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On the ionization energy of semi-relativistic Pauli-Fierz model for a single particle

By

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Abstract

A semi-relativistic Pauli-Fierz model is defined by the sum of the free Hamiltonian H_{f} of a Boson Fock space, an nuclear potential V and a relativistic kinetic energy:

$$H = \sqrt{[\boldsymbol{\sigma} \cdot (\mathbf{p} + e\mathbf{A})]^2 + M^2} - M + V + H_{\mathrm{f}}.$$

Let $-e_0 < 0$ be the ground state energy of a semi-relativistic Schrödinger operator

$$H_{\mathrm{p}} = \sqrt{\mathbf{p}^2 + M^2} - M + V.$$

It is shown that the ionization energy E^{ion} of H satisfies

$$E^{\mathrm{ion}} \geq e_0 > 0$$

for all values of both of the coupling constant $e \in \mathbb{R}$ and the rest mass $M \geq 0$. In particular our result includes the case of $M = 0$.

§ 1. Introduction

In this paper we consider a semi-relativistic Pauli-Fierz model in QED. Throughout SRPF model is a shorthand for semi-relativistic Pauli-Fierz model. This model describes a dynamics of a semi-relativistic charged particle moving in the three-dimensional Euclidean space $\mathbb{R}^3$ under the influence of a real-valued nuclear potential V and a quantized electromagnetic field.

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The Hilbert space $\mathcal{H}$ of the total system is the tensor product Hilbert space of $\mathcal{H}_{\rm part} = L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$ and the Boson Fock space $\mathcal{H}_{\rm phot}$ over $L^2(\mathbb{R}^3\times\{1,2\})$. Here $\mathcal{H}_{\rm part}$ describes the state space of a semi-relativistic charged particle with spin 1/2 and $\mathcal{H}_{\rm phot}$ that of photons. The total Hamiltonian H^V on $\mathcal{H}$ is given by a minimal coupling to a quantized electromagnetic field $\mathbf{A}$ and is of the form

$$H^V = \sqrt{[\boldsymbol{\sigma}\cdot(\mathbf{p}+e\mathbf{A})]^2+M^2} - M + V + H_{\rm f}, \tag{1.1}$$

where $\mathbf{p} = -i\nabla = (p_1,p_2,p_3)$ denotes the generalized gradient operator, $e\in\mathbb{R}$ is the coupling constant, $M\geq 0$ the rest mass, $H_{\rm f}$ the free Hamiltonian of $\mathcal{H}_{\rm phot}$, and $\boldsymbol{\sigma}=(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3)$ denote 2×2 Pauli matrices satisfying relations:

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = i\boldsymbol{\sigma}_3,\quad \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = i\boldsymbol{\sigma}_1,\quad \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 = i\boldsymbol{\sigma}_2.$$

We set $H^0 = H^{V=0}$. Let $E^V = \inf{\rm Spec}(H^V)$ be the lowest energy spectrum of H^V. Intuitively E^0 describes the energy of a particle going away from a nucleus and positivity $E^0 - E^V > 0$ suggests that ground states of H^V are stable. The existence of ground states can be indeed shown under condition $E^0 - E^V > 0$. The ionization energy is then defined by

$$E^{\rm ion} = E^0 - E^V. \tag{1.2}$$

Suppose that a semi-relativistic Schrödinger $H_{\rm p} = \sqrt{\mathbf{p}^2+M^2} - M + V$ has a negative energy ground state ϕ_0 such that $H_{\rm p}\phi_0 = -e_0\phi_0$ with $e_0>0$. Then the main result on this paper is to show that

$$E^{\rm ion} \geq e_0 > 0. \tag{1.3}$$

It is emphasized that (1.3) is shown for all $(e,M)\in\mathbb{R}\times[0,\infty)$ and that $E^{\rm ion}$ is compared with the lowest energy of $H_{\rm p}$. In order to prove (1.3), we use a natural extension of the strategy developed in [1] to a semi-relativistic case. Namely we show that the ionization energy is greater than the absolute value of the lowest energy of the semi-relativistic Schrödinger operator $H_{\rm p}$.

