x_p = 0 \, (1 \le p \le n)$$

of the Weyl group. We consider the restriction of (HO) system to the singular line

$$H_n = \{x \in \mathfrak{a} \, : \, x_1 = x_2 = \cdots = x_{n-1} = 0\}.$$

For this restriction of type (B_n, B_{n-1}), (HO) system gives ordinary differential equation of order $2n$ with the following Riemann scheme:

$$P\left\{\begin{array}{ccc}
z=1 & z=0 \;\; \text{(origin)} & z=\infty \\
0 & 0 & (n-1)k_1+k_2+\frac{1}{2}k_3-\frac{\lambda_1}{2} \\
1 & 1-nk_1 & (n-1)k_1+k_2+\frac{1}{2}k_3-\frac{\lambda_2}{2} \\
2 & 2-nk_1 & (n-1)k_1+k_2+\frac{1}{2}k_3-\frac{\lambda_3}{2} \\
\cdots & \cdots & \cdots \\
n-1 & n-1-nk_1 & (n-1)k_1+k_2+\frac{1}{2}k_3-\frac{\lambda_n}{2} \\
1-\frac{1}{2}-k_2 & 1-\frac{1}{2}-(n-1)k_1-k_2-k_3 & (n-1)k_1+k_2+\frac{1}{2}k_3+\frac{\lambda_1}{2} \\
2-\frac{1}{2}-k_2 & 2-\frac{1}{2}-(n-1)k_1-k_2-k_3 & (n-1)k_1+k_2+\frac{1}{2}k_3+\frac{\lambda_2}{2} \\
\cdots & \cdots & \cdots \\
n-\frac{1}{2}-k_2 & n-\frac{1}{2}-(n-1)k_1-k_2-k_3 & (n-1)k_1+k_2+\frac{1}{2}k_3+\frac{\lambda_n}{2}
\end{array}\right\} \tag{3.21}$$

The characteristic exponents at $z=\infty$ come from the characteristic exponents of (HO) system at infinity and the characteristic exponents at $z = 0$ are obtained by using Theorem 3.2 and Theorem 3.1. The regular singular point $z = 1$ corresponds to the pure imaginary point $x_n = \pm\sqrt{-1}$, which is not a singular point of (HO) system but of $(\mathrm{HO})_\epsilon$ system for certain ϵ. We can compute the monodromy at $z = 1$ by using representations of the Hecke algebra for W_ϵ. See the next section for $(\mathrm{HO})_\epsilon$ system. If $n = 2$, adding and subtracting $\frac{1}{2} + k_1 + k_2 + k_3$ from exponents at $z = 0$ and $z = \infty$ respectively, (3.21) becomes (3.12).

For generic k and λ, the local monodromy representation for each singular point is semisimple. Local monodromy type at a singular point is a partition of $2n$ consisting of multiplicities of the eigenvalues of the local monodromy matrix at the singular point. The local monodromy types at the singularities are (n, n), $(1, n-1, n)$, $(1, \ldots, 1)$. For these local monodromy types, the ordinary differential equation corresponds to the even family of rank $2n$ in the list (3.14). The resulting Fuchsian differential equation is free from accessory parameter.

The characteristic exponent 0 at $z=0$ is of multiplicity one and it corresponds to the restriction of the Heckman-Opdam hypergeometric function. By using this fact, we can prove that the dimension of real analytic solutions of BC_n-type (HO) system is one for generic k.

The general form of the Riemann scheme for the even family of rank $2n$ is the following:

$$P\left\{\begin{array}{ccc} z=1 & z=0 & z=\infty \\ \lambda_{2,1} & \lambda_{1,1} & \lambda_{0,1} \\ \lambda_{2,1}+1 & \lambda_{1,1}+1 & \lambda_{0,2} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \lambda_{2,1}+n-1 & \lambda_{1,1}+n-1 & \lambda_{0,n} \\ \lambda_{2,2} & \lambda_{1,2} & \lambda_{0,n+1} \\ \lambda_{2,2}+1 & \lambda_{1,2}+1 & \lambda_{0,n+2} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \lambda_{2,2}+n-1 & \lambda_{1,3} & \lambda_{0,2n} \end{array}\right\}. \tag{3.22}$$

For generic k and λ, the local monodromy representation for each singular point is semisimple. Local monodromy type at a singular point is a partition of $2n$ consisting of multiplicities of the eigenvalues of the local monodromy matrix at the singular point. The local monodromy types are (n,n), $(n,n-1,1)$, $(1,\dots,1)$.

