RIMS Kôkyûroku Bessatsu
B20 (2010), 163–171

Generalized Laplacians on classical domains

By

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Abstract

In [6], the generalized Poisson-Cauchy kernel function, which includes as special cases the Poisson kernel function and the Cauchy kernel function, was defined. Moreover the explicit formula of the generalized Poisson-Cauchy kernel function was given. In [3], we defined the generalized Laplacians on the classical domains and showed that the generalized Poisson-Cauchy transforms give rise to eigenfunctions of the generalized Laplacians. In this article, we carry out a direct computation to obtain an explicit formula of the eigenvalues.

Introduction

In this article, we consider the only classical domain of type I for simplicity. For the details of the other types and the computation, see [4], [8].

We denote by $\boldsymbol{D}$ (resp. $\boldsymbol{S}$) the classical domain of type I (resp. the Shilov boundary of type I). In [2], Hua gave the explicit formulas of the Laplace-Beltrami operator, the Poisson kernel function and the Cauchy kernel function for $\boldsymbol{D}$. In [6], the authors gave the definitions of the generalized Poisson-Cauchy transform and the generalized Poisson-Cauchy kernel function. The Poisson-Cauchy kernel function includes as special cases the Poisson kernel function and the Cauchy kernel function. In [3], the definition of the generalized Laplacian was given and it was proved that the generalized Poisson-Cauchy transforms give us eigenfunctions of the generalized Laplacian.

In the present article, an explicit formula of eigenvalues is given by following the calculation used in [8]. All computation is executed out by elementary calculus.

In section 1, we review the basic definitions and the facts. Then in section 2 we introduce the notion of the generalized Poisson-Cauchy transform and the Poisson-Cauchy kernel function. Moreover we give basic facts proved in [6]. In section 3 we

Received December 19, 2008. Accepted May 25, 2009.
2000 Mathematics Subject Classification(s): Primary 43A85, 32M15; Secondary 32A26.
Key Words: *Harmonic analysis on Symmetric spaces, Poisson-Cauchy transforms on classical domains, Eigenvalus of Laplacians*:
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introduce the notion of the generalized Laplacian which includes as a special case, the Laplace-Beltrami operator and give some basic properties. The most important fact is that the generalized Poisson-Cauchy transform gives us eigenfunctions of the generalized Laplacian. In section 4, we give explicit expressions of eigenvalues.

In this paper we follow the notation in [1],[6] and [4].

§ 1. Preliminaries

Let $m \leqq n$ and $\boldsymbol{D} = \{z \in M_{n,m}(\boldsymbol{C})\ ;\ I_m - z^*z > 0\}$. Here " > 0 " means " is positive definite". The Shilov boundary of this domain is $\boldsymbol{S} = \{u \in M_{n,m}(\boldsymbol{C})\ ;\ u^*u = I_m\}$. Let

$$\begin{aligned} &G = SU(n,m), \\ &K = S(U(n)\times U(m)) = \left\{ \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in SL(m+n,\boldsymbol{C})\ ;\ k_1 \in U(n),\ k_2 \in U(m) \right\}. \end{aligned}$$

The group G acts holomorphically on $\boldsymbol{D}$ by $g[z] = (az+b)(cz+d)^{-1}$ for $z \in \boldsymbol{D}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(n,m)$ with $a \in M_{n,n}(\mathbb{C}), b \in M_{n,m}(\mathbb{C}), c \in M_{m,n}(\mathbb{C})$ and $d \in M_{m,m}(\mathbb{C})$.