In the case of the non-relativistic Pauli-Fierz model given by

$$\frac{1}{2M}[\boldsymbol{\sigma}\cdot(\mathbf{p}+e\mathbf{A})]^2 + V + H_{\rm f},$$

the positivity of the ionization energy is shown in [1, 5]. We also refer to see the book [6] and references therein for related results. In the quantum field theory one important task is to show the existence of ground states, which is shown in general by showing or assuming the positivity of an ionization energy. See e.g.,[1, 2, 5, 9]. In this paper we are not concerned with the existence of ground states of SRPF model, but this will be done in another paper [3].

Note added in proof: In [4], M. Könenberg, O. Matte, and E. Stockmeyer also recently prove $E^{\mathrm{ion}} > 0$ of SRPF model in the case of $V(\mathbf{x}) = -\gamma/|\mathbf{x}|$, $\gamma > 0$, and $M = 1(> 0)$, but the ionization energy is not compared with the lowest energy of H_{p} but with a standard Schrödinger operator $\frac{1}{2M}\mathbf{p}^2 + V$. So it is quite different form ours. The existence of ground states of SRPF model is also shown in [4].

§ 2. Definition and Main Result

We begin with defining SRPF model in a rigorous manner. We define the Hamiltonian of SRPF model by a quadratic form.

(Photons) The Hilbert space for photons is given by

$$\mathcal{H}_{\mathrm{phot}} = \bigoplus_{n=0}^{\infty} \left[\bigotimes_{\mathrm{s}}^{n} L^2(\mathbb{R}^3 \times \{1,2\}) \right], \tag{2.1}$$

where $\otimes_{\mathrm{s}}^n$ denotes the n-fold symmetric tensor product with $\otimes_{\mathrm{s}}^0 L^2(\mathbb{R}^3 \times \{1,2\}) = \mathbb{C}$. The smeared annihilation operators in $\mathcal{H}_{\mathrm{phot}}$ are denoted by $a(f), f \in L^2(\mathbb{R}^3 \times \{1,2\})$. The adjoint of $a(f)$, $a^*(f)$, is called the creation operator. The annihilation operator and the creation operator satisfy canonical commutation relations. The Fock vacuum is defined by $\Omega_{\mathrm{phot}} = 1 \oplus 0 \oplus 0 \cdots \in \mathcal{H}_{\mathrm{phot}}$. For a closed operator T on $L^2(\mathbb{R}^3 \times \{1,2\})$, the second quantization of T is denoted by

$$d\Gamma(T) : \mathcal{H}_{\mathrm{phot}} \to \mathcal{H}_{\mathrm{phot}}. \tag{2.2}$$

Let $\omega : \mathbb{R}^3 \to [0,\infty)$ be a Borel measurable function such that $0 < \omega(\mathbf{k}) < \infty$ for almost every $\mathbf{k} \in \mathbb{R}^3$. We also denote by the same symbol ω the multiplication operator by the function ω, which acts in $L^2(\mathbb{R}^3 \times \{1,2\})$ as $(\omega f)(\mathbf{k}, j) = \omega(\mathbf{k}) f(\mathbf{k}, j)$. Then the free Hamiltonian of $\mathcal{H}_{\mathrm{phot}}$ is defined by

$$H_{\mathrm{f}} = d\Gamma(\omega). \tag{2.3}$$

(Charged particle) The Hilbert space for the particle state is defined by

$$\mathcal{H}_{\mathrm{part}} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2, \tag{2.4}$$

where $\mathbb{C}^2$ describes spin 1/2 of the particle. Then the particle Hamiltonian is given by a semi-relativistic Schrödinger operator:

$$H_{\mathrm{p}} = \sqrt{\mathbf{p}^2 + M^2} - M + V. \tag{2.5}$$

(SRPF model) The Hilbert space of the SRPF model is defined by

$$\mathcal{H} = \mathcal{H}_{\mathrm{part}} \otimes \mathcal{H}_{\mathrm{phot}}, \tag{2.6}$$

and the decoupled Hamiltonian of the system is given by

$$H_{\mathrm{p}} \otimes 1 + 1 \otimes H_{\mathrm{f}}. \tag{2.7}$$

We introduce an interaction minimally coupled to $H_{\mathrm{p}} \otimes 1 + 1 \otimes H_{\mathrm{f}}$. Let $\mathbf{e}^{(\lambda)} : \mathbb{R}^3 \to \mathbb{R}^3, \lambda = 1, 2$, be polarization vectors such that

$$\mathbf{e}^{(\lambda)}(\mathbf{k}) \cdot \mathbf{e}^{(\mu)}(\mathbf{k}) = \delta_{\lambda,\mu}, \quad \mathbf{k} \cdot \mathbf{e}^{(\lambda)}(\mathbf{k}) = 0, \quad \lambda, \mu \in \{1, 2\}. \tag{2.8}$$