Connection coefficients between the local solution at $z=0$ with the characteristic exponent $\lambda_{1,3}$ and the local solution at ∞ with the characteristic exponent $\lambda_{0,i}$ $(1\le i\le 2n)$ were given explicitly by Oshima [O5]. The connection coefficient from the local solution at $z=\infty$ with the characteristic exponent $\lambda_{0,i}$ to the local solution at $z=0$ with the characteristic exponent $\lambda_{1,3}$ is given by

$$\begin{aligned} c(\lambda_{0,i}\rightsquigarrow\lambda_{1,3}) &= \prod_{\nu=1}^{2}\frac{\Gamma(\lambda_{1,\nu}-\lambda_{1,3})}{\Gamma(\lambda_{0,i}+\lambda_{1,1}+\lambda_{2,\nu})} \\ &\quad\cdot\prod_{\substack{1\le\nu\le 2n\\ \nu\ne i}}\frac{\Gamma(\lambda_{0,i}-\lambda_{0,\nu}+1)}{\Gamma(\lambda_{0,i}+\lambda_{1,1}+\lambda_{2,1}+\lambda_{0,\nu}+\lambda_{1,2}+\lambda_{2,2}-1)} \end{aligned} \tag{3.23}$$

for $1\le i\le 2n$. There is a similar formula for $c(\lambda_{1,3}\rightsquigarrow\lambda_{0,i})$.

By using these connection formulae, the following identity for trigonometric functions follows:

$$\begin{aligned} &\sum_{i=1}^{2n}\sin(x_i+s)\sin(x_i+t)\prod_{j\in\{1,\dots,2n\}\setminus\{i\}}\frac{\sin(x_i+x_j+2u)}{\sin(x_i-x_j)} \\ &\qquad = \sin\left(2nu+\sum_{j=1}^{2n}x_j\right)\sin\left(s+t+(2n-2)u+\sum_{j=1}^{2n}x_j\right). \end{aligned} \tag{3.24}$$

By using these results, we can prove that the value of the BC-type Heckman-Opdam hypergeometric function at the origin is 1. We will explain about the proof in the next subsection.

§ 3.4. Application

Opdam [Op3] proved the Gauss summation formula for the Heckman-Opdam hypergeometric function, which asserts that $F(\lambda, k; 0) = 1$. The proof given by Opdam is complicated and indirect. One of our motivations to study ordinary differential equations by restricting (HO) system to singular sets of dimension one is to prove the Gauss summation formula for the Heckman-Opdam hypergeometric functions by using connection formulae for hypergeometric functions of one variable.

We proved that the restrictions of (HO) systems for $(A_{n-1}A_{n-2})$ and (B_n, B_{n-1}) are Fuchsian differential equations that are free from accessory parameters, which are generalized hypergeometric equation and a equation corresponding to the even family in the list (3.14) of rigid local systems respectively. In these two cases, we know some connection coefficients, as we explained in the last subsection. We can prove the Gauss summation formula for A_{n-1} and BC_n by the following way: Let $R = BC_n$ and use notations of § 3.3.2. Let u be a real analytic solution of (HO) system with $u(0) = 1$. By using the formulae for the connection coefficients $c(\lambda_{1,3} \rightsquigarrow \lambda_{0,i})$ and the method of rank-one reduction, the connection coefficient for u from 0 to ∞ is nothing but $c(\lambda, k)$, which proves that $u(x) = F(\lambda, k; x)$.

There is a proof that uses the connection coefficients $c(\lambda_{0,i} \rightsquigarrow \lambda_{1,3})$ and the trigonometric identity (3.24). This proof is essentially the same as that was described above, but it seems to be indirect. We give the proof for $R = A_2$.