Furthermore put

$$\begin{aligned} &G_c = SL(m+n,\boldsymbol{C}), \\ &K_c = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in SL(m+n,\boldsymbol{C})\ ;\ \alpha \in GL(n,\boldsymbol{C}),\ \delta \in GL(m,\boldsymbol{C}) \right\}, \\ &P_+ = \left\{ \begin{pmatrix} I_n & z \\ 0 & I_m \end{pmatrix}\ ;\ z \in M_{n,m}(\boldsymbol{C}) \right\}, P_- = \left\{ \begin{pmatrix} I_n & 0 \\ w & I_m \end{pmatrix}\ ;\ w \in M_{n,m}(\boldsymbol{C}) \right\}, \\ &U = K_cP_-, \quad \mu_0 = \begin{pmatrix} I_m & 0 & I_m \\ 0 & I_{n-m} & 0 \\ 0 & 0 & I_m \end{pmatrix}, \quad P = G \cap \mu_0 U \mu_0^{-1}. \end{aligned}$$

Then we have $G/K \cong GU/U \cong \boldsymbol{D}$ and $G/P \cong G\mu_0U/U \cong \boldsymbol{S}$.

§ 2. The generalized Poisson-Cauchy transform

In [6], the notion of the generalized Poisson-Cauchy transform was introduced. This transform includes as special cases the Poisson kernel function and the Cauchy kernel function. Before giving the definition of the Poisson-Cauchy transform, we begin with introducing following characters. Let τ_ℓ and $\eta_{\ell,s}$ be characters of U, and $\xi_{\ell,s}$ a character of P defined as follows:

$$\begin{aligned}
\tau_\ell : U \ni \begin{pmatrix} \alpha & 0 \\ \zeta & \delta \end{pmatrix} &\longmapsto (\det(\delta))^\ell \in \boldsymbol{C}^*, \\
\eta_{\ell,s} : U \ni \begin{pmatrix} \alpha & 0 \\ \zeta & \delta \end{pmatrix} &\longmapsto \left(\frac{\det(\delta)}{|\det(\delta)|}\right)^\ell |\det(\delta)|^s \in \boldsymbol{C}^*, \\
\xi_{\ell,s} : P \ni p &\longmapsto \eta_{\ell,s}(\mu_0^{-1} p \mu_0) \in \boldsymbol{C}^*,
\end{aligned}$$

where $\ell \in \boldsymbol{Z}$ and $s \in \boldsymbol{C}$. We shall write τ, η and ξ instead of τ_ℓ, $\eta_{\ell,s}$ and $\xi_{\ell,s}$ respectively for simplicity.

We regard the complex Lie group G_c as the principal fiber bundle over the complex homogeneous space G_c/U. We denote by $\tilde{E}_\tau$ the holomorphic line bundle over G_c/U associated to τ. We denote by E_τ the restriction of $\tilde{E}_\tau$ to the open submanifold $G/K \cong GU/U$ of G_c/U. Then the space of all $\boldsymbol{C}^\infty$-sections of E_τ is identified with

$$\boldsymbol{C}^\infty(E_\tau) = \left\{ h \in \boldsymbol{C}^\infty(GU) \ ; \ h(wu) = \tau(u)^{-1} h(w) \ (w \in GU, \ u \in U) \right\}.$$

We denote by $\widetilde{L}_\eta$ the $\boldsymbol{C}^\infty$-line bundle on G_c/U associated to η. We denote by L_η the restriction of $\widetilde{L}_\eta$ to the compact submanifold $G/P \cong G\mu_o U/U$ of G_c/U. Then the space of all $\boldsymbol{C}^\infty$-sections of L_η is identified with

$$\boldsymbol{C}^\infty(L_\eta) = \left\{ \psi \in \boldsymbol{C}^\infty(G\mu_o U) \ ; \ \psi(wu) = \eta(u)^{-1} \psi(w) \quad (w \in G\mu_o U, \ u \in U) \right\}.$$

Put

$$\begin{aligned}
\boldsymbol{C}^\infty(G)_\tau &= \left\{ f \in \boldsymbol{C}^\infty(G) \ ; \ f(gk) = \tau(k)^{-1} f(g) \quad (g \in G, \ k \in K) \right\}, \\
\boldsymbol{C}^\infty(G)_\xi &= \left\{ \phi \in \boldsymbol{C}^\infty(G) \ ; \ \phi(gp) = \xi(p)^{-1} \phi(g) \quad (g \in G, \ p \in P) \right\}.
\end{aligned}$$

