On the existence of ground states for the Pauli-Fierz model with a variable mass

By

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Abstract

The purpose of this paper is to review [9]. The existence of ground states of the Pauli-Fierz model with a variable mass is considered. This paper presents the outline of the proof of it under the infrared regularity condition.

§1. Introduction

The Pauli-Fierz model describes a minimal interaction between a low energy electron and a quantized radiation field, where the electron is governed by a Schrödinger operator. The Pauli-Fierz Hamiltonian is the physical quantity corresponding to the energy of the system and is realized as a self-adjoint operator on a certain Hilbert space and its bottom of the spectrum is called the ground state energy. An eigenvector associated with the ground state energy is called a ground state, if it exists.

The existence of ground states of the Pauli-Fierz Hamiltonian is investigated in [1, 2, 4, 8, 10, 12]. In [2, 8], the infrared regularity condition is not assumed. In [4, 8], the existence of ground states is shown for arbitrary values of coupling constants. The uniqueness of the ground state of the Pauli-Fierz Hamiltonian is proven in [11].

The Pauli-Fierz Hamiltonian with a variable mass is considered in this paper. It is derived from the analogy of the Nelson model on a pseudo Riemannian manifold [5, 6, 7]. Under the infrared regularity condition, this Hamiltonian has ground states for all values of a coupling constant when a variable mass decays sufficiently fast.
§ 2. Definition of the Pauli-Fierz model

§ 2.1. Hilbert space of states

We consider the Hilbert space of states of total system as

\[ \mathcal{H} := \mathcal{H}_P \otimes \mathcal{F}, \]

where

\[ \mathcal{H}_P := L^2(\mathbb{R}^3) \]

describes state space of one electron and \( \mathcal{F} \) is the boson Fock space over \( L^2(\mathbb{R}^3; \mathbb{C}^2) \) defined by

\[ \mathcal{F} := \bigoplus_{n=0}^\infty \left[ \bigotimes_{s}^{n} L^2(\mathbb{R}^3; \mathbb{C}^2) \right]. \]

Here \( \bigotimes_{s}^{n} L^2(\mathbb{R}^3; \mathbb{C}^2) \) denotes the \( n \)-fold symmetric tensor product of \( L^2(\mathbb{R}^3; \mathbb{C}^2) \) with \( \bigotimes_{s}^{0} L^2(\mathbb{R}^3; \mathbb{C}^2) = \mathbb{C} \). The inner product on \( \mathcal{F} \) is given by

\[
(\Psi, \Phi)_{\mathcal{F}} = \overline{\Psi(0)}\Phi^{(0)} + \sum_{n=1}^\infty \int_{\mathbb{R}^{3n}} \overline{\Psi(n)(k_1, \cdots, k_n)}\Phi^{(n)}(k_1, \cdots, k_n)dk_1 \cdots dk_n.
\]

The Hilbert space \( \mathcal{H} \) can be identified with

\[
\mathcal{H} \cong \int_{\mathbb{R}^3}^\oplus \mathcal{F}dx \cong L^2(\mathbb{R}^3) \oplus \left[ \bigoplus_{n=1}^\infty L_{\mathrm{sym}}^2(\mathbb{R}^{3+3n}; \mathbb{C}^2) \right].
\]

Here \( L_{\mathrm{sym}}^2(\mathbb{R}^{3+3n}; \mathbb{C}^2) \) is the set of \( L^2(\mathbb{R}^{3+3n}; \mathbb{C}^2) \)-functions such that

\[ f(x, k_1, \cdots, k_n) = f(x, k_{\sigma(1)}, \cdots, k_{\sigma(n)}) \]

for an arbitrary permutation \( \sigma \).

Let \( T \) be a densely defined closable operator on \( L^2(\mathbb{R}^3; \mathbb{C}^2) \). Then \( \Gamma(T) \) and \( d\Gamma(T) \) are defined by

\[
\Gamma(T) := \bigoplus_{n=0}^\infty \otimes^n T, \quad d\Gamma(T) := \bigoplus_{n=0}^\infty \otimes^n T^{(n)},
\]

where \( \otimes^0 T = 1, \ T^{(n)} := \sum_{k=1}^n 1 \otimes \cdots \otimes 1 \otimes T \otimes 1 \cdots \otimes 1 \) and \( T^{(0)} = 0 \). The number operator is defined by

\[ N := d\Gamma(1). \]
The annihilation operator $a(f)$ and the creation operator $a^\dagger(f)$ smeared by $f \in L^2(\mathbb{R}^3; \mathbb{C}^2)$ on $\mathcal{F}$ are defined by

\begin{align}
D(a^\dagger(f)) &= \left\{ \Psi \in \mathcal{F} \mid \sum_{n=1}^{\infty} n \left\| S_n(f \otimes \Psi^{(n-1)}) \right\|^2 < \infty \right\}, \\
(a^\dagger(f)\Psi)^{(n)} &= \sqrt{n} S_n(f \otimes \Psi^{(n-1)}), \quad n \geq 1, \quad (a^\dagger(f)\Psi)^{(0)} = 0, \\
a(f) &= (a^\dagger(\overline{f}))^*,
\end{align}

where $S_n$ denotes the symmetrization operator of degree $n$ and $D(T)$ the domain of $T$. \(\Omega:=(1,0,0,\cdots)\in \mathcal{F}\) is called the Fock vacuum.