We set $e^{(\lambda)}(\mathbf{k}) = (e_1^{(\lambda)}(\mathbf{k}), e_2^{(\lambda)}(\mathbf{k}), e_3^{(\lambda)}(\mathbf{k}))$ and suppose that each component $e_j^{(\lambda)}(\mathbf{k})$ is a Borel measurable function in $\mathbf{k}$. Let $\Lambda \in L^2(\mathbb{R}^3)$ be a function such that

$$\omega^{-1/2}\Lambda \in L^2(\mathbb{R}^3). \tag{2.9}$$

We set

$$g_j(\mathbf{k}, \lambda; \mathbf{x}) = \omega(\mathbf{k})^{-1/2}\Lambda(\mathbf{k})e_j^{(\lambda)}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{2.10}$$

For each $\mathbf{x} \in \mathbb{R}^3$, $g_j(\mathbf{x}) = g_j(\cdot, \cdot; \mathbf{x})$ can be regarded as an element of $L^2(\mathbb{R}^3 \times \{1, 2\})$. The quantized electromagnetic field at $\mathbf{x} \in \mathbb{R}^3$ is defined by

$$\mathbf{A}_j(\mathbf{x}) = \frac{1}{\sqrt{2}}\overline{[a(g_j(\mathbf{x})) + a^*(g_j(\mathbf{x}))]}, \tag{2.11}$$

where $\bar{T}$ denotes the closure of closable operator T. The quantized electromagnetic field $\mathbf{A}(\mathbf{x}) = (\mathbf{A}_1(\mathbf{x}), \mathbf{A}_2(\mathbf{x}), \mathbf{A}_3(\mathbf{x}))$ satisfies the Coulomb gage condition:

$$\sum_{j=1}^{3} \frac{\partial \mathbf{A}_j(\mathbf{x})}{\partial \mathbf{x}_j} = 0.$$

It is known that $\mathbf{A}_j(\mathbf{x})$ is self-adjoint for each $\mathbf{x} \in \mathbb{R}^3$. The Hilbert space $\mathcal{H}$ can be identified as

$$\mathcal{H} \cong \int_{\mathbb{R}^3}^{\oplus} \mathbb{C}^2 \otimes \mathcal{H}_{\mathrm{phot}} d\mathbf{x}, \tag{2.12}$$

where $\int^{\oplus} \cdots$ denotes a constant fiber direct integral [8]. The quantized electromagnetic field on the total Hilbert space is defined by the constant fiber direct integral of $\mathbf{A}_j(\mathbf{x})$:

$$\mathbf{A}_j = \int_{\mathbb{R}^3}^{\oplus} \mathbf{A}_j(\mathbf{x}) d\mathbf{x}. \tag{2.13}$$

Let $C_0^\infty(\mathbb{R}^3)$ be a set of infinite times differentiable functions with a compact support. The finite particle subspace over $C_0^\infty = C_0^\infty(\mathbb{R}^3)$ is defined by

$$\mathcal{F}_{\mathrm{fin}} = \mathcal{L}[\{a^*(f_1) \cdots a^*(f_n)\Omega_{\mathrm{phot}}, \Omega_{\mathrm{phot}} | f_j \in C_0^\infty, j = 1, \ldots, n, n \in \mathbb{N}\}], \tag{2.14}$$

where $\mathcal{L}[\cdots]$ denotes the linear hull of $[\cdots]$. We set

$$\mathcal{D} = C_0^\infty(\mathbb{R}^3;\mathbb{C}^2)\hat{\otimes}\mathcal{F}_{\mathrm{fin}}, \tag{2.15}$$

where the symbol $\hat{\otimes}$ denotes the algebraic tensor product. In what follows for notational convenience we omit $\otimes$ between $\mathcal{H}_{\mathrm{part}}$ and $\mathcal{H}_{\mathrm{phot}}$, i.e., we write p_j for $p_j \otimes 1$, H_{f} for $1 \otimes H_{\mathrm{f}}$ and V for $V \otimes 1$, etc. We define the non-negative quadratic form on $\mathcal{D} \times \mathcal{D}$ by

$$K_{\mathbf{A}}(\Psi,\Phi) = \sum_{j=1}^{3} \langle \boldsymbol{\sigma}_j(p_j + e\mathbf{A}_j)\Psi, \boldsymbol{\sigma}_j(p_j + e\mathbf{A}_j)\Phi\rangle + M^2 \langle \Psi,\Phi\rangle . \tag{2.16}$$