The generalized hypergeometric series

$$
{}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; z) = \sum_{j=0}^{\infty} \frac{(\alpha_1)_j(\alpha_2)_j(\alpha_3)_j}{(\beta_1)_j(\beta_2)_j} \frac{z^j}{j!}
$$

is a solution of (3.8) that is analytic at $z = 0$ and the exponent at $z = 1$ is $3k - 1$. Levelt [L] and Okubo et al [OTY] proved that

$$
\begin{aligned}
&\lim_{z \to 1-0} (1-z)^{\beta_3} {}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; z) \\
&\quad = \frac{\prod_{j=1}^{3} \Gamma(\beta_j)}{\prod_{j=1}^{3} \Gamma(\alpha_j)} \\
&\quad = \frac{\Gamma(1-3k)\Gamma(1+(\lambda_3-\lambda_2)/2)\Gamma(1+(\lambda_3-\lambda_1)/2)}{\Gamma(1-k)\Gamma(1-k+(\lambda_3-\lambda_2)/2)\Gamma(1-k+(\lambda_3-\lambda_1)/2)}. \qquad (3.25)
\end{aligned}
$$

Here $\beta_3 = \alpha_1 + \alpha_2 + \alpha_3 - \beta_1 - \beta_2$ and $\operatorname{Re}\beta_3 > 0$. We denote the right hand side of (3.25) by $d(\lambda, k)$. The function

$$F_{\{e_1-e_2\}}(\lambda, k, x) = c_{\{e_1-e_2\}}(\lambda, k)\Phi(\lambda, k, x) + c_{\{e_1-e_2\}}(s_{e_1-e_2}\lambda, k)\Phi(s_{e_1-e_2}\lambda, k, x)$$

is real analytic on $x_1 = x_2$ and

$$F(\lambda, k, x) = \sum_{w\in W/W_{\{e_1-e_2\}}} c^{\{e_1-e_2\}}(w\lambda, k)F_{\{e_1-e_2\}}(w\lambda, k, x). \tag{3.26}$$

Here

$$c_\Theta(\lambda, k) = \prod_{\alpha\in\langle\Theta\rangle^+} c_\alpha(\lambda, k), \quad c_\Theta(\lambda, k)c^\Theta(\lambda, k) = c(\lambda, k)$$

for $\Theta \subset \Psi$ and now $c_{\{e_1-e_2\}}(\lambda, k) = c_{e_1-e_2}(\lambda, k)$. $W_{\{e_1-e_2\}}$ is a subgroup of $W \simeq \mathfrak{S}_3$ generated by the simple reflection with respect to the root $e_1 - e_2$.

The restriction of the Heckman-Opdam hypergeometric function $F(\lambda, k, x)$ to the wall $x_1 = x_2$ is a constant multiple of

$$z^{k+\lambda_3/2}(1-z)^{1-3k}{}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3; z)$$

with $z = e^{-x_2+x_3}$. On the other hand, the boundary value of $F_{\{e_1-e_2\}}(\lambda, k, x)$ as $x_2 - x_3 \to \infty$ $(z \to 0)$ is the Gauss hypergeometric function and its value at the origin is 1. Therefore the constant is 1.

It follows from (3.26) and the above consideration that $F(\lambda, k, 0)$ is equal to

$$\sum_{w\in W/W_{\{e_1-e_2\}}} c^{\{e_1-e_2\}}(w\lambda, k)d(w\lambda, k).$$

By using

$$c^{\{e_1-e_2\}}(\lambda, k) = \frac{\tilde{c}^{\{e_1-e_2\}}(\lambda, k)}{\tilde{c}^{\{e_1-e_2\}}(\rho(k), k)}$$

with

$$\tilde{c}^{\{e_1-e_2\}}(\lambda, k) = \frac{\Gamma((\lambda_1-\lambda_3)/2)\Gamma((\lambda_2-\lambda_3)/2)}{\Gamma(k+(\lambda_1-\lambda_3)/2)\Gamma(k+(\lambda_2-\lambda_3)/2)},$$
$$\tilde{c}^{\{e_1-e_2\}}(\rho(k), k) = \frac{\Gamma(2k)\Gamma(k)}{\Gamma(3k)\Gamma(2k)} = \frac{\Gamma(k)}{\Gamma(3k)},$$

and $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$, we have

$$c^{\{e_1-e_2\}}(\lambda, k)\, d(\lambda, k) = \frac{\sin\pi k\, \sin\pi(k+(\lambda_2-\lambda_3)/2)\, \sin\pi(k+(\lambda_1-\lambda_3)/2)}{\sin 3\pi k\, \sin\pi((\lambda_2-\lambda_3)/2)\, \sin\pi((\lambda_1-\lambda_3)/2)}.$$

By the identity (3.20) for $n = 3$, we have

$$\sum_{w\in W/W_{\{e_1-e_2\}}} c^{\{e_1-e_2\}}(w\lambda, k)\, d(w\lambda, k) = 1,$$

and complete the proof of the Gauss summation formula for A_2.

