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Some abstract considerations on the homogenization problem of infinite dimensional diffusions

By

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Abstract

By generalizing the concrete formulations in [ABRY1,2,3], a general framework for the homogenization problem of infinite dimensional diffusions is proposed.

§ 1. Introduction

We present a general formulation on the homogenization problem of infinite dimensional diffusions. By discussing the problem in an abstract way we clarify some concrete considerations performed in [ABRY1,2,3], where several explicit results have been derived. We state the results without detailed proofs, because some of them are simple mathematical generalizations of the corresponding results in [ABRY1,2,3], and all of them shall be given rigorously in forthcoming papers. In the present paper, we assume that the systems we consider, infinite dimensional diffusions on $\mathbb{Z}^d$, satisfy the Poincaré inequality, which then yields their ergodicity. We then consider weak convergence for the sequence of the processes (i.e. their scaling limit). We consider the processes with the index $\mathbb{Z}^d$ as random variables taking values in a direct product space (equipped with the *direct product topology*) of continuous functions. We are also preparing a paper where a stronger form of ergodicity is assumed, namely the ergodicity which follows from a logarithemic Sobolev inequality (cf. [ABRY3]), and there we shall derive a stronger result than the one given here, and provide complete proofs.

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§ 2. Probability space $(\Theta, \overline{\mathcal{B}}, \overline{\mu})$ on which the random coefficients are defined

Suppose that we are given the following:
$\{(\Theta_{\mathbf{k}}, \mathcal{B}_{\mathbf{k}}, \lambda_{\mathbf{k}})\}_{\mathbf{k}\in\mathbb{Z}^d}$: a system of complete probability spaces, where d is a given natural number.
$(\Theta, \overline{\mathcal{B}}, \overline{\lambda})$: the probability space that is the completion of $(\prod_{\mathbf{k}} \Theta_{\mathbf{k}}, \bigotimes_{\mathbf{k}} \mathcal{B}_{\mathbf{k}}, \prod_{\mathbf{k}} \lambda_{\mathbf{k}})$, i.e., the completion of the direct product probability space.
$(\Theta, \overline{\mathcal{B}}, \mu)$: a complete probability space (corresponding to a Gibbs state) defined as follows:
for $\forall D \subset\subset \mathbb{Z}^d$ and for any bounded measurable function φ defined on $\prod_{\mathbf{k}\in D'} \Theta_{\mathbf{k}}$ with some $\forall D' \subset\subset \mathbb{Z}^d$, μ satisfies

$$(\mathbb{E}^D \varphi, \mu) = (\varphi, \mu), \tag{2.1}$$

where

$$\begin{aligned}(\mathbb{E}^D \varphi)(\theta) &\equiv \int_\Theta \varphi(\theta'_D \cdot \theta_{D^c}) \mathbb{E}^D(d\theta' | \theta_{D^c}) \\ &\equiv \int_\Theta \varphi(\theta'_D \cdot \theta_{D^c}) m_D(\theta'_D \cdot \theta_{D^c}) \overline{\lambda}(d\theta'),\end{aligned} \tag{2.2}$$

and

$$m_D(\theta'_D \cdot \theta_{D^c}) \equiv \frac{1}{Z_D(\theta_{D^c})} e^{-U_D(\theta'_D \cdot \theta_{D^c})}, \quad U_D \equiv \sum_{\mathbf{k}\in D^+} U_{\mathbf{k}}, \tag{2.3}$$

$$\Theta \ni \theta \longmapsto \theta_D \in \prod_{\mathbf{k}\in D} \Theta_{\mathbf{k}}$$

is the natural projection,
$\theta'_D \cdot \theta_{D^c}$ is the element $\theta'' \in \Theta$ such that

$$\theta''_D = \theta'_D, \qquad \theta''_{D^c} = \theta_{D^c},$$

$$D^+ = \{\mathbf{k}' | \text{support of } U_{\mathbf{k}'} \cap D \neq \emptyset\},$$

also, for each $\mathbf{k} \in \mathbb{Z}^d$, $U_{\mathbf{k}}$ is a given bounded measurable function of which support is in $\prod_{|\mathbf{k}'-\mathbf{k}|\leq L} \Theta_{\mathbf{k}'}$, where the number L (the range of interactions) does not depend on $\mathbf{k}$, and $Z_D(\theta_{D^c})$ is the normalizing constant.