Since $K_{\mathbf{A}}$ is closable, we denote its closure by $\bar{K}_{\mathbf{A}}$. The semi-bounded quadratic form $\bar{K}_{\mathbf{A}}$ defines the unique non-negative self-adjoint operator $H_{\mathbf{A}}$ such that

$$\mathrm{Dom}(H_{\mathbf{A}}^{1/2}) = Q(\bar{K}_{\mathbf{A}})$$

and

$$\bar{K}_{\mathbf{A}}(\Psi,\Phi) = (H_{\mathbf{A}}^{1/2}\Psi, H_{\mathbf{A}}^{1/2}\Phi)$$

for $\Psi,\Phi \in Q(\bar{K}_{\mathbf{A}})$. Here $Q(X)$ denotes the form domain of X. We note that

$$\begin{aligned} Q(H_{\mathbf{A}}) = \{\Psi \in \mathcal{H} | \exists \{\Psi_n\}_n \subset \mathcal{D} \text{ s.t. } \Psi_n \to \Psi, \\ K_{\mathbf{A}}(\Psi_m - \Psi_n, \Psi_m - \Psi_n) \to 0, n, m \to \infty\}, \end{aligned}$$
$$\left\langle H_{\mathbf{A}}^{1/2}\Psi, H_{\mathbf{A}}^{1/2}\Psi \right\rangle = \lim_{n\to\infty} K_{\mathbf{A}}(\Psi_n,\Psi_n) \text{ for } \Psi \in Q(H_{\mathbf{A}}).$$

Since $\mathcal{D} \subset Q(H_{\mathbf{A}})$, we have $\mathcal{D} \subset \mathrm{Dom}(H_{\mathbf{A}}^{1/2})$.

Remark 1. It is not trivial to see the essential self-adjointness or the self-adjointness of $[\boldsymbol{\sigma}\cdot(\mathbf{p} + e\mathbf{A})]^2 + M^2$. Note that however on $\mathcal{D}$ we see that

$$H_{\mathbf{A}} = [\boldsymbol{\sigma}\cdot(\mathbf{p} + e\mathbf{A})]^2 + M^2$$

and then $H_{\mathbf{A}}^{1/2}$ can be regarded as a rigorous definition of $\sqrt{[\boldsymbol{\sigma}\cdot(\mathbf{p} + e\mathbf{A})]^2 + M^2}$.

Now we define the Hamiltonian of SRPF model.

Definition 2.1. **(SRPF model)**
(1) Let us define H^0 by

$$H^0 = H_{\mathbf{A}}^{1/2} - M + H_{\mathrm{f}}. \tag{2.17}$$

(2) Let $V : \mathbb{R}^3 \to \mathbb{R}$ be such that $V \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$. Then the Hamiltonian of SRPF model is defined by

$$H^V = H^0 + V. \tag{2.18}$$

We introduce the following conditions:

(H.1) H_{p} is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$.

(H.2) H_{p} has a normalized negative energy ground state ϕ_0:

$$H_{\mathrm{p}}\phi_0 = -e_0\phi_0, \quad e_0 > 0, \quad -e_0 = \inf \mathrm{Spec}(H_{\mathrm{p}}). \tag{2.19}$$

(H.3) H^0 and H^V are essentially self-adjoint on $\mathcal{D}$. We denote the closure of $H^V\lceil_{\mathcal{D}}$ by the same symbol.

Remark 2. Although it is interested in specifying conditions for H^V to be self-adjoint or essential self-adjoint on some domain, in this paper we do not discuss it.