Remark 7. Fuchsian equations without accessory parameters form a class of ordinary differential equations whose solutions and their monodromies can be analyzed algebraically. We can see (HO) system as a "rigid" system of partial differential equation in a sense.

§ 4. Real forms of Heckman-Opdam systems

The Heckman-Opdam hypergeometric function is up to constant multiples a unique analytic solution of (HO) system on $\mathfrak{a} \simeq \mathbb{R}^n$. In this section, we study globally real analytic solutions of (HO) system on another real form $\mathfrak{a}_\epsilon$, or equivalently $(\mathrm{HO})_\epsilon$ system on $\mathfrak{a}$.

In the group case, (HO) system is the radial part of the system of the invariant differential equations for the zonal spherical function on a Riemannian symmetric space G/K with respect to the Cartan decomposition $G = KAK$. $(\mathrm{HO})_\epsilon$ system is the radial part of the system of the invariant differential equations with respect to the generalized Cartan decomposition $G = KAK_\epsilon$ in the sense of Oshima and Sekiguchi [OS2].

Oshima and Sekiguchi [OS2] constructed a basis for the space of the (K, K_ϵ)-spherical functions and gave connection formulae explicitly by using the Poisson transforms for G/K_ϵ. In this section, we generalized these results for arbitrary multiplicity k. This problem was studied by J. Sekiguchi [Sekj2] for rank two cases. A closely related problem was studied by Heckman [HS, Part III].

We begin by the simplest example $R = A_1$. Then

$$L(k) = \frac{d^2}{dx^2} + 2k \coth x \, \frac{d}{dx}$$

and the Heckman-Opdam hypergeometric function is given by

$$\begin{aligned} F(\lambda, k; x) &= {}_2F_1(\tfrac{1}{2}(k+\lambda), \tfrac{1}{2}(k-\lambda), k+\tfrac{1}{2}; -\sinh^2 x) \\ &= c(\lambda, k)\Phi(\lambda, k; x) + c(-\lambda, k)\Phi(-\lambda, k; x), \end{aligned}$$

where

$$\Phi(\lambda, k; x) = e^{(\lambda-k)x} {}_2F_1(-\lambda+k, k, -\lambda+1; e^{-2x}).$$

If $k = 1/2$, then $L(k)$ is the radial part of the Laplace-Beltrami operator and $F(\lambda, k; x)$ is the radial part of the zonal spherical function on the symmetric space $G/K = SL(2, \mathbb{R})/SO(2)$. By the change of variable $x \mapsto x + \frac{1}{2}\pi\sqrt{-1}$, $L(k)$ becomes

$$L(k)_\epsilon = \frac{d^2}{dx^2} + 2k \tanh x \, \frac{d}{dx}.$$

Then $\Phi_\epsilon(\lambda, k; x) = c(\lambda, k)\Phi(\lambda, k; x + \frac{1}{2}\pi\sqrt{-1})$ is an real analytic eigenfunction of $L(k)_\epsilon$ with the eigenvalue $\lambda^2 - \rho(k)^2$ for $\pm\lambda$, and for generic λ, $\{\Phi_\epsilon(\lambda, k; x),\ \Phi_\epsilon(-\lambda, k; x)\}$

forms a basis of real analytic solutions. The following connection formula follows from that for the Gauss hypergeometric function.

$$\Phi_\epsilon(-\lambda, k; -x) = \frac{\sin \pi\lambda}{\sin \pi(\lambda + k)} \Phi_\epsilon(\lambda, k; x) + \frac{\sin \pi k}{\sin \pi(\lambda + k)} \Phi_\epsilon(\lambda, k; -x).$$

If $k = 1/2$, then $L(k)_\epsilon$ is the radial part of the pseudo-Laplacian and the (K, K_ϵ)-spherical function on $G/K_\epsilon = SL(2, \mathbb{R})/SO_0(1, 1)$ with respect to the generalized Cartan decomposition $G = KAK_\epsilon$.