§ 3. The ergodic flow

On $(\Theta, \overline{\mathcal{B}}, \overline{\lambda})$ we are given a *measure preserving map* $T_{\mathbf{x}}$ as follows:
Suppose that

$$\exists M_1 < \infty \quad \text{and} \quad \forall \mathbf{k} \in \mathbb{Z}^d \quad \text{there exists a } d_{\mathbf{k}} \in \mathbb{R}_+ \text{ such that} \quad d_{\mathbf{k}} \leq M_1. \tag{3.1}$$

For each $\mathbf{x} \in \prod_{\mathbf{k}} \mathbb{R}^{d_{\mathbf{k}}}$ such that $\mathbf{x} = (\mathbf{x}^{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ with $\mathbf{x}^{\mathbf{k}} = (x_1^{\mathbf{k}}, \dots, x_{d_{\mathbf{k}}}^{\mathbf{k}})$
the map $T_{\mathbf{x}}$ on $(\Theta, \overline{\mathcal{B}}, \overline{\lambda})$ with values in Θ defined by i)-v) below is:
i)
$T_{\mathbf{x}}$ is a measure preserving transformation with respect to the measure $\overline{\lambda}$;
ii)

$$T_{\mathbf{0}} = \text{the identity},$$

$$\text{and for } \mathbf{x},\, \mathbf{x}' \in \mathbf{x} \in \prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}} \qquad T_{\mathbf{x}+\mathbf{x}'} = T_{\mathbf{x}} \circ T_{\mathbf{x}'},$$

where

$$\mathbf{x} + \mathbf{x}' \equiv (\mathbf{x}^{\mathbf{k}} + \mathbf{x}'^{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d},$$

with

$$\mathbf{x}^{\mathbf{k}} + \mathbf{x}'^{\mathbf{k}} = (x_1^{\mathbf{k}} + x'^{\mathbf{k}}_1, \dots, x_{d^{\mathbf{k}}}^{\mathbf{k}} + x'^{\mathbf{k}}_{d^{\mathbf{k}}}),$$

for

$$\mathbf{x} = (\mathbf{x}^{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}, \qquad \mathbf{x}^{\mathbf{k}} = (x_1^{\mathbf{k}}, \dots, x_{d_{\mathbf{k}}}^{\mathbf{k}}),$$
$$\mathbf{x}' = (\mathbf{x}^{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}, \qquad \mathbf{x}'^{\mathbf{k}} = (x'^{\mathbf{k}}_1, \dots, x'^{\mathbf{k}}_{d_{\mathbf{k}}}),$$

and

$$\mathbf{0} \equiv (\mathbf{0}^{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}, \qquad \mathbf{0}^{\mathbf{k}} = (0, \dots, 0) \in \mathbb{R}^{d_{\mathbf{k}}};$$

iii)

$$(\mathbf{x}, \theta) \in (\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}) \times \Theta \longrightarrow T_{\mathbf{x}}(\theta) \in \Theta$$

is $\mathcal{B}(\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}) \times \overline{\mathcal{B}}/\overline{\mathcal{B}}-$measurable, where $\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}$ is assumed to be the topological space with the direct product topology;
iv) A function which is $T_{\mathbf{x}}$ invariant for all $\mathbf{x} \in \prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}$ is a constant function on $(\Theta, \overline{\mathcal{B}}, \mu)$;
v) For $D \subset \mathbb{Z}^d$, let

$$\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}} \ni \mathbf{x} \longmapsto \mathbf{x}_D \in \prod_{\mathbf{k}\in D} \mathbb{R}^{d_{\mathbf{k}}}$$

be the natural *projection*. If $\mathbf{x}_{D^c} = \mathbf{0}_{D^c}$, then

$$(T_{\mathbf{x}}(\theta))_{D^c} = \theta_{D^c}, \qquad \forall\theta \in \Theta, \quad \forall D \subset\subset \mathbb{Z}^d.$$