Let $E^0 = \inf \mathrm{Spec}(H^0)$ and $E^V = \inf \mathrm{Spec}(H^V)$. The ionization energy is defined by

$$E^{\mathrm{ion}} = E^0 - E^V. \tag{2.20}$$

If $V \leq 0$, then it is trivial to see that $E^{\mathrm{ion}} \geq 0$. The main result in this paper is however as follows:

Theorem 2.2. *Assume (H.1)–(H.3). Then $E^{\mathrm{ion}} \geq e_0 > 0$ for all $(e, M) \in \mathbb{R} \times [0, \infty)$.*

§ 3. Proof of Theorem 2.2

Throughout this section we assume (H.1)–(H.3). We fix an arbitrary small $\epsilon > 0$. Let F_0 and f_0 be ϵ-minimizers of H^0 and H_{p}, respectively, i.e.,

$$\left\langle F_0, H^0 F_0 \right\rangle_{\mathcal{H}} < E^0 + \epsilon, \quad \|F_0\|_{\mathcal{H}} = 1, \quad F_0 \in \mathcal{D}, \tag{3.1}$$

$$\langle f_0, H_{\mathrm{p}} f_0 \rangle_{\mathcal{H}_{\mathrm{part}}} < -e_0 + \epsilon, \quad \|f_0\|_{\mathcal{H}_{\mathrm{part}}} = 1, \quad f_0 \in C_0^\infty(\mathbb{R}^3). \tag{3.2}$$

Since $\mathcal{D}$ and $C_0^\infty(\mathbb{R}^3)$ are cores for H^0 and H_{p}, respectively, we can choose a minimizer satisfying (3.1) and (3.2). Recall that H_{p} has a ground state ϕ_0. Clearly, its complex conjugate ϕ_0^* is also the ground state of H_{p}. Hence we may assume that ϕ_0 is real without loss of generality. Therefore we can choose a real-valued ϵ-minimizer f_0. For each $\mathbf{y} \in \mathbb{R}^3$, we set

$$U_{\mathbf{y}} = \exp(-i\mathbf{y} \cdot \mathbf{p}) \otimes \exp(-i\mathbf{y} \cdot d\Gamma(\mathbf{k})).$$

The unitary operator $U_{\mathbf{y}}$ is the parallel translation by the vector $y \in \mathbb{R}^3$. It can be shown that $U_{\mathbf{y}}\mathcal{D} = \mathcal{D}$ and H^0 is translation invariant:

$$U_{\mathbf{y}}^* H^0 U_{\mathbf{y}} = H^0. \tag{3.3}$$

Set

$$\Omega_{\mathbf{A}}(\mathbf{p}) = H_{\mathbf{A}}^{1/2}$$

Lemma 3.1. *Let $\Phi_{\mathbf{y}} = f_0(\hat{\mathbf{x}})F_{\mathbf{y}}$, where $f_0(\hat{\mathbf{x}})$ denotes the multiplication by the function $f_0(\mathbf{x})$, and $F_{\mathbf{y}} = U_{\mathbf{y}}F_0$. Then we have*

$$\int_{\mathbb{R}^3} d\mathbf{y} \left\langle \Phi_{\mathbf{y}}, H^V \Phi_{\mathbf{y}} \right\rangle = \|f_0\|^2 \left\langle F_0, H^0 F_0 \right\rangle + \langle f_0, V f_0 \rangle \langle F_0, F_0 \rangle$$
$$+\frac{1}{2}\int_{\mathbb{R}^3} d\mathbf{k} |\hat{f}_0(\mathbf{k})|^2 \langle F_0, [\Omega_{\mathbf{A}}(\mathbf{p}+\mathbf{k}) + \Omega_{\mathbf{A}}(\mathbf{p}-\mathbf{k}) - 2\Omega_{\mathbf{A}}(\mathbf{p})]F_0 \rangle .$$