Now we consider the general cases. Let $\epsilon : \Sigma \to \{\pm 1\}$ be a map such that

$$\epsilon(\alpha + \beta) = \epsilon(\alpha)\epsilon(\beta) \quad (\alpha,\, \beta,\, \alpha + \beta \in \Sigma).$$

We call ϵ a signature of roots. Define $L(k)_\epsilon$ by

$$L(k)_\epsilon = \sum_{i=1}^{n} \partial_{e_i}^2 + \sum_{\substack{\alpha \in \Sigma^+ \\ \epsilon(\alpha) > 0}} 2k_\alpha \coth \alpha\, \partial_\alpha + \sum_{\substack{\alpha \in \Sigma^+ \\ \epsilon(\alpha) < 0}} 2k_\alpha \tanh \alpha\, \partial_\alpha \tag{4.1}$$

It is given by the change of variable $x \mapsto x + \sqrt{-1} v_\epsilon$ for $L(k)$, where $v_\epsilon = \pi \sum_{\epsilon(\alpha_i) = -1} \varpi_i$. Here $\varpi_1, \ldots, \varpi_n$ are fundamental weights. There exists a commutative algebra of differential operators $\mathbb{D}(k)_\epsilon$ containing $L(k)_\epsilon$. We denote by $(\mathrm{HO})_\epsilon$ system the corresponding system of differential equations. That is, $(\mathrm{HO})_\epsilon$ system is obtained by the change of variable $x \mapsto x + \sqrt{-1} v_\epsilon$ for (HO) system (1.3).

For the group case, $2k_\alpha$ is the multiplicity of the restricted root $\alpha \in \Sigma$ of a Riemannian symmetric space and $L(k)_\epsilon$ is the radial part of the Casimir operator with respect to the generalized Cartan decomposition $G = KAK_\epsilon$. For example,

$$\begin{gathered} G = SL(n, \mathbb{R}), \quad K = SO(n), \quad K_\epsilon = SO_0(p, n-p) \quad (0 \le p \le n), \\ k_\alpha = \tfrac{1}{2}, \quad \{\alpha \in \Sigma^+_{A_{n-1}} : \epsilon(\alpha) = -1\} = \{e_i - e_j : 1 \le i \le p < j \le n\}. \end{gathered}$$

The Radial part of the Casimir operator with respect to the generalized Cartan decomposition $G = KAH$ for a semisimple symmetric spaces G/H (that is not necessarily of type K_ϵ) is of the form $L(k)_\epsilon$ for some signature of roots ϵ (cf. [Sekh], [HS, Part III], [O1]).

For a signature of roots ϵ, let $W_\epsilon = \langle s_\alpha : \epsilon(\alpha) = 1 \rangle \subset W$ and $\# W/W_\epsilon = r$. Choose a representative $\{v_1 = e,\, v_2, \ldots, v_r\} \subset W$ for the coset $W_\epsilon \backslash W$. Let $\mathcal{C} \subset \mathfrak{a}$ denote the positive Weyl chamber.

Theorem 4.1. *The dimension of the global real analytic solutions of* $(HO)_\epsilon$ *system is* r *for generic* k. *There exists a basis*

$$F_\epsilon(\lambda, k; x) = \big(F_\epsilon^{(1)}(\lambda, k; x), \ldots, F_\epsilon^{(r)}(\lambda, k; x)\big)$$

of analytic solutions of (HO)$_\epsilon$ system such that

$$F_\epsilon(\lambda,k;x)_i = \sum_{w\in W} c(w\lambda,k)A^\epsilon_w(\lambda,k)_{i\text{-}th\ row}\Phi_\epsilon(w\lambda,k,x)_i \quad (1\le i\le r,\ x\in\mathcal{C}).$$