□

§ 4. The core

We assume that an existence of a *core* $\mathcal{D}^{\Theta}$. Namely, there exists $\mathcal{D}^{\Theta}$ which is a dense subset of both $L^2(\mu)$ and $L^1(\mu)$, and $\forall\varphi \in \mathcal{D}^{\Theta}$ satisfies

($\mathcal{D}$-1) φ is a bounded measurable function having only a finite number of variables θ_D for some $D \subset\subset \mathbb{Z}^d$,
($\mathcal{D}$-2)

$$\varphi(T_{\mathbf{x}_D}(\theta)) \in C^\infty(\prod_{\mathbf{k}\in D} \mathbb{R}^{d_\mathbf{k}} \to \mathbb{R}), \quad \forall\theta \in \Theta,$$

(cf. v) in the previous section) where we identify $\mathbf{x}_D \in \prod_{\mathbf{k}\in D} \mathbb{R}^{d_\mathbf{k}}$ with an $\mathbf{x} \in (\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_\mathbf{k}})$ of which projection to $\prod_{\mathbf{k}\in D} \mathbb{R}^{d_\mathbf{k}}$ is $\mathbf{x}_D$,
($\mathcal{D}$-3) in ($\mathcal{D}$-2) for each $\theta \in \Theta$, all the partial derivatives of all orders of the function $\varphi(T_\cdot(\theta))$ (with the variables $\mathbf{x}_D$) are bounded and

$$\forall\varphi \in \mathcal{D},\ \exists M < \infty; \quad |\nabla_\mathbf{k}\varphi(T_\mathbf{x}(\theta))| < M, \quad \forall\theta \in \Theta,\ \forall\mathbf{x},\ \forall\mathbf{k} \in \mathbb{Z}^d, \tag{4.1}$$

where

$$\nabla_\mathbf{k} = (\frac{\partial}{x_1^\mathbf{k}}, \dots, \frac{\partial}{x_{d_\mathbf{k}}^\mathbf{k}}).$$

□

§ 5. Probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ on which the infinite dimensional diffusions are defined

Suppose that we are given a family of functions $a_{ij}^\mathbf{k}$, $\mathbf{k} \in \mathbb{Z}^d$, $1 \le i,j \le d_\mathbf{k}$ on $(\Theta, \overline{\mathcal{B}}, \overline{\mu})$ such that for each $\mathbf{k} \in \mathbb{Z}^d$ and each $1 \le i,j \le d_\mathbf{k}$, $a_{ij}^\mathbf{k}$ is a measurable function on $\Theta_\mathbf{k}$ and there exists $M_2 \in (0,\infty)$ and

$$M_2{}^{-1}\,\|\mathbf{x}\|^2 \le \sum_{1\le i,j\le d_\mathbf{k}} a_{ij}^\mathbf{k}(\theta_\mathbf{k}) x_i\, x_j \le M_2\,\|\mathbf{x}\|^2, \quad \forall\mathbf{k} \in \mathbb{Z}^d,\ \forall\theta_\mathbf{k} \in \Theta_\mathbf{k},$$

$$\forall\mathbf{x} = (x_1, \dots, x_{d_\mathbf{k}}) \in \mathbb{R}^{d_\mathbf{k}}, \tag{5.1}$$

also

$$a_{ij}^\mathbf{k}(\cdot) = a_{ji}^\mathbf{k}(\cdot).$$

We assume that

$$U_\mathbf{k},\ a_{ij}^\mathbf{k} \in \mathcal{D}^\Theta, \quad \mathbf{k} \in \mathbb{Z}^d,\ 1 \le i,j \le d_\mathbf{k}.$$

Also, we assume that there exists a common $M < \infty$ by which the bound (4.1) holds for all $a_{i,j}^\mathbf{k}$ and $U_\mathbf{k}$.