\(\Omega\) is defined by

\begin{align}
(a(k)\Psi)^{(n)}(k_1, \cdots, k_n) &= \sqrt{n+1}\Psi^{(n+1)}(k, k_1, \cdots, k_n) \quad \text{for } \Psi \in D(N^{1/2}).
\end{align}

Then for almost every $k$, $a(k)\Psi \in \mathcal{F}$.

\section*{§ 2.2. Definition of the Pauli-Fierz model}

Let $v$ be a multiplication operator on $L^2(\mathbb{R}^3)$. We introduce assumptions on $v$.

\textbf{Assumption 1.} \hspace{1cm}

(1) $\sigma_P(-\Delta + v) \subset (0, \infty)$;

(2) $v(x) \leq \text{const.} \langle x \rangle^{-\beta}$ with $\beta > 3$, where $\langle x \rangle = \sqrt{1+|x|^2}$.

Here $\sigma_P(T)$ denotes the set of eigenvalues of $T$.

Then there exists a unique function $\Psi(k, x)$ such that for $k \neq 0$,

\begin{align}
(-\Delta_x + v(x))\Psi(k, x) &= |k|^2\Psi(k, x)
\end{align}

and $\Psi(k, x)$ satisfies the Lippman-Schwinger equation:

\begin{align}
\Psi(k, x) &= e^{ikx} - \frac{1}{4\pi} \int \frac{e^{ik|y|}}{|x-y|} v(y) \Psi(k, y) dy.
\end{align}

We will use the regularity properties of $\Psi(k, x)$ below to show the existence of ground states.

\textbf{Lemma 2.1.} \hspace{1cm} \textbf{Suppose Assumption 1. Then}

\begin{align}
|\Psi(k, x) - e^{ikx}| &\leq \text{const.} \langle x \rangle^{-1}
\end{align}

(a)
holds.

(b) $\Psi(k, x)$ is continuously differentiable in $x$ for each fixed $k$ but $k \neq 0$ and

$$\frac{\partial}{\partial x_\mu} \Psi(k, x) - ik_\mu e^{ikx} = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \left( \frac{e^{i|k||x-y|}(x_\mu - y_\mu)}{|x-y|^3} - \frac{i|k|e^{i|k||x-y|}(x_\mu - y_\mu)}{|x-y|^2} \right) v(y) \Psi(k, y) dy.$$  

In particular, for any compact set $D$ but $0 \notin D$, $\sup_{k \in D, x} \left| \frac{\partial \Psi}{\partial x_\mu}(k, x) \right| < \infty$.

(c) For $k \neq 0$ and $k + h \neq 0$,

$$\frac{1}{|h|} \left| \Psi(k + h, x) - \Psi(k, x) \right| \leq \text{const.} (1 + |x|),$$

$$\frac{1}{|h|} \left| \frac{\partial}{\partial x_\nu} \Psi(k + h, x) - \frac{\partial}{\partial x_\nu} \Psi(k, x) \right| \leq \text{const.} (1 + |k| + |x| + |k||x|)$$

hold, and $\Psi(k, x)$ and $\frac{\partial}{\partial x_\nu} \Psi(k, x)$ are differentiable in $k \in \mathbb{R}^3 \setminus \{0\}$ for each fixed $x$.

Let us introduce the dispersion relation and the quantized radiation field with a variable mass $v$.

**Definition 2.2.** The dispersion relation with a variable mass is given by

$$\omega := \sqrt{-\Delta + v}$$

on $L^2(\mathbb{R}^3; \mathbb{C}^2)$, where $v$ is called a variable mass. The free Hamiltonian is defined by the second quantization of $\omega$:

$$H_f = d\Gamma(\omega).$$

Let $m \geq 0$ and $\omega_m := \sqrt{-\Delta + v + m^2}$. We set

$$H_f(m) = d\Gamma(\omega_m).$$

In order to define the quantized radiation field, we introduce a cutoff functions: $\hat{\varphi}_j^\mu$, $j = 1, 2$, $\mu = 1, 2, 3$.

