Weighted Besov-Morrey spaces and Triebel-Lizorkin spaces

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Introduction

The aim of this note is to review and unify the authors' recent papers [7, 20, 22], concerning the Besov spaces, the Triebel-Lizorkin spaces, the Morrey spaces, the Besov-Morrey spaces and the Triebel-Lizorkin-Morrey spaces. From the works [7, 20, 22] we conclude that the proof does not depend heavily on the structure of the function spaces. The authors begin to be aware of the fact that in order to obtain some (atomic) decomposition, we have only to require some elementary axioms about the function spaces. One of such axims is the boundedness of the powered maximal operator. As an example, in the present paper we develop the theory of the Besov-Morrey spaces and the Triebel-Lizorkin-Morrey spaces coming with an $A_p^{\rm loc}$-weight. Analogous results will be obtained, for example, the Orlicz-Besov spaces and so on as long as the function space satisfies the (local) maximal inequality. We define the weighted Besov-Morrey spaces and the weighted Triebel-Lizorkin-Morrey spaces with the underlying weight w in $A_p^{\rm loc}$. After defining the function spaces, we formulate the atomic decomposition. Here we content ourselves with the formulation of the atomic decomposition. The precise proof will be published elsewhere.

This paper consists of four parts. The first part is devoted to the review of [20, 22]. The second part is the weighted version of the first part. In the third part, which is the heart of this paper, we consider the Besov-Morrey spaces and the Triebel-Lizorkin-Morrey spaces coming with $A_\infty^{\rm loc}$-weights. As a preliminary step we investigate the function spaces coming with $A_\infty^{\rm loc}$-weights. $A_\infty^{\rm loc}$ will be of importance in the various field of mathematics such as differential geometry and computational science, because it contains weights of exponential order. Finally the fourth part contains two open problems on Morrey spaces.

Part I

Unweighted Besov-Morrey spaces and unweighted Triebel-Lizorkin spaces

1 Introduction

The Besov-Morrey space emerged originally in [11]. H. Kozono and M. Yamazaki investigated time-local solutions of the Navier-Stokes equations. Later it was investigated by A. Mazzucato. Mazzucato investigated the atomic decomposition and the molecular decomposition [13, 14]. In [11, 13, 14] the authors developed a theory of the function space $\mathcal{N}^s_{pqr}$ with $1 \le q \le p < \infty$, $1 \le r \le \infty$ and $s \in \mathbb{R}$. L. Tang and J. Xu defined the function spaces $\mathcal{N}^s_{pqr}$ and $\mathcal{E}^s_{pqr}$ with $0 < q \le p < \infty$, $0 < r \le \infty$ and $s \in \mathbb{R}$ (see [25]). The present authors developed a theory of decompositions in $\mathcal{N}^s_{pqr}$ and $\mathcal{E}^s_{pqr}$ with $0 < q \le p < \infty$, $0 < r \le \infty$ and $s \in \mathbb{R}$. This type of decomposition results will be of more importance because it provides us with a convenient way of analysis. For example, the synthesis result covers a part of theory of wavelet analysis. For more details of this approach we refer to [8].

Before we go into the definitions of the Besov-Morrey spaces and the Triebel-Lizorkin-Morrey spaces, let us recall the definitions of the Besov spaces and the Triebel-Lizorkin spaces, which are prototypes of the Besov-Morrey spaces and the Triebel-Lizorkin-Morrey spaces respectively. To describe these function spaces, we fix some notations. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Define the Fourier transform and its inverse by

$$\mathcal{F}f(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx,\ \mathcal{F}^{-1}f(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}\,d\xi$$

for $f \in L^1(\mathbb{R}^n)$. Denote by χ_E the indicator function of a set E. $B(r)$ means the open ball centered at the origin of radius $r > 0$. Let $\{f_j\}_{j\in\mathbb{N}_0}$ be a sequence of functions. Then define

$$\|\{f_j\}_{j\in\mathbb{N}_0} \,:\, l_q(L_p)\| := \left(\sum_{j\in\mathbb{N}_0} \|f_j \,:\, L_p\|^q\right)^{\frac{1}{q}},\ \|\{f_j\}_{j\in\mathbb{N}_0} \,:\, L_p(l_q)\| := \left\|\left(\sum_{j\in\mathbb{N}_0} |f_j|^q\right)^{\frac{1}{q}} \,:\, L_p\right\|$$

for $0 < p, q \le \infty$. Here a natural modification is made if $q = \infty$. Next we fix a sequence of smooth functions $\{\varphi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ so that

$$\chi_{B(2)} \le \varphi_0 \le \chi_{B(4)},\ \chi_{B(4)\setminus B(2)} \le \varphi_1 \le \chi_{B(8)\setminus B(1)},\ \varphi_j = \varphi_1(2^{-j+1}\cdot)$$

for $j \in \mathbb{N}$. Given $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\tau \in \mathcal{S}(\mathbb{R}^n)$, we define $\tau(D)f := \mathcal{F}^{-1}(\tau \cdot \mathcal{F}f)$.