Finally, suppose that we are given a complete probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, $(t \in \mathbb{R}_+)$ with a filtration $\mathcal{F}_t$. On $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ suppose that there exists a system of independent 1-dimensional $\mathcal{F}_t$-adapted Brownian motion processes

$$\{(B^{\mathbf{k},i}(t))_{t\ge0}\}_{\mathbf{k}\in\mathbb{Z}^d,\,1\le i\le d_\mathbf{k}}.$$

Now, for each $\theta \in \Theta$, let

$$X^\theta \equiv \{(X^{\theta,\mathbf{k},i}(t))_{t\geq 0}\}_{\mathbf{k}\in\mathbb{Z}^d,\, 1\leq i\leq d_{\mathbf{k}}}.$$

be the unique solution of

$$\begin{aligned} X^{\theta,\mathbf{k},i}(t) = X^{\theta,\mathbf{k},i}(0) &+ \int_0^t \sum_{1\leq j\leq d_{\mathbf{k}}} \Big\{ \frac{\partial}{\partial x_j^{\mathbf{k}}} a_{ij}^{\mathbf{k}}(T_{X^{\theta,\mathbf{k}}(s)}(\theta)) \\ &-a_{ij}^{\mathbf{k}}(T_{X^{\theta,\mathbf{k}}(s)}(\theta))(\frac{\partial}{\partial x_j^{\mathbf{k}}}(\sum_{\mathbf{k}'\in\{\mathbf{k}\}^+} U_{\mathbf{k}'}(T_{X^\theta(s)}(\theta)))) \Big\} ds \\ &+ \int_0^t \sum_{1\leq j\leq d_{\mathbf{k}}} \sigma_{ij}^{\mathbf{k}}(T_{X^{\theta,\mathbf{k}}(s)}(\theta)) dB^{\mathbf{k},j}(s), \qquad t\geq 0, \end{aligned} \tag{5.2}$$

where, *in the matrix sense*,

$$(\sigma_{ij}^{\mathbf{k}}) = (2a_{ij}^{\mathbf{k}})^{\frac{1}{2}},$$

and

$$X^{\theta,\mathbf{k}}(t) = (X^{\theta,\mathbf{k},1}(t),\ldots,X^{\theta,\mathbf{k},d_{\mathbf{k}}}(t)), \quad \{\mathbf{k}\}^+ = \{\mathbf{k}'|\text{ support of } U_{\mathbf{k}'}\cap\{\mathbf{k}\}\neq\emptyset\},$$

also, by $X^\theta(t)$ we denote the vector

$$(X^{\theta,\mathbf{k}}(t))_{\mathbf{k}\in\mathbb{Z}^d} \in \prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}.$$

To get the unique solution for (5.2) we assume the following:

Assumption 1. *All the coefficients appeared in (5.2) are uniformly bounded and equi-continuous for all* $1\leq i,j\leq d_{\mathbf{k}}$ *and* $\mathbf{k}\in\mathbb{Z}^d$.

□

Then, for each $\theta\in\Theta$, the random variable X^θ on $(\Omega,\mathcal{F},P;\mathcal{F}_t)$ is the one taking values in

$$C([0,\infty)\to\prod_{\mathbf{k}\in\mathbb{Z}^d}\mathbb{R}^{d_{\mathbf{k}}}).$$

Moreover we have

Proposition 5.1. *For each* $\theta\in\Theta$ *and* $n\in\mathbb{N}$, *let* $X_{[n]}^\theta$ *be the unique solution of the SDE which is a modification of (5.2) such that it is the same equation as (5.2) for* $|\mathbf{k}|\leq n$, *and for* $|\mathbf{k}|>n$

$$X_{[n]}^{\theta,\mathbf{k},i}(t) = X^{\theta,\mathbf{k},i}(0), \qquad \forall t\geq 0, \quad 1\leq i\leq d_{\mathbf{k}}.$$

Then $C([0,\infty) \to \prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}})$*-valued random variable* X^θ *on* $(\Omega,\mathcal{F},P;\mathcal{F}_t)$ *is the limit of the sequence of* $C([0,\infty) \to \prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}})$*-valued random variables* $\{X^\theta_{[n]}\}_{n\in\mathbb{N}}$ *on* $(\Omega,\mathcal{F},P;\mathcal{F}_t)$ *such that*

$$\lim_{n\to\infty} E[\rho(X^\theta, X^\theta_{[n]})] = 0,$$

where the metric ρ *on* $C([0,\infty) \to \prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}})$ *is given by*

$$\rho(\mathbf{x}(\cdot),\mathbf{x}'(\cdot)) = \sum_{n\in\mathbb{N}} \frac{1}{2^n}\Big\{\sup_{0\le t\le n}\Big\{\sum_{\mathbf{k}\in\mathbb{Z}^d}\frac{1}{M^{|\mathbf{k}|}}\sum_{1\le i\le d_{\mathbf{k}}}|x^{\mathbf{k},i}(t)-x'^{\mathbf{k},i}(t)|\Big\}\wedge 1\Big\}.$$

