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# Existence and absence of ground states for a particle interacting through the quantized scalar field on a static spacetime

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**Abstract**

The Nelson model describes a quantum particle interacting with a scalar Bose field on the Minkowski spacetime. It is extended to models on a static spacetime. In a series of papers [2, 3, 4] we prove the existence and absence of ground states of these extended models. In this paper, we illustrate the results obtained in [3, 4] only for a special case where the boson has a variable mass. If the variable mass decays sufficiently slowly, the ground state exists, but if it decays sufficiently fast, it does not exist. Furthermore we prove the absence of the ground state of the model studied in [2] under a weaker condition on a variable mass than that of [2].

## § 1. Introduction

We consider a confined quantum particle interacting with a scalar Bose field, whose Hamiltonian is given by

$$H = K \otimes I + I \otimes d\Gamma(\omega) + \phi(\omega^{-1/2}\rho_X). \tag{1.1}$$

Here the particle is governed by the Schrödinger operator:

$$K = -\Delta_X + V(X) \quad \text{acting on } \mathcal{K} = L^2(\mathbb{R}^3; dX) \tag{1.2}$$

with a confining potential $V : \mathbb{R}^3 \to \mathbb{R}$ such that $K$ has a compact resolvent, while one boson Hamiltonian $\omega$ is given by

$$\omega = h^{1/2},$$

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where

$$h = -\Delta_x + v(x) \quad \text{acting on } \mathfrak{h} = L^2(\mathbb{R}^3; dx) \tag{1.3}$$

with a function $v$ such that $h$ is a positive operator. The constant

$$m_{\text{rest}} := \inf \sigma(\omega)$$

is viewed as the rest mass of a boson. We say that $H$ is *massless* (resp. *massive*) if $m_{\text{rest}} = 0$ (resp. $m_{\text{rest}} > 0$). The second quantization of $\omega$, $d\Gamma(\omega)$, which acts on the boson Fock space $\Gamma(\mathfrak{h})$ over $\mathfrak{h}$, is the free Hamiltonian of the scalar Bose field. The Segal field operator $\phi(f)$ ($f \in \mathfrak{h}$) is given by $\phi(f) = (a(f)+a^*(f))/\sqrt{2}$, where the annihilation operator $a(f)$ and the creation operator $a^*(f)$ satisfy canonical commutation relations: $[a(f), a^*(g)] = \langle f, g\rangle_{\mathfrak{h}}$ and $[a^\sharp(f), a^\sharp(g)] = 0$ ($a^\sharp = a$ or $a^*$). $\rho : \mathbb{R}^3 \to [0, \infty)$ is the charge distribution and introduces an ultraviolet cutoff. We assume that $\rho \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ ($\rho \not\equiv 0$). We set $\rho_X(x) = \rho(x - X)$ and $g := \int \rho(x)dx > 0$. Here $X$ is the position of the quantum particle and the charge $g$ describes the strength of the interaction.

## § 1.1. The standard Nelson model

If $v$ is a constant: $v(x) \equiv m_{\mathrm{b}}^2$ ($m_{\mathrm{b}} \geq 0$), then $H$ becomes the Hamiltonian of the standard Nelson model on the Minkowski spacetime. Let $\mathscr{F}$ be the Fourier transform on $L^2(\mathbb{R}^3)$ and $\hat{\rho}(k) = (\mathscr{F}\rho)(k)$. Then $H$ is unitarily transformed to

$$H_{\text{Nelson}} = K \otimes I + I \otimes d\Gamma(w_{m_{\mathrm{b}}}) + \phi\left(\overline{\psi_0(\cdot, X)} w_{m_{\mathrm{b}}}^{-1/2} \hat{\rho}\right), \tag{1.4}$$

by the second quantization of $\mathscr{F}$, $\Gamma(\mathscr{F})$, where $\psi_0(k, X) = e^{ik\cdot X}$ is the plane wave. Here $w_{m_{\mathrm{b}}}(k) = \sqrt{|k|^2 + m_{\mathrm{b}}^2}$ is the dispersion relation of the boson with momentum $k \in \mathbb{R}^3$ and the mass $m_{\mathrm{b}} \geq 0$. We observe that $m_{\text{rest}} = m_{\mathrm{b}}$. Assumption:

$$\int dk (w_{m_{\mathrm{b}}}(k)^{-1} + w_{m_{\mathrm{b}}}(k)^{-2}) |\hat{\rho}(k)|^2 < \infty, \quad j = 1, 2, \tag{1.5}$$

yields that $H_{\text{Nelson}}$ is a (well-defined) self-adjoint operator bounded below. In [9], Nelson proved that the ultraviolet cutoff can be removed at the expense of infinite energy renormalization, i.e., $H_{\text{Nelson}}$ can be defined in the point charge limit $\rho(x) \to g\delta(x)$ by subtracting an infinite energy.