Then we obtain the following four onto-isomorphisms :

$$\begin{aligned}
&\boldsymbol{C}^\infty(E_\tau) \ni h \longmapsto f \in \boldsymbol{C}^\infty(G)_\tau, \quad f(g) = h(g) \quad (g \in G), \\
&\boldsymbol{C}^\infty(E_\tau) \ni h \longmapsto F \in \boldsymbol{C}^\infty(\boldsymbol{D}), \quad F(z) = h \begin{pmatrix} I_n & z \\ 0 & I_m \end{pmatrix} \quad (z \in \boldsymbol{D}), \\
&\boldsymbol{C}^\infty(L_\eta) \ni \psi \longmapsto \phi \in \boldsymbol{C}^\infty(G)_\xi, \quad \phi(g) = \psi(g\mu_o) \quad (g \in G), \\
&\boldsymbol{C}^\infty(L_\eta) \ni \psi \longmapsto \Phi \in \boldsymbol{C}^\infty(\boldsymbol{S}), \quad \Phi(u) = \psi\left(\begin{pmatrix} I_n & u \\ 0 & I_m \end{pmatrix} \mu_0 \right) \quad (u \in \boldsymbol{S}).
\end{aligned}$$

Now, we define *the generalized Poisson-Cauchy transform.*

Definition 2.1. We define

$$\boldsymbol{P}_{\tau,\xi} \ : \ \boldsymbol{C}^\infty(G)_\xi \ni \phi \longmapsto f \in \boldsymbol{C}^\infty(G)_\tau$$

by

$$f(g) = \int_K \tau(k)\phi(gk)dk \quad (g \in G).$$

Moreover we define $\mathcal{P}_{\tau,\eta}$ in such a way that the following diagram is commutative.

$$\begin{array}{ccccc} & \boldsymbol{C}^\infty(G)_\xi & \cong & \boldsymbol{C}^\infty(\boldsymbol{S}) & \\ \boldsymbol{P}_{\tau,\xi} & \downarrow & & \downarrow & \mathcal{P}_{\tau,\eta} \\ & \boldsymbol{C}^\infty(G)_\tau & \cong & \boldsymbol{C}^\infty(\boldsymbol{D}) & \end{array}$$

We call $\mathcal{P}_{\tau,\eta}$ the generalized Poisson-Cauchy transform (with respect to the pair $(\boldsymbol{D}, \boldsymbol{S})$).

The following theorem was proved in [6].

Theorem 2.2. *For any $\Phi \in \boldsymbol{C}^\infty(\boldsymbol{S})$, we have*

$$\left(\mathcal{P}_{\tau,\eta}(\Phi)\right)(z) = \int_S \boldsymbol{K}_{\tau,\eta}(z,u)\Phi(u)du \quad (z \in \boldsymbol{D}, u \in \boldsymbol{S}),$$

where

$$\boldsymbol{K}_{\tau,\eta}(z,u) = \frac{1}{(\det(I_m - u^*z))^\ell}\left(\frac{\det(I_m - z^*z)}{|\det(I_m - u^*z)|^2}\right)^{n-(\ell+s)/2}.$$

In [2], Hua gave explicit expressions of the Poisson kernel function and the Cauchy kernel function for the classical domains.

Comparing our formula with [2], we check immediately that the generalized Poisson-Cauchy kernel function coincides with the Poisson kernel function (resp. the Cauchy kernel function) in the case $\ell = s = 0$ (resp. $\ell = s = n$).

§ 3. The generalized Laplacian

We begin with the definitions of representations π_τ and T_τ of G.