□

Through $X^\theta = (X^\theta(t))_{t\ge 0}$ we define a Θ valued process such that

$$\{T_{X^\theta(t)}(\theta)\}_{t\ge 0}.$$

Proposition 5.2. *i) If* $T_{X^\theta(0)}(\theta) = T_{X^{\theta'}(0)}(\theta'), \quad P-a.e. \quad \omega\in\Omega,$ *then*

$$(X^\theta(t)-X^\theta(0))_{t\ge 0} = (X^{\theta'}(t)-X^{\theta'}(0))_{t\ge 0}, \qquad P-a.e. \quad \omega\in\Omega,$$

ii) For

$$\theta' = T_{X^\theta(0)}(\theta), \tag{5.3}$$

$$(T_{X^\theta(t)}(\theta))_{t\ge 0} = (T_{X_0^{\theta'}(t)}(\theta'))_{t\ge 0}, \qquad P-a.e. \quad \omega\in\Omega, \tag{5.4}$$

where $X_0^{\theta'}(t)$ *is the diffusion defined by (5.2) with* $X_0^{\theta'}(0) = \mathbf{0}$ *and replacing* θ *by* θ' *in it.*
For the case where (5.3) holds, setting

$$(Y_{\theta'}(t))_{t\ge 0} \equiv (T_{X_0^{\theta'}(t)}(\theta'))_{t\ge 0}$$

we have:
iii) The process $(Y_{\theta'}(t))_{t\ge 0}$ *satisfies* $Y_{\theta'}(0) = \theta'$*,* $\forall\theta'\in\Theta$*, and*

$$\int_\Theta E[\varphi(Y_{\theta'}(t))]\,\mu(d\theta') = \int_\Theta \varphi(\theta')\,\mu(d\theta'), \qquad \forall t\ge 0,\ \forall\varphi\in\mathcal{D}^\Theta.$$

□

For each $\theta\in\Theta$ let us define two σ-fields $\sigma(T_\cdot(\theta))$ and $(T_\cdot(\theta))^{-1}(\overline{\mathcal{B}})$ which are sub σ-fields of $\mathcal{B}(\prod_{\mathbf{k}\in D}\mathbb{R}^{d_{\mathbf{k}}})$ as follows:

$$\sigma(T_\cdot(\theta)) \equiv \text{the totality of } A\in\mathcal{B}(\textstyle\prod_{\mathbf{k}\in D}\mathbb{R}^{d_{\mathbf{k}}}) \text{ such that}$$

$$\exists B \in \Theta \quad \text{and} \quad A = \{\mathbf{x} \,|\, T_{\mathbf{x}}(\theta) \in B\}.$$

$$(T_{\cdot}(\theta))^{-1}(\overline{\mathcal{B}}) \equiv \{A \in \mathcal{B}(\prod_{\mathbf{k}\in D} \mathbb{R}^{d_{\mathbf{k}}}) \,|\, \exists B \in \overline{\mathcal{B}} \text{ and } T_{\mathbf{x}}(\theta) \in B\}.$$

Proposition 5.3. *By Proposition 5.2, and the Markov property of* $(X^\theta(t))_{t\geq 0}$ *(noting that this process is the unique (strong) solution of the SDE (5.2)), it is possible to define Markovian semi-groups* $p_t^{X,\theta}$ *resp.* p_t^Y *corresponding to the processes* $(X^\theta(t))_{t\geq 0}$ *resp.* $(Y_{\cdot}(t))_{t\geq 0}$ *as follows:*
i) Let $\mathcal{D}^X$ *be the totality of* f *such that* $f \in C_b^2(\prod_{\mathbf{k}\in D} \mathbb{R}^{d_{\mathbf{k}}} \to \mathbb{R})$ *,with some bounded* $D \subset\subset \mathbb{Z}^d$*, on which all partial derivatives of* f *are bounded.*
For $f \in \mathcal{D}^X$*,*