Assuming the ultraviolet cutoff, Lőrinczi, Minlos and Spohn [6] studied the infrared behavior. This is of interest only for massless case $m_{\text{rest}} = m_{\mathrm{b}} = 0$, since the strictly positive mass $m_{\text{rest}} = m_{\mathrm{b}} > 0$ plays the role of an infrared cutoff, i.e.,

$$\int_{\mathbb{R}^3} \frac{|\hat{\rho}(k)|^2}{w_{m_{\mathrm{b}}}(k)^3} dk < \infty.$$

They showed that the massless Nelson model is infrared divergent, i.e., $H_{\mathrm{Nelson}}$ has no ground state, which is done from

$$\int_{\mathbb{R}^3} \frac{|\hat{\rho}(k)|^2}{w_0(k)^3} dk = \infty. \tag{1.6}$$

Note that $\hat{\rho}(0) = (2\pi)^{-3/2} g > 0$ and $w_0(k) = |k|$. Then (1.5) and (1.6) imply that

(a) $\rho_X \in D(w_0^{-j/2})$ and $\sup_{X \in \mathbb{R}^3} \|w_0^{-j/2} \hat{\rho}_X\| < \infty$ for $j = 1, 2$;

(b) $\rho_X \notin D(w_0^{-3/2})$.

The condition (a) ensures the self-adjointness of $H_{\mathrm{Nelson}}$ and the condition (b) is equal to (1.6).

## § 1.2. Extensions of the Nelson model on a static spacetime

In [3, 4], the ultraviolet and infrared behavior of a general class of Hamiltonians,

$$H_{\mathrm{G}} = K_{\mathrm{G}} \otimes I + I \otimes d\Gamma(\omega_{\mathrm{G}}) + \phi(\omega_{\mathrm{G}}^{-1/2} \rho_X) \tag{1.7}$$

is studied, where both $K_{\mathrm{G}}$ and $h_{\mathrm{G}} = \omega_{\mathrm{G}}^2$ are the second order elliptic operators. A typical example of (1.7) is a Hamiltonian of a quantum particle interacting with a scalar Bose field on a *static spacetime*. In this case, the function $v$ is determined by the mass of the boson and the metric of a given static spacetime (see [1, 2, 3] for details).

The Hamiltonian

$$H = K \otimes I + I \otimes d\Gamma(\omega) + \phi(\omega^{-1/2} \rho_X) \tag{1.8}$$

defined in (1.1) is included as a special case of $H_{\mathrm{G}}$. In this paper, we review the results obtained in [3, 4] for a simple version (1.8). Recall that $\omega = (-\Delta_x + v(x))^{1/2}$. We call the function

$$m(x) = |v(x)|^{1/2}$$

the *variable mass*. In this paper, imposing an ultraviolet cutoff, we treat only the infrared behavior of $H$. We also assume that the function $v$ satisfies $0 \le v \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ and decays at infinity: $\lim_{|x| \to \infty} v(x) = 0$. Then $v$ is relatively compact with respect to $h_0 = -\Delta_x$ and the essential spectrum of $h_0$ is invariant under the perturbation $v$. We also see that $\omega$ has no zero eigenvalue and that the spectrum of the one-boson Hamiltonian $\omega$ is

$$\sigma(\omega) = [0, \infty), \quad \sigma_{\mathrm{p}}(\omega) \not\ni 0.$$

In particular, although the boson has a variable mass $m(x)$, $m_{\mathrm{rest}} = 0$ and hence $H$ is massless. We clarify whether the infrared divergence occurs or not, i.e., whether $H$ has a ground state or not. See Lemmas 2.1 and 3.1 below. For sufficiently fast decaying $m(x)$, the one-boson Hamiltonian $\omega$ satisfies the same condition as (b) above. This derives the infrared divergence. On the other hand, for sufficiently slowly decaying $m(x)$, $\omega$ has a nice property such that $\rho_X \in D(\omega^{-3/2})$ and $\|\omega^{-3/2}\rho_X\|_{\mathfrak{h}}$ grows at most polynomially in $X$. This yields the existence of the ground state of $H$. Indeed, we have the results below (see Theorems 2.2 and 3.2):

1. If $m(x)$ decays like $O(|x|^{-1-\epsilon})$ ($\epsilon > 0$), then $H$ has no ground state;
2. If $m(x)$ decays like $O(|x|^{-1+\epsilon})$ ($0 \le \epsilon < 1$), then $H$ has a ground state.