Proof. Clearly, $\Phi_{\mathbf{y}} \in \mathcal{D}$. By using (3.3), we have

$$\begin{aligned} \left\langle \Phi_{\mathbf{y}}, H^V \Phi_{\mathbf{y}} \right\rangle &= \left\langle f_0(\hat{\mathbf{x}})F_{\mathbf{y}}, f_0(\hat{\mathbf{x}})H^V F_{\mathbf{y}} \right\rangle - \frac{1}{2}\left\langle F_{\mathbf{y}}, [f_0(\hat{\mathbf{x}}), [f_0(\hat{\mathbf{x}}), H^V]]F_{\mathbf{y}} \right\rangle \\ &= \left\langle f_0(\hat{\mathbf{x}}+\mathbf{y})^2 F_0, H^0 F_0 \right\rangle + \left\langle f_0(\hat{\mathbf{x}}+\mathbf{y})^2 F_0, V(\mathbf{x}+\mathbf{y})F_0 \right\rangle \\ &\quad -\frac{1}{2}\langle F_0, [f_0(\hat{\mathbf{x}}+\mathbf{y}), [f_0(\hat{\mathbf{x}}+\mathbf{y}), \Omega_{\mathbf{A}}(\mathbf{p})]]F_0 \rangle . \end{aligned} \tag{3.4}$$

It should be noted that $\mathcal{D}$ is invariant by the unitary operator $e^{i\mathbf{k}\cdot\mathbf{x}}$ and

$$\begin{aligned} \left\langle \Omega_{\mathbf{A}}(\mathbf{p})e^{i\mathbf{k}\cdot\mathbf{x}}\Psi, \Omega_{\mathbf{A}}(\mathbf{p})e^{i\mathbf{k}\cdot\mathbf{x}}\Phi \right\rangle &= \sum_{j=1}^{3} \langle \boldsymbol{\sigma}_j(p_j + k_j + e\mathbf{A}_j)\Psi, \boldsymbol{\sigma}_j(p_j + k_j + e\mathbf{A}_j)\Phi \rangle \\ &= \langle \Omega_{\mathbf{A}}(\mathbf{p}+\mathbf{k})\Psi, \Omega_{\mathbf{A}}(\mathbf{p}+\mathbf{k})\Phi \rangle \end{aligned}$$

for $\Psi \in \mathcal{D}$. Hence by the definition of $\Omega_{\mathbf{A}}(\mathbf{p})$ we have

$$\Omega_{\mathbf{A}}(\mathbf{p}+\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{x}}\Omega_{\mathbf{A}}(\mathbf{p})e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{3.5}$$

Thus the last term (3.4) can be computed. We have by the inverse Fourier transformation,

$$\begin{aligned} &-\frac{1}{2}\langle F_0, [f_0(\hat{\mathbf{x}}+\mathbf{y}), [f_0(\hat{\mathbf{x}}+\mathbf{y}), \Omega_{\mathbf{A}}(\mathbf{p})]]F_0 \rangle \\ &= -\frac{1}{2(2\pi)^3}\int_{\mathbb{R}^6} d\mathbf{k}_1 d\mathbf{k}_2 \hat{f}_0(\mathbf{k}_1)\hat{f}_0(\mathbf{k}_2) e^{i\mathbf{k}_1\cdot\mathbf{y}} e^{i\mathbf{k}_2\cdot\mathbf{y}} \left\langle F_0, [e^{i\mathbf{k}_1\cdot\mathbf{x}}, [e^{i\mathbf{k}_2\cdot\mathbf{x}}, \Omega_{\mathbf{A}}(\mathbf{p})]]F_0 \right\rangle . \end{aligned}$$

Using (3.5) twice, we see that

$$\begin{aligned} &= -\frac{1}{2(2\pi)^3}\int_{\mathbb{R}^6} d\mathbf{k}_1 d\mathbf{k}_2 \hat{f}_0(\mathbf{k}_1)\hat{f}_0(\mathbf{k}_2) e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{y}} \left\langle F_0, [e^{i\mathbf{k}_1\cdot\mathbf{x}}, \Omega_{\mathbf{A}}(\mathbf{p}-\mathbf{k}_2) - \Omega_{\mathbf{A}}(\mathbf{p})]F_0 \right\rangle \\ &= -\frac{1}{2(2\pi)^3}\int_{\mathbb{R}^6} d\mathbf{k}_1 d\mathbf{k}_2 \hat{f}_0(\mathbf{k}_1)\hat{f}_0(\mathbf{k}_2) e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{y}} \\ &\quad \times \langle F_0, [\Omega_{\mathbf{A}}(\mathbf{p}-\mathbf{k}_2-\mathbf{k}_1) - \Omega_{\mathbf{A}}(\mathbf{p}-\mathbf{k}_2) - \Omega_{\mathbf{A}}(\mathbf{p}-\mathbf{k}_1) + \Omega_{\mathbf{A}}(\mathbf{p})]F_0 \rangle . \end{aligned}$$