Definition 3.1. For any $g \in G$ and $h \in \boldsymbol{C}^\infty(E_\tau)$, we define

$$(\pi_\tau(g)h)(w) = h(g^{-1}w) \quad (w \in GU).$$

For any $g \in G$, we define $T_\tau(g)$ in such a way that the following diagram is commutative:

$$\begin{array}{ccccc} & \boldsymbol{C}^\infty(\boldsymbol{E}_\tau) & \cong & \boldsymbol{C}^\infty(\boldsymbol{D}) & \\ \pi_\tau(g) & \downarrow & & \downarrow & T_\tau(g) \\ & \boldsymbol{C}^\infty(\boldsymbol{E}_\tau) & \cong & \boldsymbol{C}^\infty(\boldsymbol{D}). & \end{array}$$

Then we have

Lemma 3.2. *For any $g \in G$ and $F \in \boldsymbol{C}^\infty(\boldsymbol{D})$ we have*

$$T_\tau(g)F(z) = (\det(cz+d))^{-\ell} F(g^{-1}[z]),$$

where $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$

In [4], the canonical Riemannian metrics g on the classical domains are introduced. For our domain $\boldsymbol{D}$, it is defined by $g_z(u,v) = \frac{1}{2}\mathrm{Tr}((I_n - zz^*)^{-1}u(I_m - z^*z)^{-1}v^*)$ where $z \in \boldsymbol{D}, u, v \in M_n(\mathbb{C})$. We modify the Laplace-Beltrami operator with respect to this metric so that modified operator is invariant with respect to the representation T_τ. The outline of the method is explained as follows (for details see [4]).

Firstly we show that the Laplace-Beltrami operator Δ can be written as the form $D_z \cdot h(z) \cdot D_z^*$, where $D_z = \left(\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_N}\right), N = mn, z_{i+(j-1)n} = z_{ij}$ and $h(z)$ is an $N \times N$ matrix valued function which satisfies the following assumption (here the symbol " $\cdot$ " means that any function between two dots should not be differentiated[1]).

Assumption 1. For any $g \in G$ and $z \in \boldsymbol{D}$,

$$h(w) = \frac{\partial(w)}{\partial(z)} h(z) \left(\frac{\partial(w)}{\partial(z)}\right)^* \quad (w = g[z]),$$

where $\frac{\partial(w)}{\partial(z)}$ is the Jacobian matrix.

Secondly, we assume a non-zero $\boldsymbol{C}^\infty$ function r_τ satisfies the following assumption.

Assumption 2. For any $g \in G$ and $z \in \boldsymbol{D}$,

$$r_\tau(w) = |\rho_\tau(g,z)|^{-2} r_\tau(z) \quad (w = g^{-1}[z]),$$

where $\rho_\tau(g,z) = \det(cz+d)^\ell$ (see Lemma 1 in [4]).

Then we define the generalized Laplacian Δ_τ by

$$\Delta_\tau(z) = r_\tau(z)^{-1} D_z r_\tau(z) \cdot h(z) \cdot D_z^*.$$

Actually it can be shown that the function $r_\tau(z)$ always exists and is unique up to constant multiple. Thus Δ_τ is uniquely determined by the definition above.

The operator Δ_τ is invariant with respect to the representation T_τ. In fact, from the definition of Δ_τ we have

$$\begin{aligned}
\Delta_\tau(w) &= r_\tau(w)^{-1} D_w r_\tau(w) \cdot h(w) \cdot D_w^* \\
&= |\rho_\tau(g,z)|^2 r_\tau(z)^{-1} D_z |\rho_\tau(g,z)|^{-2} r_\tau(z) \\
&\quad \times \cdot \left(\frac{\partial(w)}{\partial(z)}\right)^{-1} h(w) \left(\left(\frac{\partial(w)}{\partial(z)}\right)^*\right)^{-1} \cdot D_z^* \\
&= \rho_\tau(g,z) r_\tau(z)^{-1} D_z r_\tau(z) \cdot h(z) \cdot D_z^* \rho_\tau(g,z)^{-1}.
\end{aligned}$$

[1] If the reader is unfamiliar with this, see Appendix.