## § 1.3. Alternate extension

There is an alternate extension of the standard Nelson model onto a static spacetime. We define the Hamiltonian $\tilde{H}$ by

$$\tilde{H} = K \otimes I + I \otimes d\Gamma(w_0) + \phi\left(\overline{\psi(\cdot, X)}w_0^{-1/2}\chi\right), \tag{1.9}$$

where $\psi(k, x)$ is a distorted plane wave that is a solution to the Schrödinger equation:

$$(-\Delta_x + v(x))\psi(k, x) = |k|^2\psi(k, x),$$

and $\chi$ is a smooth function. If $v \equiv 0$ and $\chi = \hat{\rho}$, then $\tilde{H}$ becomes $H_{\text{Nelson}}$ and hence is unitarily transformed to $H$ by the Fourier transform. Under suitable conditions on $v$ decaying at infinity one can define the generalized Fourier transform

$$\mathscr{F}_\sharp : f \mapsto (2\pi)^{-3/2} \int dx \overline{\psi(\cdot, x)} f(x)$$

such that $\mathscr{F}_\sharp$ is unitary and $\mathscr{F}_\sharp \omega \mathscr{F}_\sharp^{-1} = w_0$. It is noticed that $\tilde{H}$ can not be unitarily transformed to $H$ by $\mathscr{F}_\sharp$ in general. Indeed, the generalized Fourier transform of $\omega^{-1/2}\rho_X$ is equal to $w_0^{-1/2}\mathscr{F}_\sharp\rho_X$ but

$$w_0^{-1/2}\mathscr{F}_\sharp\rho_X \neq \overline{\psi(\cdot, X)}w_0^{-1/2}\mathscr{F}_\sharp\rho.$$

We are also interested in the infrared behavior of $\tilde{H}$. Note that, in the standard Nelson model, the infrared divergence comes from (1.6). In [2] we assume that $\chi(0) > 0$ and prove that $\tilde{H}$ has no ground state if $m(x) = O(|x|^{-3/2-\epsilon})$ with $\epsilon > 0$ and $m(x)$ is sufficiently shallow. In this paper we weaken this condition. From (ii) above we deduce that $\tilde{H}$ has no ground if $m(x)$ decays like $O(|x|^{-1-\epsilon})$ ($\epsilon > 0$). We prove it for sufficiently shallow $v$.

This paper is organized as follows: In Section 2, we show that $H$ has no ground state for $m(x) = O(|x|^{-1-\epsilon})$ ($\epsilon > 0$). Section 3 is devoted to proving that $H$ has a ground state if $m(x)$ decays like $O(|x|^{-1+\epsilon})$ $(0 \le \epsilon < 1)$. Finally in Section 4 we prove that $\tilde{H}$ has no ground state for $m(x) = O(|x|^{-1-\epsilon})$ $(\epsilon > 0)$.

## § 2. Absence of the ground state of $H$

For the particle Hamiltonian $K$ we suppose the following:

**(K)** $V \in L^2_{\rm loc}(\mathbb{R}^3)$ and $V(X) \ge c_0|X|^{2q} - c_1$ with some $c_0 > 0$, $c_1 \ge 0$ and $q > 0$.

Then $K = -\Delta_X + V(X)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$ with a compact resolvent and hence $K$ has a ground state $\varphi_K$: $K\varphi_K = E_K\varphi_K$ with $E_K := \inf\sigma(K)$. Moreover the ground state $\varphi_K$ is unique, i.e., $\dim\ker(K - E_K) = 1$, and strictly positive $\varphi_K(x) > 0$.

Let $\langle x\rangle = \sqrt{1+|x|^2}$. We also introduce the following condition for the one-boson Hamiltonian $\omega = h^{1/2}$:

**(QD$_1$)** $0 \le v \in L^2_{\rm loc}(\mathbb{R}^3)$ and

$$m(x) \le \frac{a}{\langle x\rangle^{1+\epsilon}} \quad \text{for } |x| \ge R$$

with some $R \ge 0$, $a > 0$ and $\epsilon > 0$.