Under identification $\mathcal{H} \cong \int_{\mathbb{R}^3}^{\oplus} \mathbb{C}^2 \otimes \mathcal{H}_{\rm phot} d\mathbf{x}$, F_0 can be regarded as a $\mathbb{C}^2 \otimes \mathcal{H}_{\rm phot}$-valued L^2-function. Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} d\mathbf{y} \left\langle \Phi_{\mathbf{y}}, H^V \Phi_{\mathbf{y}} \right\rangle \\
&= \int_{\mathbb{R}^3} d\mathbf{y} \int_{\mathbb{R}^3} d\mathbf{x} f_0(\mathbf{x}+\mathbf{y})^2 \left(\left\langle F_0(\mathbf{x}), (H^0 F_0)(\mathbf{x}) \right\rangle + V(\mathbf{x}+\mathbf{y}) \left\langle F_0(\mathbf{x}), F_0(\mathbf{x}) \right\rangle \right) \\
&\quad - \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} d\mathbf{y} \int_{\mathbb{R}^6} d\mathbf{k}_1 d\mathbf{k}_2 \hat{f}_0(\mathbf{k}_1) \hat{f}_0(\mathbf{k}_2) e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot \mathbf{y}} \\
&\quad \times \left\langle F_0, [\Omega_{\mathbf{A}}(\mathbf{p}-\mathbf{k}_2-\mathbf{k}_1) - \Omega_{\mathbf{A}}(\mathbf{p}-\mathbf{k}_2) - \Omega_{\mathbf{A}}(\mathbf{p}-\mathbf{k}_1) + \Omega_{\mathbf{A}}(\mathbf{p})] F_0 \right\rangle,
\end{aligned}$$

where we used the fact that $\hat{f}_0(-\mathbf{k}) = \hat{f}_0(\mathbf{k})^*$. Hence we have

$$\begin{aligned}
\int_{\mathbb{R}^3} d\mathbf{y} \left\langle \Phi_{\mathbf{y}}, H^V \Phi_{\mathbf{y}} \right\rangle &= \|f_0\|^2 \left\langle F_0, H^0 F_0 \right\rangle + \langle f_0, V f_0 \rangle \langle F_0, F_0 \rangle \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^3} d\mathbf{k} |\hat{f}_0(\mathbf{k})|^2 \left\langle F_0, [\Omega_{\mathbf{A}}(\mathbf{p}+\mathbf{k}) + \Omega_{\mathbf{A}}(\mathbf{p}-\mathbf{k}) - 2\Omega_{\mathbf{A}}(\mathbf{p})] F_0 \right\rangle.
\end{aligned}$$

Then the lemma follows. □

The following inequality is the key to the proof of Theorem 2.2

Lemma 3.2. *For all $M \geq 0$ and $\mathbf{k} \in \mathbb{R}^3$, the operator inequality*

$$\frac{1}{2} \left\{ \Omega_{\mathbf{A}}(\mathbf{p}+\mathbf{k}) + \Omega_{\mathbf{A}}(\mathbf{p}-\mathbf{k}) - 2\Omega_{\mathbf{A}}(\mathbf{p}) \right\} \leq \sqrt{\mathbf{k}^2 + M^2} - M, \tag{3.6}$$

holds on $\mathrm{Dom}(\Omega_{\mathbf{A}}(\mathbf{p}))$.

Proof. Note that the domains of $\Omega_{\mathbf{A}}(\mathbf{p}+\mathbf{k})$, $\Omega_{\mathbf{A}}(\mathbf{p}-\mathbf{k})$ and $\Omega_{\mathbf{A}}(\mathbf{p})$ are identical. (3.6) is equivalent to

$$\Omega_{\mathbf{A}}(\mathbf{p}+\mathbf{k}) + \Omega_{\mathbf{A}}(\mathbf{p}-\mathbf{k}) \leq 2(\sqrt{\mathbf{k}^2 + M^2} - M + \Omega_{\mathbf{A}}(\mathbf{p})). \tag{3.7}$$