Under these notations, we define the Besov norm and the Triebel-Lizorkin norm. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. We define

$$\|f \,:\, B^s_{pq}(\mathbb{R}^n)\| := \|\{2^{js}\phi_j(D)f\}_{j\in\mathbb{N}_0} \,:\, l_q(L_p)\|,\ 0 < p \le \infty,\ 0 < q \le \infty,\ s \in \mathbb{R}$$
$$\|f \,:\, F^s_{pq}(\mathbb{R}^n)\| := \|\{2^{js}\phi_j(D)f\}_{j\in\mathbb{N}_0} \,:\, L_p(l_q)\|,\ 0 < p < \infty,\ 0 < q \le \infty,\ s \in \mathbb{R}.$$

Different admissible choices of ϕ_0 and ϕ_1 will yield equivalent quasi-norms. To unify the formulation in the sequel, as was defined in [30], for example, we use $A^s_{pq}(\mathbb{R}^n)$ to denote either $B^s_{pq}(\mathbb{R}^n)$ or $F^s_{pq}(\mathbb{R}^n)$ with $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$.

Let us describe the function spaces $\mathcal{N}^s_{pqr}$ and $\mathcal{E}^s_{pqr}$ briefly. Suppose that the parameters p, q, r, s satisfy

$$0 < q \le p < \infty,\ 0 < r \le \infty,\ s \in \mathbb{R}.$$

Define the Morrey norm of a measurable function f by

$$\|f\ :\ \mathcal{M}^p_q\| := \sup_{x \in \mathbb{R}^n,\, r>0} r^{\frac{n}{p}-\frac{n}{q}} \left(\int_{B(x,r)} |f|^q \right)^{\frac{1}{q}}. \tag{1}$$

The Besov-Morrey spaces and the Triebel-Lizorkin-Morrey spaces are obtained by replacing the L_p norm with the Morrey norm $\| \cdot\ :\ \mathcal{M}^p_q\|$ given by (1). Given a sequence of measurable functions $\{f_j\}_{j\in\mathbb{N}_0}$, we define

$$\|\{f_j\}_{j\in\mathbb{N}_0}\ :\ l_r(\mathcal{M}^p_q)\| := \left(\sum_{j\in\mathbb{N}_0} \|f_j\ :\ \mathcal{M}^p_q\|^r \right)^{\frac{1}{r}}$$

$$\|\{f_j\}_{j\in\mathbb{N}_0}\ :\ \mathcal{M}^p_q(l_r)\| := \left\| \left(\sum_{j\in\mathbb{N}_0} |f_j|^r \right)^{\frac{1}{r}}\ :\ \mathcal{M}^p_q \right\|,$$

for $0 < q \le p < \infty$, $0 < r \le \infty$.

Definition 1.1. [11, 25] Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then define

$$\|f\ :\ \mathcal{N}^s_{pqr}(\mathbb{R}^n)\| := \|\{2^{js}\phi_j(D)f\}_{j\in\mathbb{N}_0}\ :\ l_r(\mathcal{M}^p_q)\|$$
$$\|f\ :\ \mathcal{E}^s_{pqr}(\mathbb{R}^n)\| := \|\{2^{js}\phi_j(D)f\}_{j\in\mathbb{N}_0}\ :\ \mathcal{M}^p_q(l_r)\|$$

for $0 < q \le p < \infty$, $0 < r \le \infty$, $s \in \mathbb{R}$. $\mathcal{N}^s_{pqr}$ and $\mathcal{E}^s_{pqr}$ are the set of all Schwartz distributions f for which the norms are finite. $\mathcal{A}^s_{pqr}$ denotes either $\mathcal{N}^s_{pqr}$ or $\mathcal{E}^s_{pqr}$.