Under the condition (QD$_1$) we observe that $h = -\Delta_x + v(x)$ is a positive self-adjoint operator on $D(-\Delta_x)$ with the spectrum $\sigma(\omega) = [0,\infty)$. Moreover $h$ has no zero eigenvalue. Indeed, if 0 is assumed to be an eigenvalue with an eigenvector $\varphi \in \mathfrak{h}$, then $0 \ge -\langle\varphi, v\varphi\rangle_{\mathfrak{h}} = \langle\varphi,(-\Delta_x)\varphi\rangle_{\mathfrak{h}} \ge 0$, which implies that $\varphi = 0$. In particular, $\omega^{-1}$ is an unbounded operator.

For the charge distribution $\rho$ we assume that

**(IR$_1$)** $0 \le \rho \in C_0^\infty(\mathbb{R}^3)$.

**Lemma 2.1.** *Assume (QD$_1$) and (IR$_1$). Then*

*(a) for $j = 1, 2$, we have $\rho_X \in D(\omega^{-j/2})$ and*

$$\sup_{X\in\mathbb{R}^3} \|\omega^{-j/2}\rho_X\|_{\mathfrak{h}} < \infty;$$

*(b) $\rho_X \notin D(\omega^{-3/2})$.*

*Proof.* By the Laplace transform we observe that for $j = 1, 2, 3$,

$$\omega^{-j/2} = \gamma_j \int_0^\infty e^{-t\omega^2} t^{(j-4)/4} dt \tag{2.1}$$

with some $\gamma_j > 0$. For the integral kernel $e^{-th}(x, y)$ of the Schrödinger semigroup $e^{-th}$, it holds that

$$Ce^{-t(-\Delta_x)}(x, y) \le e^{-th}(x, y) \le e^{-t(-\Delta_x)}(x, y) \tag{2.2}$$

with some $C > 0$, where $e^{-t(-\Delta_x)}(x, y)$ is the integral kernel of the semigroup $e^{-t(-\Delta_x)}$. The right hand side of (2.2) is derived from the positivity of $v$, and the left hand side from the condition ($\mathrm{QD}_1$) (see [12]). Combining (2.1) and (2.2) and using the condition ($\mathrm{IR}_1$), we obtain the desired result. □

For a self-adjoint operator $L$ bounded below, we say that $L$ has a ground state if $E_0 := \inf \sigma(L) > -\infty$ is an eigenvalue of $L$ and that $L$ has no ground state if $E_0$ is not an eigenvalue of $L$.

**Theorem 2.2.** *Suppose (K), ($QD_1$) and ($IR_1$). Then*

*(i) $H$ is a self-adjoint operator bounded below;*

*(ii) $H$ has no ground state.*

*Proof.* In a similar way to the standard Nelson model, (i) is proven by (a) in Lemma 2.1. (ii) is shown in [4] by means of functional integral methods. □

## § 3. Existence of the ground state of $H$

In this section we assume (K) and show the existence of ground state. Instead of ($\mathrm{QD}_1$), we introduce the condition:

**(SD)** $v \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$, $\lim_{|x|\to\infty} v(x) = 0$ and

$$m(x) \ge \frac{a}{\langle x \rangle} \tag{3.1}$$

with some $a > 0$.

*Remark.* (1) We note that if $m(x) \ge \frac{a}{\langle x \rangle^\beta}$ with some $a > 0$ and $0 < \beta < 1$, then $m$ satisfies (3.1).

(2) Under the condition (SD), $h = -\Delta_x + v(x)$ is a positive self-adjoint operator on $D(-\Delta_x)$ with the spectrum $\sigma(\omega) = [0, \infty)$ and does not have zero eigenvalue. Hence $\omega^{-1}$ is an unbounded operator.

The following lemma gives a difference between ($\mathrm{QD}_1$) and (SD), and compare with Lemma 2.1.