By the Kato-Rellich Theorem, (3.7) follows from

$$\|[\Omega_{\mathbf{A}}(\mathbf{p}+\mathbf{k}) + \Omega_{\mathbf{A}}(\mathbf{p}-\mathbf{k})]\Psi\|^2 \leq \left\| 2[\sqrt{\mathbf{k}^2 + M^2} - M + \Omega_{\mathbf{A}}(\mathbf{p})]\Psi \right\|^2 \tag{3.8}$$

for $\Psi \in \mathrm{Dom}(\Omega_{\mathbf{A}}(\mathbf{p}))$. We have the bound

$$\begin{aligned}
\|[\Omega_{\mathbf{A}}(\mathbf{p}+\mathbf{k}) + \Omega_{\mathbf{A}}(\mathbf{p}-\mathbf{k})]\Psi\|^2 &\leq 2[\|\Omega_{\mathbf{A}}(\mathbf{p}+\mathbf{k})\Psi\|^2 + \left\|\Omega_{\mathbf{A}}(\mathbf{p}-\mathbf{k})^2\Psi\right\|^2] \\
&= 4\mathbf{k}^2 \|\Psi\|^2 + 4\|\Omega_{\mathbf{A}}(\mathbf{p})\Psi\|^2,
\end{aligned} \tag{3.9}$$

for all $\Psi \in \mathrm{Dom}(\Omega_{\mathbf{A}}(\mathbf{p}))$. While we have

$$\begin{aligned}
&4[\sqrt{\mathbf{k}^2 + M^2} - M + \Omega_{\mathbf{A}}(\mathbf{p})]^2 \\
&= 4[\mathbf{k}^2 + [\boldsymbol{\sigma} \cdot (\mathbf{p} + e\mathbf{A})]^2 + M^2 + 2(\sqrt{\mathbf{k}^2 + M^2} - M)(\Omega_{\mathbf{A}}(\mathbf{p}) - M)],
\end{aligned} \tag{3.10}$$

in the sense of form on $\mathrm{Dom}(\Omega_{\mathbf{A}}(\mathbf{p}))$. Since $\Omega_{\mathbf{A}}(\mathbf{p}) - M$ is positive, inequality (3.9) and equality (3.10) imply (3.8). Therefore inequality (3.6) holds. □

Corollary 3.3. *It follows that*

$$\int_{\mathbb{R}^3} d\mathbf{y} \left\langle \Phi_{\mathbf{y}}, H^V \Phi_{\mathbf{y}} \right\rangle \leq \left\langle F_0, H^0 F_0 \right\rangle + \langle f_0, H_{\mathrm{p}} f_0 \rangle .$$

Proof. By using Lemmas 3.1 and 3.2, we have

$$\begin{aligned} \int_{\mathbb{R}^3} d\mathbf{y} \left\langle \Phi_{\mathbf{y}}, H^V \Phi_{\mathbf{y}} \right\rangle &\leq \left\langle F_0, H^0 F_0 \right\rangle + \langle f_0, V f_0 \rangle + \int_{\mathbb{R}^3} d\mathbf{k} (\sqrt{\mathbf{k}^2 + M^2} - M) |\hat{f}_0(\mathbf{k})|^2 \\ &= \left\langle F_0, H^0 F_0 \right\rangle + \langle f_0, H_{\mathrm{p}} f_0 \rangle . \end{aligned}$$

Then the corollary follows. □

Proof of Theorem 2.2: By Corollary 3.3 and the definitions of F_0 and f_0, we have

$$\int_{\mathbb{R}^3} d\mathbf{y} \left[- \left\langle \Phi_{\mathbf{y}}, H^V \Phi_{\mathbf{y}} \right\rangle + (E^0 - e_0 + 2\epsilon) \left\| \Phi_{\mathbf{y}} \right\|^2 \right] > 0.$$

Therefore, there exists a vector $y \in \mathbb{R}^3$ such that $\Phi_{\mathbf{y}} \neq 0$ and

$$E^V \left\| \Phi_{\mathbf{y}} \right\|^2 \leq \left\langle \Phi_{\mathbf{y}}, H^V \Phi_{\mathbf{y}} \right\rangle < (E^0 - e_0 + 2\epsilon) \left\| \Phi_{\mathbf{y}} \right\|^2 .$$

Since ϵ is arbitrary, this yields (3.6) and completes the proof of the theorem.

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