Thus for any $g \in G$ and $F \in \boldsymbol{C}^\infty(\boldsymbol{D})$, we obtain

$$(T_\tau(g)\Delta_\tau F)(z) = \rho_\tau(g,z)^{-1}\Delta_\tau(w)F(w) = \Delta_\tau(z)T_\tau(g)F(z).$$

The generalized Laplacian for $\boldsymbol{D}$ is given explicitly as follows (see [3],[4]):

$$\Delta_\tau(z) = \mathrm{Tr}\left(\det\left(I_m - z^*z\right)^{-\ell}\left(I_m - z^*z\right)\partial_z \det\left(I_m - z^*z\right)^{\ell} \cdot \left(I_n - zz^*\right)\cdot \partial_z^*\right),$$

where ∂_z is the matrix valued differential operator $\partial_z = {}^t\left(\frac{\partial}{\partial z_{ij}}\right)$.

In [2], Hua considered the invariant differential operator

$$\Delta = \mathrm{Tr}\left(\left(I_m - z^*z\right)\partial_z \cdot \left(I_n - zz^*\right)\cdot \partial_z^*\right),$$

which turned out to be the Laplace-Beltrami operator Δ with respect to canonical Riemannian metric (see [4, section 3]). The expression of Δ_τ makes clear the relation between Hua's Δ and our generalized Laplacian Δ_τ. Namely, if $\ell = 0$, then the generalized Laplacian coincides with the Laplace-Beltrami operator.

In the following section we prove that the Poisson-Cauchy kernel function is an eigenfunction of the generalized Laplacian and also compute the corresponding eigenvalue.

§ 4. Eigenvalues of the generalized Laplacian

To compute the eigenvalue we use the following proposition.

Proposition 4.1. *Put $c_{\tau,\eta} = \Delta_\tau \boldsymbol{K}_{\tau,\eta}(z,u)|_{z=0}$. If $c_{\tau,\eta}$ is independent of $u \in \boldsymbol{S}$. Then $\Delta_\tau \boldsymbol{K}_{\tau,\eta}(z,u) = c_{\tau,\eta}\boldsymbol{K}_{\tau,\eta}(z,u)$.*

Proof. We prove this proposition, making use of the representation T_τ as follows. For any $z \in \boldsymbol{D}$ and $u \in \boldsymbol{S}$, we choose $g \in G$ such that $g^{-1}[z] = 0$. Then

$$\begin{aligned} T_\tau(g)\Delta_\tau(z)\boldsymbol{K}_{\tau,\eta}(z,u) &= \rho_\tau(g,z)^{-1}\Delta_\tau(w)\boldsymbol{K}_{\tau,\eta}(w,u) \\ &= c_{\tau,\eta}\rho_\tau(g,z)^{-1}\boldsymbol{K}_{\tau,\eta}(w,u) \\ &= c_{\tau,\eta}T_\tau(g)\boldsymbol{K}_{\tau,\eta}(z,u), \end{aligned}$$

where we use the change of variable $w = g^{-1}[z]$. If we apply $T_\tau(g^{-1})$ to the both sides of this equation, the proposition follows. □

In the rest of the paper, we give an explicit expression of the eigenvalue $c_{\tau,\eta}$.

We first observe

$$\Delta_\tau \boldsymbol{K}_{\tau,\eta}(z,u)|_{z=0} = \mathrm{Tr}\left(\partial_z\partial_z^*\right)\boldsymbol{K}_{\tau,\eta}(z,u)|_{z=0}.$$