Here $F^{(j)}_\epsilon(\lambda,k;x)$ is a column vector of r components whose i-th row giving the value at $v_ix\in v_i\mathcal{C}$, $A^\epsilon_w(\lambda,k)$ are intertwining matrices of size r which satisfy

$$A^\epsilon_{wv}(\lambda,k) = A^\epsilon_w(v\lambda,k)A^\epsilon_v(\lambda,k) \qquad (\forall w,\ v\in W), \tag{4.2}$$

and $\Phi_\epsilon(\lambda,k,x)$ is a column vector of r components whose i-th row giving the series solution of (HO)$_\epsilon$ on $v_ix\in v_i\mathcal{C}$ with

$$\Phi_\epsilon(\lambda,k,x)_i \sim e^{\langle\lambda-\rho(k),x\rangle}+\cdots \quad (x\to\infty).$$

If s_α is a simple reflection with respect to $\alpha\in\Psi$, $A^\epsilon_{s_\alpha}(\lambda,k)$ is a suitable direct product of matrices and scalars of the form

$$A(s,k) = \begin{pmatrix} \frac{\sin\pi k}{\sin\pi(s+k)} & \frac{\sin\pi s}{\sin\pi(s+k)} \\ \frac{\sin\pi s}{\sin\pi(s+k)} & \frac{\sin\pi k}{\sin\pi(s+k)} \end{pmatrix}, \quad \frac{\cos\frac12\pi(s-k)}{\cos\frac12\pi(s+k)} \quad \textit{and} \quad 1. \tag{4.3}$$

Moreover there is a functional equation

$$F_\epsilon(\lambda,k;x) = F_\epsilon(w\lambda,k;x)A^\epsilon_w(\lambda,k) \qquad (\forall w\in W).$$

We did not give precise definition of the $r\times r$ matrix $A_w(\lambda,k)$ in the statement of the above theorem. We first define $A_w(\lambda,k)$ for simple reflections. Then $A_w(\lambda,k)$ for general $w\in W$ are given by the product formula (4.2). We give rank two examples.

Example 4.2 (Case A_2)**.** $\Psi=\{e_1-e_2,\ e_2-e_3\}$. Let ϵ be the signature of roots defined by $\epsilon(e_1-e_2)=1$, $\epsilon(e_2-e_3)=-1$. Then $W_\epsilon=\{1,\ s_1\}$ and $\#W/W_\epsilon=3$. In this case,

$$A^\epsilon_{s_1}(\lambda) = \begin{pmatrix} 1 & \\ & A(\lambda_1-\lambda_2,k)\end{pmatrix}, \quad A^\epsilon_{s_2}(\lambda) = \begin{pmatrix} A(\lambda_2-\lambda_3,k) & \\ & 1\end{pmatrix},$$

where $A(s,k)$ is the 2×2 matrix in (4.3).

Example 4.3 (Case $B^{(1)}_2$)**.** $\Psi=\{\alpha_1=e_1-e_2,\alpha_2=e_2\}$, $\epsilon(e_1-e_2)=-1$, $\epsilon(e_2)=1$.

$W_\epsilon=\{1,s_2\}$, $W_\epsilon\backslash W=\{1,s_1,s_1s_2,s_1s_2s_1\}$.

$$A^\epsilon_{s_1}(\lambda) = \begin{pmatrix} A(\lambda_1-\lambda_2,k_1) & \\ & A(\lambda_1-\lambda_2,k_1)\end{pmatrix} \in GL(4,\mathbb{C}),$$

$$A^{\epsilon}_{s_2}(\lambda) = \begin{pmatrix} 1 & & \\ & A(2\lambda_2, k_2) & \\ & & 1 \end{pmatrix} \in GL(4, \mathbb{C}),$$

where $A(s,k)$ is the 2×2 matrix in (4.3).

Example 4.4 (Case $B_2^{(2)}$)**.** $\Psi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2\}$, $\epsilon(e_1 - e_2) = 1$, $\epsilon(e_2) = -1$. $W_\epsilon = \{1, s_1, s_2s_1s_2, s_1s_2s_1s_2\}$, $W_\epsilon \backslash W = \{1, s_2\}$.

$$\begin{aligned} A_{s_1}(\lambda) &= I_2, \\ A_{s_2}(\lambda) &= \begin{pmatrix} \frac{\sin \pi k_2}{\sin \pi(2\lambda_2 + k_2)} & \frac{\sin \pi(2\lambda_2)}{\sin \pi(2\lambda_2 + k_2)} \\ \frac{\sin \pi(2\lambda_2)}{\sin \pi(2\lambda_2 + k_2)} & \frac{\sin \pi k_2}{\sin \pi(2\lambda_2 + k_2)} \end{pmatrix} = A(2\lambda_2, k_2). \end{aligned}$$