**Assumption 2.**

(1) The support of $\hat{\varphi}_j^\mu$ is compact;

(2) $\hat{\varphi}_j^\mu$ is differentiable and the derivative function is bounded;

(3) (infrared regularity condition)

It holds that

$$\int_{\mathbb{R}^3} \frac{|\hat{\varphi}_j^\mu(k)|^{2p}}{|k|^{5p}} dk < \infty \quad \text{for all} \quad 0 < p < 1.$$
Let the test function $\rho_x^{\mu} = (\rho_x^{\mu,1}, \rho_x^{\mu,2}) \in L^2(\mathbb{R}^3; \mathbb{C}^2) \in L^2(\mathbb{R}^3; \mathbb{C}^2)$ be such that

$$\rho_x^{\mu, j}(y) := (2\pi)^{-3/2} \int \overline{\Psi(k, x)} \Psi(k, y) \hat{\varphi}_j^{\mu}(k) dk.$$ 

The quantized radiation field with a variable mass is given by

$$A_\mu(x) := \frac{1}{\sqrt{2}} (a^\dagger (\hat{\omega}^{-1/2} \rho_x^{\mu}) + a (\overline{\hat{\omega}^{-1/2} \rho_x^{\mu}})) , \quad \mu = 1, 2, 3,$$

for each $x \in \mathbb{R}^3$.

**Definition 2.3.** Let $V$ be a multiplication operator, and $V_+$ and $V_-$ the positive part and the negative part of $V$, respectively. Then the quadratic form $q_m^V$ is defined by

$$q_m^V(\Psi, \Phi) = \frac{1}{2} \sum_{\mu=1}^{3} ((p_\mu + \sqrt{\alpha} A_\mu) \Psi, (p_\mu + \sqrt{\alpha} A_\mu) \Phi)$$

$$+ \left( \hat{H}_t^{1/2}(m) \Psi, \hat{H}_t^{1/2}(m) \Phi \right) + \left( V_+^{1/2} \Psi, V_+^{1/2} \Phi \right) - \left( V_-^{1/2} \Psi, V_-^{1/2} \Phi \right)$$

with the form domain

$$Q(q_m^V) = D(|p|) \cap D(H_t^{1/2}(m)) \cap D(|V|^{1/2}).$$

Here $\alpha$ is a coupling constant. When $m = 0$, we denote $q^V$ for $q_0^V$.

§ 2.3. Generalized Fourier transformation

By [14], under Assumption 1, the generalized Fourier transformation is defined by

$$f \mapsto \mathcal{F} f(\cdot) := (2\pi)^{-3/2} \int f(x) \overline{\Psi(\cdot, x)} dx,$$

which is a unitary transformation on $L^2(\mathbb{R}^3)$. By $1 \otimes \Gamma(\mathcal{F}) : \mathcal{H} \rightarrow \mathcal{H}$, the quadratic form $q_m^V$ is transformed as

$$q_m^V(\Psi, \Phi) = q_m^V(1 \otimes \Gamma(\mathcal{F}) \Psi, 1 \otimes \Gamma(\mathcal{F}) \Phi)$$

$$= \frac{1}{2} \sum_{\mu=1}^{3} \left( (p_\mu + \sqrt{\alpha} \hat{A}_\mu) \Psi, (p_\mu + \sqrt{\alpha} \hat{A}_\mu) \Phi \right) + \left( \hat{H}_t^{1/2}(m) \Psi, \hat{H}_t^{1/2}(m) \Phi \right)$$

$$+ \left( V_+^{1/2} \Psi, V_+^{1/2} \Phi \right) - \left( V_-^{1/2} \Psi, V_-^{1/2} \Phi \right)$$

with the form domain

$$Q(q_m^V) = D(|p|) \cap D(H_t^{1/2}(m)) \cap D(|V|^{1/2}).$$
Here
\begin{equation}
\hat{A}_\mu(x) := \frac{1}{\sqrt{2}} \sum_{j=1,2} \left( a^\dagger \left( \frac{\hat{\phi}_j^\mu \Psi(\cdot, x)}{\sqrt{\omega}} \right) + a \left( \frac{\hat{\phi}_j^\mu \Psi(\cdot, x)}{\sqrt{\omega}} \right) \right), \quad \omega(k) = |k|,
\end{equation}
and
\begin{equation}
\hat{H}_f(m) := d \Gamma(\omega_m), \quad \omega_m(k) := \sqrt{k^2 + m^2}.
\end{equation}
We introduce following assumptions on $V$:

**Assumption 3.**

1. $V$ is a measurable function and for almost every $x \in \mathbb{R}^3$, $-\infty < V(x) < \infty$;
2. For all $\epsilon > 0$, there exists a positive constant $C_\epsilon$ such that for $\Psi \in D(|p|),$
\begin{equation}
\| V^{1/2} \Psi \|^2 \leq \epsilon \| p |\Psi| \|^2 + C_\epsilon \| \Psi \|^2;
\end{equation}
3. $Q(\hat{q}_m^V)$ is dense.