Therefore we need to calculate $\frac{\partial^2}{\partial z_{ij}\partial \overline{z}_{ij}}\boldsymbol{K}_{\tau,\eta}(z,u)|_{z=0}$. By straightforward calculation we have

$$\begin{aligned}&\frac{\partial}{\partial \overline{z}_{ij}}\boldsymbol{K}_{\tau,\eta}(z,u) = \det\left(I_m - u^*z\right)^{-\ell-n+\alpha}\frac{\partial}{\partial \overline{z}_{ij}}\left(\frac{\det\left(I_m - z^*z\right)}{\det(I_m - z^*u)}\right)\\ &= (n-\alpha)\det(I_m - u^*z)^{-\ell-n+\alpha}\det(I_m - z^*z)^{-1+n-\alpha}\det(I_m - z^*u)^{-1-n+\alpha}\gamma_{ij},\end{aligned}$$

where $\alpha = (\ell+s)/2$ and

$$\gamma_{ij} = \left(\frac{\partial}{\partial \overline{z}_{ij}}\det(I_m - z^*z)\right)\det(I_m - z^*u) - \det(I_m - z^*z)\left(\frac{\partial}{\partial \overline{z}_{ij}}\det(I_m - z^*u)\right).$$

Furthermore

$$\begin{aligned}&\frac{\partial^2}{\partial z_{ij}\partial \overline{z}_{ij}}\boldsymbol{K}_{\tau,\eta}(z,u)|_{z=0}\\ &= (n-\alpha)\frac{\partial}{\partial z_{ij}}\left(\det(I_m - u^*z)^{-\ell-n+\alpha}\det(I_m - z^*z)^{-1+n-\alpha}\gamma_{ij}\right)|_{z=0}\\ &= (n-\alpha)\left\{(-\ell-n+\alpha)\left(\frac{\partial}{\partial \overline{z}_{ij}}\det(I_m - u^*z)\right)\gamma_{ij} + \frac{\partial}{\partial z_{ij}}\gamma_{ij}\right\}|_{z=0}.\end{aligned}$$

After the following lemma, we obtain an explicit formula of the eigenvalue. Since the proof of the lemma is straightforward, we omit it.

Lemma 4.2. *For $z \in \boldsymbol{D}, u \in \boldsymbol{S}$ we have the following formulas:*

(1) $\frac{\partial}{\partial \overline{z}_{ij}}\det(I_m - z^*z)|_{z=0} = 0$,
(2) $\frac{\partial^2}{\partial z_{ij}\partial \overline{z}_{ij}}\det(I_m - z^*z)|_{z=0} = -1$,
(3) $\frac{\partial}{\partial \overline{z}_{ij}}\det(I_m - u^*z)|_{z=0} = -\overline{u}_{ij}$,
(4) $\frac{\partial}{\partial \overline{z}_{ij}}\det(I_m - z^*u)|_{z=0} = -u_{ij}$,
(5) $\frac{\partial}{\partial z_{ij}}\det(I_m - z^*z)|_{z=0} = 0$,
(6) $\gamma_{ij}|_{z=0} = u_{ij}$,
(7) $\frac{\partial}{\partial z_{ij}}\gamma_{ij}|_{z=0} = -1$.

By this lemma and the computation of $\frac{\partial^2}{\partial z_{ij}\partial \overline{z}_{ij}}\boldsymbol{K}_{\tau,\eta}(z,u)|_{z=0}$, we have

$$\Delta_\tau \boldsymbol{K}_{\tau,\eta}(z,u)|_{z=0} = \sum_{\substack{1\le i\le m\\ 1\le j\le n}}(n-\alpha)\left\{(-\ell-n+\alpha)\left(-\overline{u}_{ij}u_{ij}\right) - 1\right\}.$$