For the group case, that is, the case of a pseudo-Riemannian symmetric space G/K_ϵ due to Oshima and Sekiguchi [OS2], the product formula (4.2) was proved by using intertwining operators, which can be viewed as a generalization of the Gindikin-Karpelevič product formula, and the formulae for simple reflections were obtained by computing integrals. For generic parameter k, we define $A_w(\lambda, k)$ for simple reflections w by generalizing Lemma 4.14 of [OS2]. (Compare above rank two examples with those of [OS1].) Then define $A_w(\lambda, k)$ for general $w \in W$ by (4.2). We have to show that $A_w(\lambda, k)$ is well-defined, which can be proved by direct computations for rank two cases. For example, for A_2 case, we can prove by direct computations that

$$A_{s_1}(s_2s_1\lambda)A_{s_2}(s_1\lambda)A_{s_1}(\lambda) = A_{s_2}(s_1s_2\lambda)A_{s_1}(s_2\lambda)A_{s_2}(\lambda)$$

corresponding to minimal expressions $w^* = s_1s_2s_1 = s_2s_1s_2$ of the longest element of $W \simeq \mathfrak{S}_3$. Analytic continuation of the function $F_\epsilon^{(j)}(\lambda, k; x)$ through the walls of the Weyl group can be proved by the method of rank one reduction, which was employed by Heckman and Opdam for $\epsilon \equiv 1$.

Acknowledgement

The authors wish to thank the referee for helpful comments. In particular, the referee pointed out mistakes in Theorem 3.2 and Example 3.3 of earlier version of this paper.

References

[B] Beerends R. J., Some special values for the BC type hypergeometric function, *Contemp. Math.* **138** (1992), 27–49.

[D] Deligne P., Equations Differentielles á Points Singulier Regulier, *LNM* **163**, Springer-Verlag, 1970.

[E] P. Etingof, Whittaker functions on quantum groups and q-deformed Toda operators, *Differential Topology, Infinite-Dimensional Lie Algebras, and Applications, Amer. Math. Soc. Transl. Ser. 2* **194**, Amer. Math. Soc., Providence. 1999, 9–25.

[GP] Geck M. and Pfeiffer G., Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras, *Oxford University Press*, 2000.

[G] Gleizer O. A., Some explicit solutions of the additive Deligne-Simpson problem and their applications, *Adv. Math.* **178** (2003), 311–374.

[Har] Haraoka Y., Hypergeometric Functions (in Japanese), *Asakura Publishing Co.*, 2002.

[Has] Hashizume M., Whittaker functions on semisimple Lie groups, *Hiroshima Math. J.* **12** (1982), 259–293.

[Hec1] Heckman G. J., Root systems and hypergeometric functions II, *Comp. Math.* **64** (1987), 353–373.

[Hec2] ———, Hecke algebras and hypergeometric functions, *Invent. Math.* **100** (1990), 403–417.

[Hec3] ———, A remark on the Dunkl differential-difference operators, *"Harmonic Analysis on Reductive Groups", W. Barker and P. Sally, eds.*, 181–191, Birkhauser, Boston, 1991.

[Hec4] ———, Dunkl operators, *Séminare BOURBAKI*, **39** (1996-1997), Expose No. 828, 223–246.

[HO] Heckman G. and Opdam E. M., Root systems and hypergeometric functions I, *Comp. Math.*, **64** (1987), 329–352.

[HS] Heckman G. and Schlichtkrull H., Harmonic Analysis and Special Functions on Symmetric Spaces, *Perspectives in Mathematics* Vol. 16, Academic Press, 1994.

[HIO] Hirano M., Ishii T., and Oda T., Confluence from Siegel-Whittaker functions to Whittaker functions on $Sp(2,\mathbb{R})$, *Math. Proc. Cambridge Philos. Soc.* **141** (2006), 15–31.

[In] V. I. Inozemtsev, The finite Toda lattices, *Comm. Math. Phys.* **121** (1989), 629–638.

[Is] Ishii T., Siegel-Whittaker functions on $SO_0(2,q)$ for class one principal series representations, *Compos. Math.* **140** (2004), 827–854.