$$(p_t^{X,\theta} f)(\mathbf{x}) \equiv E[f(X^\theta(t)) \,|\, X^\theta(0) = \mathbf{x}], \qquad \mathbf{x} \in \prod_{\mathbf{k}} \mathbb{R}^{d_{\mathbf{k}}},$$

defines a $\mathcal{C}_0$ *semi-group on* $\mathcal{D}^X$*.*
ii) For each $\theta \in \Theta$*, if*

$$\sigma(T_{\cdot}(\theta)) \subset (T_{\cdot}(\theta))^{-1}(\overline{\mathcal{B}}),$$

then there exists a $\overline{\mathcal{B}}$*-measurable function* $(p_t^Y \varphi)(\cdot)$ *on* $(\Theta, \overline{\mathcal{B}}, \overline{\mu})$ *such that for* $\varphi \in \mathcal{D}^\Theta$

$$(p_t^Y \varphi)(T_{X^\theta(s)}(\theta)) = E[\varphi(T_{X^\theta(s+t)}(\theta)) \,|\, T_{X^\theta(s)}(\theta)], \quad \forall s,\, t \geq 0,\ \forall \theta \in \Theta,$$

$$P - a.s.\, \omega \in \Omega,$$

and $\{p_t^Y\}_{t\geq 0}$ *is a a* $\mathcal{C}_0$ *semi-group on* $L^2(\mu)$*.*

□

§ 6. Statement of the problem on $(\Omega \times \Theta, \mathcal{F} \times \overline{\mathcal{B}}, P \times \overline{\mu}; \mathcal{F}_t \times \{\Theta, \emptyset\})$

Consider the probability space $(\Omega \times \Theta, \mathcal{F} \times \overline{\mathcal{B}}, P \times \overline{\mu}; \mathcal{F}_t \times \{\Theta, \emptyset\})$, then $\{(X_0^\theta(t))_{t\geq 0}, (Y_\theta(t))_{t\geq 0})\}$ is a random variable on this probability space (cf. Proposition 5.2).

Problem. For each $\theta \in \Theta$, $\mu - a.s.$, we are concerning the scaling limit of $(X_0^\theta(t))_{t\geq 0}$ such that

$$\lim_{\epsilon\downarrow 0} \{\epsilon X_0^\theta(\frac{t}{\epsilon^2})\}_{t\geq 0} \tag{6.1}$$

More precisely, we consider the weak convergence of (6.1), where the sequence of the processes $\{\epsilon X_0^\theta(\frac{t}{\epsilon^2})\}_{t\geq 0}$ is understood as the sequence of random variables on $(\Omega\times\Theta, \mathcal{F}\times\overline{\mathcal{B}}, P\times\overline{\mu}; \mathcal{F}_t\times\{\Theta,\emptyset\})$ taking values in the direct product space $\prod_{\mathbf{k}\in\mathbb{Z}^d} C([0,\infty)\to\mathbb{R}^{d_{\mathbf{k}}})$ equipped with the direct product topology.

□

Definition 6.1. For each $\mathbf{k}\in\mathbb{Z}^d$ and $i=1,\dots,d_{\mathbf{k}}$, define an operator $D^{\mathbf{k},i}:\mathcal{D}^\Theta\to\mathcal{D}^\Theta$ such that

$$(D^{\mathbf{k},i}\varphi)(\theta)\equiv\frac{\partial}{\partial x_i^{\mathbf{k}}}\varphi(T_{\mathbf{x}}(\theta))|_{\mathbf{x}=0},\qquad \varphi\in\mathcal{D}^\Theta,\quad \theta\in\Theta.$$

Also, define a quadratic form $\mathcal{E}$ on $L^2(\mu)$ such that

$$\mathcal{E}(\varphi,\psi)\equiv\sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{1\leq i,j\leq d_{\mathbf{k}}}\int_\Theta(D^{\mathbf{k},i}\varphi)(\theta)\,a_{i,j}^{\mathbf{k}}(\theta)(D^{\mathbf{k},j}\psi)(\theta)\,\mu(d\theta),\qquad \varphi,\psi\in\mathcal{D}^\Theta.$$