To apply Proposition 4.1 we need to prove that the right-hand side of this equation is independent of $u \in \boldsymbol{S}$. It is shown as follows. By taking trace of the both sides of the condition $u^*u = I_m$ we get $\sum_{\substack{1\le i\le m\\1\le j\le n}} \overline{u}_{ij}u_{ij} = m$. Therefore we finally obtain

$$\begin{aligned}\Delta_\tau \boldsymbol{K}_{\tau,\eta}(z,u)|_{z=0} &= (n-\alpha)\left\{(\ell+n-\alpha)\sum_{\substack{1\le i\le m\\1\le j\le n}} \overline{u}_{ij}u_{ij} - \sum_{\substack{1\le i\le m\\1\le j\le n}} 1\right\}\\ &= (n-\alpha)\left\{m(\ell+n-\alpha)-mn\right\}\\ &= m\,(n-\alpha)\,(\ell-\alpha)\\ &= \frac{m\,(2n-\ell-s)\,(\ell-s)}{4}.\end{aligned}$$

Combining Proposition 4.1 and the calculation above, we get

$$c_{\tau,\eta} = \frac{m(2n-\ell-s)(\ell-s)}{4}.$$

The computation for each type of the classical domain proceeds along the same line as that of type I. The following is our main theorem:

Theorem 4.3 ([4, Theorem 2]). *For each type of the classical domain, the eigenvalue $c_{\tau,\eta}$ is given as follows:*
(type I) $\dfrac{m(2n-\ell-s)(\ell-s)}{4}$,
(type II, n:even) $\dfrac{n\,(n-1-\ell-s)\,(\ell-s)}{4}$,
(type II, n:odd) $\dfrac{(n-1)(n-\ell-s)(\ell-s)}{4}$,
(type III) $\dfrac{n\,(n+1-\ell-s)\,(\ell-s)}{4}$,
(type IV) $(n-\ell-s)\,(\ell-s)$.

§ 5. Remarks

We have seen that the computation of the eigenvalue $c_{\tau,\eta}$ is carried out by direct way. There is another way to obtain the formula for the eigenvalue $c_{\tau,\eta}$. In fact, Δ_τ is equal to the operator $\alpha\, dT_\tau(\Omega) + \beta I_{C^\infty(D)}$, where Ω is the Casimir operator of $\mathfrak{su}(n,m)$, and α, β are constants explicitly given in [4, section 6]. Therefore the calculation of $c_{\tau,\eta}$ is reduced to the observation of the infinitesimal character of the generalized principal series representation realized on $C^\infty(G)_\xi$ (see Shimeno [7] and also Okamoto-Ozeki [5]).

In 1935, E. Cartan proved that there exist only six types of irreducible homogeneous bounded symmetric domains. Besides the classical four types, there exist only two; their dimensions are 16 and 27. In this paper, we gave the formulas of $c_{\tau,\eta}$ for the four types. We do not know any explicit expressions of the generalized Poisson-Cauchy kernel functions and the generalized Laplacians for the exceptional cases. It may be interesting problem to give them.

§ 6. Appendix

We explain the meaning of the dot "·" here (see §3). As mentioned in §3, this symbol means that any function between two dots should not be differentiated. Let us illustrate with following matrix example. Put

$$\partial_z = \begin{pmatrix} \partial_{11} \cdots \partial_{n1} \\ \vdots \quad \ddots \quad \vdots \\ \partial_{1n} \ldots \partial_{nn} \end{pmatrix}, A(z) = \begin{pmatrix} A_{11}(z) \cdots A_{n1}(z) \\ \vdots \quad \ddots \quad \vdots \\ A_{1n}(z) \ldots A_{nn}(z) \end{pmatrix}.$$

If each entry of $A(z)$ is differentiated by ∂_z, then the (i, j)-entry of $\partial_z A(z) \partial_z^*$ becomes

$$\underline{\sum_l \sum_k (\partial_{li} A_{lk}(z)) \overline{\partial}_{kj}} + \sum_l \sum_k A_{lk}(z) \partial_{l1} \overline{\partial}_{k1}.$$

If we add two dots as $\partial_z \cdot A(z) \cdot \partial_z^*$, then it signifies the omission of underlined terms.

Acknowledgements

I am indebted to Professors Hideyuki Ishi and Kiyosato Okamoto who read the manuscript and suggested many improvements. The author also acknowledges the referee for helpful advices.

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