**Proposition 2.4.** Suppose Assumptions 1, 2 and 3. Then there exists the unique self-adjoint operator $\hat{H}_m^V$ such that $Q(\hat{q}_m^V) = D(|\hat{H}_m^V|^{1/2})$ and for all $\Psi$ and $\Phi \in Q(\hat{q}_m^V)$,
\begin{equation}
\hat{q}_m^V(\Psi, \Phi) - E^V(m)(\Psi, \Phi) = \left( (\hat{H}_m^V - E^V(m))^{1/2} \Psi, (\hat{H}_m^V - E^V(m))^{1/2} \Phi \right).
\end{equation}
Here we denote the ground state energy of $\hat{q}_m^V$ by
\begin{equation}
E^V(m) := \inf_{\Psi \in Q(\hat{q}_m^V), \| \Psi \|=1} \hat{q}_m^V(\Psi, \Psi).
\end{equation}

Formally, the Pauli-Fierz Hamiltonian $H_m^V$ is given by
\begin{equation}
H_m^V := \frac{1}{2} \sum_{\mu, \nu} (p_\mu + \sqrt{\alpha} A_\mu) a_{\mu \nu} (p_\nu + \sqrt{\alpha} A_\nu) + H_f(m) + V.
\end{equation}
Here $\{a_{\mu, \nu}\}_{\mu, \nu=1,2,3} = \{a_{\mu, \nu}(x)\}_{\mu, \nu=1,2,3}$ is positive definite. We consider only the case of $a_{\mu, \nu}(x) = \delta_{\mu, \nu}$ for simplicity.

§ 3. Binding condition

We introduce functions $\phi_R$ and $\tilde{\phi}_R$ below. Let $\phi \in C^\infty(\mathbb{R}^3)$ be such that for all $x \in \mathbb{R}^3$, $0 \leq \phi(x) \leq 1$ and
\begin{equation}
\phi(x) = \begin{cases} 
1 & \text{if } |x| < 1, \\
0 & \text{if } |x| > 2.
\end{cases}
\end{equation}
Let $\tilde{\phi} \in C^\infty(\mathbb{R}^3)$ be such that for all $x \in \mathbb{R}^3$, $0 \leq \tilde{\phi}(x) \leq 1$ and

$$\phi(x)^2 + \tilde{\phi}(x)^2 = 1.$$ 

We set for $R > 0$,

(3.1) \hspace{1cm} \phi_R(x) := \phi(x/R), \quad \tilde{\phi}_R(x) := \phi(x/R).

Let

(3.2) \hspace{1cm} E^V(R, m) = \inf_{\|\phi_R \Psi\| = 1, \Psi \in D(H_m^V)} (\phi_R \Psi, \hat{H}_m^V \tilde{\phi}_R \Psi).

$$\lim_{R \to \infty} E^V(R, m) - E^V(m)$$ formally describes ionization energy by definition, it is expected that positive ionization energy yields ground state.

**Assumption 4** (Binding condition).

(3.3) \hspace{1cm} E^V(m) < \lim_{R \to \infty} E^V(R, m).

§ 4. Massive case

The existence of ground states in the case of $m > 0$ is considered in this section.

**Theorem 4.1.** Let $m > 0$. Suppose Assumptions 1-4. Then ground states of $\hat{H}_m^V$ exist for all values of a coupling constant.

**Outline of Proof.** Let $\{\Psi^j\}_j \subset Q(\hat{q}_m^V)$ be a sequence such that weakly converges to 0. It suffices to show that

(4.1) \hspace{1cm} \lim_{j \to \infty} \inf \tilde{q}_m^V(\Psi^j, \Psi^j) > E^V(m).

We can suppose that $\sup_j \tilde{q}_m^V(\Psi^j, \Psi^j) < \infty$. Let $\phi_R$ and $\tilde{\phi}_R$ be in (3.1).

$$\tilde{q}_m^V(\Psi^j, \Psi^j) = \tilde{q}_m^V(\Psi^j_R, \Psi^j_R) + \tilde{q}_m^V(\tilde{\Psi}^j_R, \tilde{\Psi}^j_R)$$

(4.2) \hspace{1cm} -\frac{1}{2} \|(|\nabla \phi_R| \otimes 1)\Psi^j\|^2 - \frac{1}{2} \|(|\nabla \tilde{\phi}_R| \otimes 1)\Psi^j\|^2.

holds. Here $\Psi^j_R = \phi_R \Psi^j$ and $\tilde{\Psi}^j_R = \tilde{\phi}_R \Psi^j$. Let $j_1$ and $j_2$ be nonnegative, smooth functions on $\mathbb{R}^3$ such that