□

By (2.1), (2.2), (2.3), (5.2) and the above definition we can show that the following holds:

Proposition 6.2. *The probability law of $(Y_\theta(t))_{t\geq 0})$ is identical with the probability law of a Markov process that corresponds to a Dirichlet form, a Markovian closed extention $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ of the quadratic form $\mathcal{E}$ on $L^2(\mu)$ defined on the dense domain $\mathcal{D}^\Theta$ (cf. Def. 6.1). The analytic property of the semigroup p_t^Y can be investigated through $\mathcal{E}$ on $L^2(\mu)$.*

□

Next, we put a key *assumption* yielding the ergodicity of the processes that we are discussing:

Assumption 2. *There exist $c>0$, $\alpha>1$ and $K>0$ such that*

$$\|p_t^Y\varphi-<\varphi,\mu>\|_{L^2(\mu)}\leq K(c+t)^{-\alpha}\|\varphi\|_{L^2(\mu)},\quad \forall t>0, \forall\varphi\in\mathcal{D}^\Theta. \tag{6.2}$$

□

(6.2) has the form of a Poincaré inequality and was also a key assumption in [ABRY1,2].

Proposition 6.3. *Assume that Assumptions 1,2 and the assumption of Proposition 5.3 hold, then the following holds:*
For each $\mathbf{k} \in \mathbb{Z}^d$, $i = 1, \dots, d_{\mathbf{k}}$ *let*

$$b_i^{\mathbf{k}}(\theta) \equiv \sum_{1 \le j \le d_{\mathbf{k}}} \Big\{ (D^{\mathbf{k},j} a_{ij}^{\mathbf{k}})(\theta) - a_{ij}^{\mathbf{k}}(\theta)(D^{\mathbf{k},j}(\sum_{\mathbf{k}' \in \{\mathbf{k}\}^+} U_{\mathbf{k}'})(\theta)) \Big\}, \quad \theta \in \Theta,$$

then (cf. Prop. 5.3 and Prop. 6.1)
i)

$$\chi_i^{\mathbf{k}}(\cdot) \equiv \int_0^\infty (p_t^Y b_i^{\mathbf{k}})(\cdot) dt \in \mathcal{D}(\mathcal{E}).$$

$\chi_i^{\mathbf{k}}(\theta)$ *satisfies*
ii)

$$\mathcal{E}(\chi_i^{\mathbf{k}}, \varphi) = \int_\Theta b_i^{\mathbf{k}}(\theta)\, \varphi(\theta)\, \mu(d\theta), \qquad \forall \varphi \in \mathcal{D}^\Theta. \tag{6.3}$$

iii) *For each* $\mathbf{k} \in \mathbb{Z}^d$ *and each* $0 \le i \le d_{\mathbf{k}}$ *let*

$$\rho_j^{\mathbf{k},i}(\theta) = \sigma_{ij}^{\mathbf{k}}(\theta) + \sum_{1 \le p \le d_{\mathbf{k}}} \sigma_{pj}^{\mathbf{k}}(\theta)(D^{\mathbf{k},p}\chi_i^{\mathbf{k}})(\theta), \quad 1 \le j \le d_{\mathbf{k}}, \tag{6.4}$$

and

$$\rho_{\mathbf{k}',j}^{\mathbf{k},i}(\theta) = \sum_{1 \le p \le d_{\mathbf{k}'}} \sigma_{p,j}^{\mathbf{k}'}(\theta)(D^{\mathbf{k}',p}\chi_i^{\mathbf{k}})(\theta), \qquad 1 \le j \le d_{\mathbf{k}'}, \tag{6.5}$$

then there exists $M_3 < \infty$ *and*

$$\sum_{j=1}^{d_{\mathbf{k}}} \|\rho_j^{\mathbf{k},i}\|_{L^2(\mu)} + \sum_{\mathbf{k}' \ne \mathbf{k}} \sum_{j=1}^{d_{\mathbf{k}'}} \|\rho_{\mathbf{k}',j}^{\mathbf{k},i}\|_{L^2(\mu)} \le M_3, \qquad \forall \mathbf{k} \in \mathbb{Z}^d. \tag{6.6}$$