**Lemma 3.1.** *Assume (SD) and ($IR_1$). Then*

*(a) for $j = 1, 2$, we have $\rho_X \in D(\omega^{-j/2})$ and*

$$\sup_{X\in\mathbb{R}^3} \|\omega^{-j/2}\rho_X\|_{\mathfrak{h}} < \infty;$$

*(b) $\rho_X \in D(\omega^{-3/2})$ and*

$$\|\omega^{-3/2}\rho_X\|_{\mathfrak{h}} \leq C\langle X\rangle^{\mu}$$

*for any $\mu > 3/2$ and some $C \geq 0$ depending only on $\mu$.*

*Proof.* We note that the upper bound in (2.2) still holds because of $v \geq 0$. Hence we can see (a) in a similar way to the proof of (a) in Lemma 2.1.

By (SD), we have

$$e^{-th}(x, y) \leq C\Phi_\alpha(x, t)\Phi_\alpha(y, t)e^{-ct(-\Delta_x)}(x, y) \tag{3.2}$$

with some constants $C$ and $c \geq 0$. Here the function $\Phi_\alpha$ is given by

$$\Phi_\alpha(x, t) = \left[\frac{\langle x\rangle^2}{t + \langle x\rangle^2}\right]^\alpha$$

with some strictly positive $\alpha > 0$. Using (2.1) and (3.2) we obtain (b). □

*Remark.* In [11], Zhang obtains a similar bound to (3.2) but $\alpha = 0$ and for sufficiently small $a > 0$. Using an abstract theorem in [8] we obtain (3.2) with strictly positive $\alpha > 0$. Moreover one can take $\alpha > 0$ sufficiently small. See [3] for details.

Let $h_\nu := h + \nu^2$ and $\omega_\nu = h_\nu^{1/2}$. We define the operator $H_\nu$ by replacing $\omega$ in (1.1) with $\omega_\nu$. Then, by a standard argument, one can show that $H_\nu$ has a normalized ground state $\Psi_\nu$. Since the unit ball in a Hilbert space is compact under the weak topology, there exists a sequence $\nu_j \to 0$ ($j \to \infty$) and a vector $\Psi_0$ such that $\Psi_{\nu_j}$ tends weakly to $\Psi_0$. To prove that $H$ has a ground state, it suffices to show that

$$\Psi_0 \neq 0. \tag{3.3}$$

(3.3) is derived from the so-called boson number bound: $\sup_\nu\langle\Psi_\nu, N\Psi_\nu\rangle < \infty$. This is shown by (b) in Lemma 3.1 and by the fact that $\|\langle X\rangle^q(K + \eta)^{-1/2}\| \leq C$ with some $\eta > 0$ and $C > 0$.

**Theorem 3.2.** *Assume (K), (SD) and ($IR_1$). Then*

*(i) $H$ is a bounded below self-adjoint operator;*

*(ii) in addition, if (K) holds for some $q > 3/2$, then $H$ has a ground state.*

*Proof.* (i) is obtained from (a) of Lemma 3.1 and (ii) is proven in [3]. □

## § 4. Absence of the ground state of $\tilde{H}$

We assume (K) throughout this section. We introduce the following condition:

**($QD_2$)** $v(x) = v_a(x)$ satisfies

$$|v_a(x)| \leq \frac{a}{\langle x \rangle^{2+\epsilon}}$$

with some $\epsilon > 0$ and $a \geq 0$.

We set $m_a(x) := \sqrt{|v_a(x)|}$ and hence $m_a(x) = O(|x|^{-1-\epsilon})$ $(\epsilon > 0)$. By [5], the condition ($QD_2$) yields that

(i) there exists a generalized eigenfunction $\psi(k, x) = \psi_a(k, x)$ satisfying

$$(-\Delta_x + v_a(x) - |k|^2)\psi_a(k, x) = 0, \tag{4.1}$$

$$\psi_a(k, x) = e^{ik\cdot x} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}v_a(y)}{|x-y|}\psi_a(k, x)dy; \tag{4.2}$$

(ii) there is no positive eigenvalues.

Moreover, by the Lieb-Thirring inequality [7], we see that

(iii) for sufficiently small $a > 0$, $\sigma(h) = [0, \infty)$ and $0 \notin \sigma_{\mathrm{p}}(h)$.

Although ($QD_2$) does not require the positivity of $v$, the condition ($QD_2$) is somewhat stronger than ($QD_1$). We have the following lemma which is the key to prove the absence of ground state of $\tilde{H}$.

**Lemma 4.1.** *Assume ($QD_2$). Then there exist positive constants $a_0 > 0$ and $C_\epsilon > 0$ such that, for any $a \leq a_0$,*

$$\sup_{x,k\in\mathbb{R}^3} |e^{ik\cdot x} - \psi_a(k, x)| \leq aC_\epsilon. \tag{4.3}$$

*In particular,* $\sup_{x,k\in\mathbb{R}^3} |\psi_a(k, x)| < \infty$.