[Katz] Katz N. M., Rigid Local Systems, Annals of Mathematics Studies **139**, *Princeton University Press* 1995.

[Koh] Kohno M., Global analysis in linear differential equations, *Kluwer Academic Publishers*, 1999.

[Koo] Koornwinder T. H., Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent differential operators, I-IV. *Indag. Math* **36** (1974) 48–66 and 358–381.

[KO] B. Krötz and E. Opdam, Analysis on the crown domain, *GAFA* **18** (2008) 1326–1421.

[L] Levelt A. H. M., Hypergeometric functions III, *Indag. Math.* **23** (1961), 386–403.

[Oa] Oaku T., Algorithmic methods for Fuchsian systems of linear partial differential equations. *J. Math. Soc. Japan* **47** (1995), 297–328.

[OTY] Okubo K., Takano K., and Yoshida S., A connection problem for the generalized hypergeometric equation, *Funkcialaj Ekvacioj*, **31** (1988), 483–495.

[Op1] Opdam E. M., Root systems and hypergeometric functions III, *Comp. Math.* **67** (1988), 21–49.

[Op2] ———, Root systems and hypergeometric functions IV, *Comp. Math.* **67** (1988), 191–209.

[Op3] ———, An analogue of the Gauss summation formula for hypergeometric functions related to root systems, *Math. Z.* **212** (1993), 313–336.

[Op4] ———, Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group, *Comp. Math.* **85** (1993), 333–373.

[Op5] ———, Lectures on Dunkl Operators for Real and Complex Reflection Groups, *MSJ Memoirs* **8**, Mathematical Society of Japan, 2000.

[O1] Oshima T., A class of completely integrable quantum systems associated with classical root systems, *Indag. Math.* **16** (2005), 655-677.

[O2] ———, Completely integrable quantum systems associated with classical root systems, *SIGMA* **3-071** (2007), pp. 50.

[O3] ———, Heckman-Opdam hypergeometric functions and their specializations, *Harmonische Analysis und Darstellungstheorie Topologischer Gruppen, Mathematisches Forschungsinstitut Oberwolfach*, Report No. 49/2007, 38-40.

[O4] ———, Commuting differential operators with regular singularities, *Algebraic Analysis of Differential Equations — from Microlocal Analysis to Exponential Asymptotics — Festschrift in Honor of Takahiro Kawai*, Springer-Verlag, Tokyo, 2007, 195–224.

[O5] ———, Classification of Fuchsian systems and their connection problem, arXiv:0811.2916.

[OS1] Oshima T. and Sekiguchi J., Harmonic analysis on affine symmetric spaces (in Japanese), *RIMS Kôkyûroku* **287** (1977), 70–87.

[OS2] ———, Eigenspaces of invariant differential operators on an affine symmetric space, *Invent. Math.* **57** (1980), 1–81.

[Sc] Schapira B., Contributions to the hypergeometric function theory of Heckman and Opdam: sharp estimates, Schwartz space, heat kernel, *Geom. Funct. Anal.* **18**, (2008), 222–250.

[Sekj1] Sekiguchi J., Zonal spherical functions on some symmetric spaces, *Publ. RIMS. Kyoto Univ.* **12** Suppl. (1977), 455–464.

[Sekj2] ———, Connection relations for a generalization of spherical functions, *Proceeding of the symposium on representation theory at Kakegawa* (2005), 70–83.

[Sekh] Sekiguchi H., Radial parts of Casimir operators on semisimple symmetric spaces (in Japanese), *RIMS Kôkyûroku* **816** (1992), 155–168.

[Sh1] Shimeno N., Boundary value problems for the Shilov boundary of a bounded symmetric domain of tube type, *J. of Funct. Anal.* **140** (1996), 124–141.

[Sh2] ———, Boundary value problems for various boundaries of Hermitian symmetric spaces, *J. of Funct. Anal.* **170** (2000), 265–285.

[Sh3] ———, A limit transition from the Heckman-Opdam hypergeometric functions to the Whittaker functions associated with root systems, arXiv:0812.3773.

[Si] Simpson C., Products of matrices, *Amer. Math. Soc. Proc.* **1** (1992), 157–185.

[WW] Whittaker E. T. and Watson G. N., A Course of Modern Analysis, 4th edition, *Cambridge University Press*, 1955.