□

§ 7. The result

By Proposition 6.2 we can show that for μ-a.s. $\theta \in \Theta$, the following holds:

$$X_0^{\theta,\mathbf{k},i}(\cdot) = \big(\chi_i^{\mathbf{k}}(T_{X_0^\theta(\cdot)}(\theta)) - \chi_i^{\mathbf{k}}(\theta)\big) + M_i^{\mathbf{k}}(\cdot), \qquad \forall \mathbf{k} \in \mathbb{Z}^d,\ 1 \le i \le d_{\mathbf{k}},\ P-a.s., \tag{7.1}$$

where $\{M_i^{\mathbf{k}}(t)\}_{t \ge 0}$ is an $\{\mathcal{F}_t\}$ martimgale such that
(7.2)

$$M_i^{\mathbf{k}}(t) = \sum_{1 \le i \le d_{\mathbf{k}}} \int_0^t \rho_j^{\mathbf{k},i}(T_{X_0^\theta(s)}(\theta)) dB_j^{\mathbf{k}}(s) + \sum_{\mathbf{k}' \ne \mathbf{k}} \sum_{1 \le j \le d_{\mathbf{k}'}} \int_0^t \rho_{\mathbf{k}',j}^{\mathbf{k},i}(T_{X_0^\theta(s)}(\theta)) dB_j^{\mathbf{k}'}(s).$$

Let

$$\overline{\rho_j^{\mathbf{k},i}} =< \rho_j^{\mathbf{k},i}(\cdot), \mu >, \qquad \overline{\rho_{\mathbf{k}',j}^{\mathbf{k},i}} =< \rho_{\mathbf{k}',j}^{\mathbf{k},i}(\cdot), \mu > \tag{7.3}$$

then, again by Propsition 6.2 and 6.3 we can define the following:

(7.4)

$$\overline{X^{\mathbf{k},i}}(t) = \sum_{1\le i\le d_{\mathbf{k}}} \overline{\rho_j^{\mathbf{k},i}} B_j^{\mathbf{k}}(t) + \sum_{\mathbf{k}'\neq\mathbf{k}} \sum_{1\le j\le d_{\mathbf{k}'}} \overline{\rho_{\mathbf{k}',j}^{\mathbf{k},i}} B_j^{\mathbf{k}'}(t), \quad t\ge 0,\, \mathbf{k}\in\mathbb{Z}^d,\, 1\le i\le d_{\mathbf{k}}.$$

We denote

$$\overline{X} = \left\{ \{\overline{X^{\mathbf{k},i}}(t)\}_{t\ge 0} \right\}_{\mathbf{k}\in\mathbb{Z}^d,\, 1\le i\le d_{\mathbf{k}}}. \tag{7.5}$$

In order to state the final result, we put an assumption:

Assumption 3. *There exists a number $M_4 < \infty$ such that the following holds:*

$$|\chi_i^{\mathbf{k}}(\theta)| \le M_4, \qquad \forall \mathbf{k}\in\mathbb{Z}^d,\, 1\le i\le d_{\mathbf{k}}, \quad \forall\theta\in\Theta.$$

□

By using Assumption 2, Propositions 5.2, 6.2 and the expressions (7.1), (7.2) and (7.4), since boundedness Assumptions 1 and 3 are assumed here, through the same discussions on $\{\epsilon X_0^\theta(\frac{t}{\epsilon^2})\}_{t\ge 0}$ performed in [ABRY1,2,3], we are able to derive the following result:

Theorem 7.1. *Assume that Assumptions 1,2,3 hold. Then for μ-a.e. $\theta\in\Theta$, the scaling process $\{\epsilon X_0^\theta(\frac{t}{\epsilon^2})\}_{t\ge 0}$ with the scaling parameter $\epsilon > 0$ converges weakly, as $\epsilon\downarrow 0$, to $\overline{X}$ defined by (7.4) and (7.5). $\overline{X}$ is a Gaussian process with a constant covariance matrix characterized by $\sigma_{ij}^{\mathbf{k}}$, $\chi_i^{\theta,\mathbf{k}}$, $\mathbf{k}\in\mathbb{Z}^d$, $1\le i,j\le d_{\mathbf{k}}$.*

□

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