(4.3) \hspace{1cm} j_1(k) = \begin{cases} 1 \text{ if } |k| < 1, \\ 0 \text{ if } |k| > 2 \end{cases} \quad \text{and} \quad j_1(k)^2 + j_2(k)^2 = 1.
We set \( \hat{j}_{1,P} = j_{1}(-i\nabla_{k}/P) \), \( l = 1, 2 \), and

\[
\hat{j}_{P}\Psi = \hat{j}_{1,P}\Psi \oplus \hat{j}_{2,P}\Psi,
\]

for \( \Psi \in L^{2}(\mathbb{R}^{3};\mathbb{C}^{2}) \). Let us define the isometric operator from \( \mathcal{F} \) to \( \mathcal{F} \otimes \mathcal{F} \) by

\[
d\tilde{\Gamma}(\hat{j}_{P})a^{\dagger}(h_{1})\cdots a^{\dagger}(h_{n})\Omega
\]

\[
= a^{\dagger}(\hat{j}_{1,P}h_{1})\cdots a^{\dagger}(\hat{j}_{1,P}h_{n})\Omega \oplus a^{\dagger}(\hat{j}_{2,P}h_{1})\cdots a^{\dagger}(\hat{j}_{2,P}h_{n})\Omega.
\]

By the localization argument (see [8]), it holds that

\[
\lim_{j\to\infty}\inf_{\infty}\hat{q}_{m}^{V}(\Psi_{R}^{j}, \Psi_{R}^{j}) \geq (E^{V}(m) + m)\lim_{j\to\infty}\inf_{\infty}\Vert\Psi_{R}^{j}\Vert^{2} + o_{R}(P^{0})
\]

(4.6)

and

\[
\hat{q}_{m}^{V}(\Psi_{R}^{j}, \Psi_{R}^{j}) \geq E_{R,m}^{V}\Vert\Psi_{R}^{j}\Vert^{2} + o(R^{0}).
\]

Here \( o_{R}(P^{0}) \) goes to zero as \( P \to \infty \) for each fixed \( R > 0 \). By (4.2), (4.6) and (4.7), we can see that

\[
\lim_{j\to\infty}\inf_{\infty}\hat{q}_{m}^{V}(\Psi^{j}, \Psi^{j}) \geq E^{V}(m) + \min\{m, E^{V}(R, m) - E^{V}(m)\}.
\]

(4.8)

By the binding condition, we obtain (4.1). ☐

§ 5. The case of \( m = 0 \)

Throughout in this section, we suppose Assumptions 1, 2, 3 and Assumption 4 with \( m = 0 \). \( \Phi_{m} \) denotes the normalized ground state of \( \hat{H}_{m}^{V} \). Similarly to the case of \( v = 0 \), the following lemma holds.

**Lemma 5.1.** Let \( \{m_{j}\}_{j=1}^{\infty} \) be a sequence converging to 0. Then

\[
\lim_{j\to\infty}E^{V}(m_{j}) = E^{V}(0)
\]

and for sufficiently small \( 0 < m \), the binding condition holds.

The pull through formula below leads to a photon number bound (Lemma 5.3 and Corollary 5.4) and a photon derivative bound (Lemma 5.6).

**Lemma 5.2** (Pull through formula). Let \( f \in D(\omega_{m}) \). Then \( a(f)\Phi_{m} \in Q(\hat{q}_{m}^{V}) \) and for all \( \eta \in Q(\hat{q}_{m}^{V}) \),

\[
\hat{q}_{m}^{V}(\eta, a(f)\Phi_{m}) - E^{V}(m)(\eta, a(f)\Phi_{m})
\]

\[
= -\sqrt{\alpha}(\eta, (\bar{f}, \bar{G}) \cdot (p + \sqrt{\alpha A})\Phi_{m}) + \frac{i\sqrt{\alpha}}{2}(\eta, (\bar{f}, \nabla_{x} \cdot \bar{G})\Phi_{m}) - (\eta, a(\omega_{m}f)\Phi_{m}).
\]
holds. Here
\[ G_{j}^{\mu}(k, x) := \frac{\hat{\Phi}_{j}^{\mu}(k)\Psi(k, x)}{\sqrt{2\omega(k)}}. \]

Lemma 5.3. Let \( \theta = (\theta_{1}, \theta_{2}) \in L^{\infty}(\mathbb{R}^{3}; \mathbb{R}^{2}) \). Then
\[
\| d\Gamma(\theta^{2})^{1/2} \Phi_{m} \|^2 \leq C\alpha \sum_{\mu,j} \int \frac{\hat{\Phi}_{j}^{\mu}(k)^2 \theta_{j}(k)^2}{\omega(k)\omega_{m}(k)^2} dk,
\]
where \( C \) is a constant independent of \( \alpha \) and sufficiently small \( m \).