*Proof.* In general, for any $\alpha > 0$ and $\beta > 0$ satisfying $\alpha < n$ and $\alpha + \beta > n$, it holds that

$$\sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{dy}{|x-y|^\alpha \langle y \rangle^\beta} < \infty. \tag{4.4}$$

Indeed, since $|x-y|<\langle x\rangle/2$ implies that $1+2|y|\le|x|$ and $\langle x\rangle^2<8\langle y\rangle^2$, we have

$$I_1\equiv\int_{|x-y|<\langle x\rangle/2}\frac{dy}{|x-y|^\alpha\langle y\rangle^\beta}\le\frac{C_\beta}{\langle x\rangle^\beta}\int_{|y|<\langle x\rangle/2}|y|^{-\alpha}dy=C_{\alpha,\beta,n}\langle x\rangle^{n-(\alpha+\beta)},\tag{4.5}$$

where constant $C_\beta>0$ (resp. $C_{\alpha.\beta,n}>0$) depends only on $\beta$ (resp. on $\alpha,\beta$ and $n$). On the other hand, since $|x-y|\ge\langle x\rangle$ implies that $|x-y|\ge|x|/2$ and $5|x-y|\ge|x|+|y|$ from $|x-y|\ge|y|-|x|$, we have $5^\alpha|x-y|^\alpha\ge|x|^{\alpha-\epsilon}|y|^\epsilon$ for $\epsilon\in[0,\alpha]$ and

$$I_2\equiv\int_{|x-y|\ge\langle x\rangle/2}\frac{dy}{|x-y|^\alpha\langle y\rangle^\beta}\le\frac{C_\alpha}{|x|^{\alpha-\epsilon}}\int|y|^{-\epsilon}\langle y\rangle^{-\beta}dy$$

for $\epsilon\in[0,\alpha]$ satisfying $\epsilon+\beta>n$, with some positive $C_\alpha>0$ depending only on $\alpha$. In particular, taking $\epsilon=\alpha$, we observe that

$$I_2\le C'_{\alpha,\beta,n}\tag{4.6}$$

with some positive $C'_{\alpha,\beta,n}>0$. Combining (4.5) and (4.6) we have (4.4).

Iterating (4.2), we see that by $(\mathrm{QD}_2)$

$$|e^{ik\cdot x}-\psi_a(k,x)|\le\sum_{n=1}^\infty\left(\frac{a}{4\pi}\right)^n\int\cdots\int\frac{dy_1\cdots dy_n}{\prod_{j=1}^n|y_j-y_{j-1}|\langle y_j\rangle^{2+\epsilon}}\le\frac{aC}{4\pi-aC}$$

for $a<4\pi/C$, where $C>0$ is a constant depending on $\epsilon$ and we have used (4.4) in the last inequality. This completes the lemma. □

For the coupling function $\chi$ we suppose that

**($\mathbf{IR}_2$)** $\chi\in C_0^\infty(\mathbb{R}^3)$ is real-valued and satisfies $\chi(0)>0$.

**Lemma 4.2.** *Suppose ($QD_2$) and ($IR_2$). Then for sufficiently small $a$ we have*

*(a)* $\overline{\psi_a(\cdot,X)}w_0^{-j/2}\chi\in L^2(\mathbb{R}^3;dk)$ *($j=1,2$) and*

$$\|\overline{\psi_a(\cdot,X)}w_0^{-j/2}\chi\|^2\le C_a\int_{\mathbb{R}^3}\frac{|\chi(k)|^2}{w_0(k)^j}dk<\infty,\quad j=1,2;$$

*(b)* $\overline{\psi_a(\cdot,X)}w_0^{-3/2}\chi\notin L^2(\mathbb{R}^3;dk)$.

We are now in a position to state the main result in this section.

**Theorem 4.3.** *Suppose (K), ($QD_2$) and ($IR_2$). Then for sufficiently small $a>$ 0,*

*(i)* $\tilde H$ *is a self-adjoint operator bounded below;*

*(ii)* $\tilde{H}$ *has no ground state.*

*Proof.* (i) is proven by (a) of Lemma 4.2, and (ii) by Lemma 4.1 in a similar way to that of [2, Theorem 4.7]. □

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