The crucial property is as follows:

Theorem 1.2. [25] *The function space $\mathcal{A}^s_{pqr}$ does not depend on the particular choice of $\{\phi_j\}_{j\in\mathbb{N}_0}$. The function space $\mathcal{A}^s_{pqr}$ is a quasi-Banach space.*

2 Some elementary properties

Concrete spaces The function space $\mathcal{A}^s_{pqr}(\mathbb{R}^n)$ covers many families of function spaces such as the Hölder-Zygmund space $\mathcal{C}^s(\mathbb{R}^n)$, the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$, the Sobolev-Morrey space, the Besov space $B^s_{pq}(\mathbb{R}^n)$ and the Triebel-Lizorkin space $F^s_{pq}(\mathbb{R}^n)$. As for the Sobolev-Morrey space, we refer to [15, 16, 17]. Recall that the Hölder space $\mathcal{C}^s(\mathbb{R}^n)$, $0 < s < 1$ is a set of all continuous functions normed by

$$\|f\ :\ \mathcal{C}^s(\mathbb{R}^n)\| := \|f\ :\ L_\infty\| + \sup_{\substack{x,y\in\mathbb{R}^n \\ x\ne y}} \frac{|f(x)-f(y)|}{|x-y|^s}.$$

Proposition 2.1. *Suppose that $0 < q \le p < \infty$, $0 < r \le \infty$ and $s \in \mathbb{R}$.*

1. *Let $k \in \mathbb{N}$. Then $f \in \mathcal{A}^s_{pqr}(\mathbb{R}^n)$ if and only if $f \in \mathcal{A}^{s-k}_{pqr}(\mathbb{R}^n)$ and $\partial_j{}^k f \in \mathcal{A}^{s-k}_{pqr}(\mathbb{R}^n)$ for every $j = 1, 2, \ldots, n$. Furthermore, we have the following norm equivalence*

$$\|f : \mathcal{A}^s_{pqr}(\mathbb{R}^n)\| \simeq \|f : \mathcal{A}^{s-k}_{pqr}(\mathbb{R}^n)\| + \sum_{j=1}^{n} \|\partial_j{}^k f : \mathcal{A}^{s-k}_{pqr}(\mathbb{R}^n)\|$$

   for $f \in \mathcal{A}^s_{pqr}(\mathbb{R}^n)$.
2. *$\mathcal{A}^s_{ppr}(\mathbb{R}^n) = A^s_{pr}(\mathbb{R}^n)$.*
3. *$\mathcal{E}^0_{pq2}(\mathbb{R}^n) = \mathcal{M}^p_q(\mathbb{R}^n)$, if $1 < q \le p < \infty$.*
4. *$B^s_{\infty\infty}(\mathbb{R}^n) = \mathcal{C}^s(\mathbb{R}^n)$, if $s \in (0, 1)$.*

Assertions *1,3,4* can be found in [25, Proposition 2.15], [13, Proposition 4.1] and [24] respectively, while the assertion *2* is immediate from the definition.

Proposition 2.2. *Let the parameters $p, q, r, r_1, r_2, s, \varepsilon$ satisfy*

$$0 < q \le p < \infty,\ 0 < r, r_1, r_2 \le \infty,\ s \in \mathbb{R},\ \varepsilon > 0.$$

Then we have

1. *$\mathcal{N}^{s+\varepsilon}_{pqr_1}(\mathbb{R}^n) \subset \mathcal{E}^s_{pqr_2}(\mathbb{R}^n)$ and $\mathcal{E}^{s+\varepsilon}_{pqr_1}(\mathbb{R}^n) \subset \mathcal{N}^s_{pqr_2}(\mathbb{R}^n)$.*
2. *$\mathcal{A}^s_{pqr_1}(\mathbb{R}^n) \subset \mathcal{A}^s_{pqr_2}(\mathbb{R}^n)$, if $r_1 \le r_2$.*
3. *$\mathcal{N}^s_{pq\min(q,r)}(\mathbb{R}^n) \subset \mathcal{E}^s_{pqr}(\mathbb{R}^n) \subset \mathcal{N}^s_{pq\infty}(\mathbb{R}^n)$.*

The proof of this proposition is straightforward.

Proposition 2.3. *Let $0 < q \le p < \infty$, $0 < r \le \infty$ and $s > \dfrac{n}{p}$. Then we have $\mathcal{A}^s_{pqr}(\mathbb{R}^n) \subset BUC(\mathbb{R}^n)$, where $BUC(\mathbb{R}^n)$ denotes a set of all uniformly continuous and bounded functions.*

Proof. A minor modification of the proof of [22, Proposition 3.7] readily gives us the inclusion $\mathcal{A}^s_{pqr}(\mathbb{R}^n) \subset B^{s-\frac{n}{p}}_{\infty\infty}(\mathbb{R}^n)$. Since it is known in [27, Chapter 2] that $B^{s-\frac{n}{p}}_{\infty\infty}(\mathbb{R}^n) \subset BUC(\mathbb{R}^n)$ (the proof is simple), we obtain the desired result. ∎

3 Decompositions of the function spaces

Atomic decomposition

To describe the atomic decomposition, we introduce some notations, which are based on those in [29, 30]. Now we follow [22, Section 4] to formulate the atomic decomposition.