Outline of proof of Lemma 5.3. Inserting \( \eta = a(f)\Phi_{m} \) into (5.2), we have
\[
(a(f)\Phi_{m}, a(\omega_{m}f)\Phi_{m}) \leq -\sqrt{\alpha} \left(a(f)\Phi_{m}, (\overline{f}, \overline{G}) \cdot (p + \sqrt{\alpha}A) \Phi_{m}\right)
\]
\[ + \frac{\sqrt{\alpha}}{2} (a(f)\Phi_{m}, (\overline{f}, \nabla_{x} \cdot \overline{G}) \Phi_{m}). \]

Let \( f := \omega_{m}\theta g_{i} \). Here \( \{g_{i}\}_{i=1}^{\infty} \) is a complete orthonormal system such that each \( g_{i} \in D(\omega_{m}^{1/2}) \). Note that
\[
\sum_{i=1}^{\infty} \left(a(\omega_{m}^{-1/2} \theta g_{i})\Phi_{m}, a(\omega_{m}^{1/2} \theta g_{i})\Phi_{m}\right)
\]
\[ = \sum_{j=1,2} \int_{\mathbb{R}^{3}} \theta_{j}(k)^2 \| a_{j}(k)\Phi_{m} \|^2 dk = \| d\Gamma(\theta^{2})^{1/2} \Phi_{m} \|^2. \]

Then by (5.3) and (5.4),
\[
\| d\Gamma(\theta^{2})^{1/2} \Phi_{m} \|^2 \leq 2\alpha \int_{\mathbb{R}^{3}} \omega_{m}(k)^{-2} \| \theta(k)G(k) \cdot (p + \sqrt{\alpha}A) \Phi_{m} \|^2 dk
\]
\[ + \frac{\alpha}{2} \int_{\mathbb{R}^{3}} \omega_{m}(k)^{-2} \| \theta(k)\nabla_{x} \cdot G(k) \Phi_{m} \|^2 dk. \]

can be estimated. Since \( \Psi(k, x) \) and \( \hat{\varphi}(k)\frac{\partial}{\partial x_{\mu}}\Psi(k, x) \) are bounded in \( k \) and \( x \), we can see that the lemma follows. \( \square \)

From Lemma 5.3, we can see that following facts hold.

Corollary 5.4. It holds that
\[
(1) \sup_{m < m_{0}} \| N^{1/2} \Phi_{m} \| < \infty,
\]
\[
(2) \text{supp} \Phi_{m}^{(n)}(x, \cdot) \subset \Pi_{k=1}^{n} \left[ \bigcup_{j, \mu} \text{supp} \hat{\Phi}_{j}^{\mu} \right].
\]

We can show the spatial exponentially decay of \( \Phi_{m} \) for many external potentials. See [13].
Assumption 5.

1) For sufficiently large $|x|$, $V(x) > \text{const.}|x|^{2n}$.

2) $\liminf_{|x| \to \infty} V(x) > \inf \sigma(H_p)$ and for all $t > 0$, $e^{-tH_p} : L^2 \to L^\infty$ with

$$\|e^{-tH_p}f\|_{L^\infty(\mathbb{R}^3)} \leq \text{const.}\|f\|_{L^2(\mathbb{R}^3)};$$

where $H_p = -\frac{1}{2}\Delta + V$.

Theorem 5.5. Suppose Assumption 5. Then for some $c$ and $m_0 > 0$,

$$(5.6) \sup_{0 < m < m_0} \|\exp(c|x|)\Phi_m\| < \infty.$$ holds.

Outline of Proof. Since $\Phi_m = e^{tE}e^{-t\hat{H}^V_m}\Phi_m$, by the functional integral representation of $e^{-t\hat{H}^V_m}$, we can see that for all $t \geq 0$,

$$\|\Phi_m(x)\| \leq Ce^{tE^V(m)}E^{x}[e^{-\int_{0}^{t}V(B_s)ds}]$$

holds. Here $(B_t)_{t \geq 0}$ denotes Brownian motion starting from $x$. $C$ is a constant independent of $x$ and $m$.

$$(5.7) e^{t(x)E^V(m)}E^{x}[e^{-\int_{0}^{t(x)}V(B_s)ds}] \leq C_1 \exp(-C_2|x|^{n+1})$$

and

$$(5.8) e^{t'(x)E^V(m)}E^{x}[e^{-\int_{0}^{t'(x)}V(B_s)ds}] \leq C'_1 \exp(-C'_2|x|)$$

hold. Here $t(x) = |x|^{1-n}$, $t'(x) = \beta|x|$. (5.8) and (5.9) are called Carmona’s estimate [3]. By (5.7), (5.8) and (5.9), the theorem can be proven.

Lemma 5.6. Suppose Assumption 5. Let $1 \leq p < 2$. Then

(a) $\Phi_m^{(n)} \in H^1(\mathbb{R}^{3+3n})$ for all $n \geq 0$;

(b) $\{\|\Phi_m^{(n)}\|_{W^{1,p}(\Omega)}\}_{0 < m \leq m_0}$ is bounded, where $m_0$ is sufficiently small number and $\Omega$ is any measurable and bounded set in $\mathbb{R}^{3+3n}$.

Here $W^{1,p}(\Omega)$ is the Sobolev space.

Outline of proof. Let $f = \omega_m^{-1/2}\theta g_i$. By the pull through formula with $f(x)$ replaced by $f(x+h) - f(x)$, similarly to the proof of Lemma 5.3, we can see that for
almost every \( k \) and sufficiently small \( h \),

\[
(5.10) \| |h|^{-1}(a(k + h) - a(k))\Phi_m \|^2 \\
\leq \text{const.} \frac{\omega_m(k)^2}{\omega_m(k)^2} \left( \sum_{\mu=1}^{3} \left( \| |h|^{-1}(\delta_h G_{\mu})(k)\Phi_m \|^2 + \| |h|^{-1}(\nabla_x \delta_h G_{\mu})(k)\Phi_m \|^2 \right) \\
+ \| |h|^{-1}(\nabla_x \cdot \delta_h G)(k)\Phi_m \|^2 + \sum_{j, \mu} \frac{\hat{\varphi}^{(n)}_j(k + h)^2}{\omega_m(k + h)^2 \omega(k + h)} \frac{|\omega(k + h) - \omega(k)|^2}{|h|^2} \right).
\]

Here \((\delta_h G_{\mu})(k) = G_{\mu}(k + h, x) - G_{\mu}(k, x)\). By Lemma 2.1 (c) and Assumption 5,

\[
(5.11) \| |h|^{-1}(a(k + h) - a(k))\Phi_m \|^2 \\
\leq C \omega_m(k)^{-2} \sum_{\nu, j} \left( (1 + |k|^{-3})\hat{\varphi}^{(n)}_j(k)^2 + |k|^{-1} \sum_{\lambda} |\partial_{\lambda}\hat{\varphi}^{(n)}_j(k)|^2 \right)
\]

holds for almost every \( k \) and sufficiently small \(|h|\). Let \( e_1 = (1, 0, 0) \), \( e_2 = (0, 1, 0) \), \( e_3 = (0, 0, 1) \). Thus by Alaoglu theorem, for almost every \( k \), there exists the sequence \( \{h_l(k)\}_{l=1}^{\infty} \) depending on \( k \) so that

\[
\lim_{l \to \infty} h_l(k) = 0
\]

and \( |h_l(k)|^{-1}(a(k - |h_l(k)|e_\mu) - a(k))\Phi_m \) weakly converges to some vector \( v_\mu(k) \):

\[
v_\mu(k) := \text{w- lim}_{l \to \infty} |h_l(k)|^{-1}(a(k - |h_l(k)|e_\mu) - a(k))\Phi_m.
\]

It can be proven that \( v^{(n)}_\mu(k)(x, k_1, \cdots, k_n) \) is the weak derivative \( \Phi^{(n+1)}_m(x, k, k_1, \cdots, k_n) \) with respect to \( k_\mu \). Thus by (5.11), (a) and (b) are proven directly.

**Theorem 5.7.** Let \( m = 0 \). Suppose Assumption 5. Then ground states of \( \hat{H}^V \) exist for all values of a coupling constant.

By Lemmas 5.3, 5.6 and Theorem 5.5, Theorem 5.7 can be proven similarly to [8, Theorem2.1].

**§ 6. Remarks on infrared cutoffs**

We assumed the infrared regularity condition, but in the case of \( v = 0 \), we can show the existence of ground states of \( \hat{H} \) without the infrared regularity condition. In the case of \( v \neq 0 \),

\[
(6.1) \Psi(k, x) - e^{ikx} = \sum_{n=1}^{\infty} \left( \frac{1}{4\pi} \right)^n \int_{\mathbb{R}^3} e^{ik|\Sigma_{j=1}^{n} |y_{j-1} - y_j|} v(y_j) dy_1 \cdots dy_n
\]
and

\[
\nabla_x \Psi(k, x) - ike^{ikx} = \frac{1}{4\pi} \int_{\mathbb{R}^3} \left( \frac{e^{i|k||x-y|}(x-y)}{|x-y|^3} - \frac{i|k|e^{i|k||x-y|}(x-y)}{|x-y|} \right) v(y) \Psi(k, y) \, dy
\]

hold. Here \( y_0 := x \). The right hand side of (6.1) is not \( O(|k|) \), \( k \to 0 \). This is the reason that we assumed the infrared regularity condition. To see this, let us consider the case of \( v = 0 \). Set \( v = 0 \) and \( \hat{\varphi}_j^\mu(k) = \chi_\Lambda(k)e_j^\mu(k) \), \( j = 1, 2, \mu = 1, 2, 3 \), where \( \chi_\Lambda \) is the characteristic function of the set \( \{ k \mid |k| < \Lambda \} \) and \( e_1(k) \) and \( e_2(k) \), \( k \in \mathbb{R}^3 \setminus \{0\} \) are polarization vectors given by

(6.2) \[ e_1(k) := \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}} \quad \text{and} \quad e_2(k) := \frac{k \times e_1(k)}{|k|}. \]

Note that the infrared regularity condition is not assumed in this case. Define the unitary operator \( U \) as

(6.3) \[ U := \exp[i\sqrt{\alpha}x \cdot \hat{A}(0)]. \]

Put

(6.4) \[ \hat{q}_m^V(\Psi, \Phi) := \hat{q}_m^V(U\Psi, U\Phi) \]

and

(6.5) \[ \tilde{A}(x) := \hat{A}(x) - \hat{A}(0). \]

Then

(6.6) \[ \hat{q}_m^V(\Psi, a(f)\Phi_m) - E_v^V(m)(\Psi, a(f)\Phi_m) = -\sqrt{\alpha}(\Psi, (\overline{f}, \overline{G})(p + \sqrt{\alpha}\tilde{A})\Phi_m) - (\Psi, a(\omega_m f)\Phi_m) + i(\Psi, (\overline{f}, \omega_m \omega)\Phi_m) \]

follows. Here \( \Phi_m = U\Phi_m \), \( w_j := \frac{\chi_\Lambda(k)e_j(k) \cdot x}{\sqrt{\omega(k)}} \) and \( \tilde{G}_j^\mu := \frac{\chi_\Lambda(k)e_j^\mu(k)(e^{ikx} - 1)}{\sqrt{2\omega(k)}} \). Similarly to the proof of Theorem 5.3, we have

(6.7) \[ \| a(k)\Phi_m \|^2 \leq \text{const}. \omega(k)^{-2} \left\{ \| \tilde{G}\Phi_m \|^2 + \| \nabla_x \tilde{G}\Phi_m \|^2 + \omega(k)^2 \| w\Phi_m \|^2 \right\} \chi_\Lambda(k). \]

Since \( |e^{ikx} - 1| \leq |k||x| \) and \( |\nabla_x e^{ikx}| = |k| \), by the exponential decay of \( \Phi_m \), it holds that

(6.8) \[ \| d\Gamma(\theta^2)^{1/2}\Phi_m \|^2 \leq C\alpha \sum_j \int \frac{\chi_\Lambda(k)\theta_j(k)^2}{\omega(k)} \, dk. \]
Here C is a constant independent of $\alpha$ and $m$ for sufficiently small $m$. Also by (6.6), for almost every $k$ and sufficiently small $h$,

\begin{equation}
(6.9) \quad \| (a(k+h) - a(k)) \tilde{\Phi}_m \|^2 \\
\leq \text{const.} \left\{ \| \delta_h \tilde{C} \tilde{\Phi}_m \|^2 + \| \nabla_x \delta_h \tilde{C} \tilde{\Phi}_m \|^2 + \omega(k)^2 \| \delta_h (\omega w) \tilde{\Phi}_m \|^2 \\
+ \frac{1}{|k+h|} \| x \| \tilde{\Phi}_m \|^2 \omega(k+h) - \omega(k) \chi_{\Lambda}(k+h) \right\}
\end{equation}

can be proven. Since $|e^{ikx} - 1| \leq |k||x|$ and $|\nabla_x e^{ikx}| = |k|$, by Assumption 5, we can see that

\begin{equation}
(6.10) \quad |h|^{-1} \| (a(k+h) - a(k)) \tilde{\Phi}_m \|^2 \\
\leq \text{const.} \left\{ \frac{1}{|k|(|k_1|^2 + |k_2|^2)} + \frac{1}{|k-h|((k_1-h_1)^2 + (k_2-h_2)^2)} \right\}
\end{equation}

holds. This inequality implies that $\{\| \tilde{\Phi}^{(n)}_m \|_{W^{1,p}(\Omega)} \}_{0 < m \leq m_0}$ with $1 \leq p < 2$ is bounded, where $\Omega$ is a bounded set and $m_0$ a sufficiently small number. Therefore in this case, the existence of ground states can be proven without the infrared regularity condition. Key inequalities are

\begin{equation}
(6.11) \quad |e^{ikx} - 1| \leq |k||x|
\end{equation}

and

\begin{equation}
(6.12) \quad |\nabla_x e^{ikx}| = |k|.
\end{equation}

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**References**