We define two indices σ_q and σ_{qr} by

$$\sigma_q := \frac{1}{\min(1, q)} - 1,\ \sigma_{qr} := \max(\sigma_q, \sigma_r).$$

Notation . 1. Let $\nu \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Then define $Q_{\nu m} := \prod_{j=1}^{n} \left[\dfrac{m_j}{2^\nu}, \dfrac{m_j + 1}{2^\nu}\right]$. Call $Q_{\nu m}$ a dyadic cube in $\mathbb{R}^n$ for each $\nu \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$.

2. Let $0 < p < \infty$, $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. $\chi_{\nu m}^{(p)}$ denotes the p-normalized indicator given by

$$\chi_{\nu m}^{(p)} := 2^{\frac{\nu n}{p}} \chi_{Q_{\nu m}}.$$

3. Given a doubly indexed sequence $\lambda = \{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$, define

$$\|\lambda \,:\, \mathbf{n}_{pqr}(\mathbb{R}^n)\| := \left\| \left\{ \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{\nu m}^{(p)} \right\}_{\nu \in \mathbb{N}_0} \,:\, l_r(\mathcal{M}_q^p) \right\|, \; 0 < q \le p \le \infty, \; 0 < r \le \infty$$

$$\|\lambda \,:\, \mathbf{e}_{pqr}(\mathbb{R}^n)\| := \left\| \left\{ \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{\nu m}^{(p)} \right\}_{\nu \in \mathbb{N}_0} \,:\, \mathcal{M}_q^p(l_r) \right\|, \; 0 < q \le p \le \infty, \; 0 < r \le \infty.$$

In order to unify the formulations in the sequel, denote by $\mathbf{a}_{pqr}(\mathbb{R}^n)$ either $\mathbf{n}_{pqr}(\mathbb{R}^n)$ or $\mathbf{e}_{pqr}(\mathbb{R}^n)$. We rule out $\mathbf{a}_{pqr}(\mathbb{R}^n)$ with $p = \infty$ and $r < \infty$.

Following [29], let us recall the definition of atoms.

Definition 3.1. Let $0 < p < \infty$ and $s \in \mathbb{R}$. Fix $K \in \mathbb{N}_0$, $L \in \mathbb{Z} \cap [-1, \infty)$ and $d > 1$.

1. Suppose further that $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. A C^K-function a is said to be an atom centered at $Q_{\nu m}$, if it is supported on $d\,Q_{\nu m}$ and satisfies the differential inequality and the moment condition given below:

$$\|\partial^\alpha a \,:\, L_\infty\| \le 2^{-\nu\left(s - \frac{n}{p}\right) + \nu|\alpha|} \text{ for } \alpha \in {\mathbb{N}_0}^n \text{ with } |\alpha| \le K \tag{2}$$

and

$$\int_{\mathbb{R}^n} x^\beta a(x)\, dx = 0 \text{ for } \beta \in {\mathbb{N}_0}^n \text{ with } |\beta| \le L, \nu \ge 1. \tag{3}$$

Here condition (3) means no condition, if $L = -1$.

2. Define

$$\mathbf{Atom}_0 := \{\{a_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \,:\, \text{ each } a_{\nu m} \text{ is an atom centered at } Q_{\nu m}\},$$
$$\mathbf{Atom} := \{\{a_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \,:\, \{c\, a_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{Atom}_0 \text{ for some } c > 0\}.$$

With this definition in mind, we formulate our atomic decomposition theorem.

Theorem 3.2 (Atomic decomposition). *Suppose that the parameters $K, L \in \mathbb{Z}$ and $p, q, r, s, d \in \mathbb{R}$ satisfy*

$$0 < q \le p < \infty, \; 0 < r \le \infty, \; d > 1, \; K \ge (1 + [s])_+, \; L \ge \max(-1, [\sigma_q - s])$$

for the $\mathcal{N}$-scale and

$$0 < q \le p < \infty, \; 0 < r \le \infty, \; d > 1, \; K \ge (1 + [s])_+, \; L \ge \max(-1, [\sigma_{qr} - s])$$

for the $\mathcal{E}$-scale.

1. *Assume that $\{a_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{Atom}_0$ and $\lambda = \{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{a}_{pqr}(\mathbb{R}^n)$. Then the sum*

$$f := \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and belongs to $\mathcal{A}_{pqr}^s(\mathbb{R}^n)$ with the norm estimate

$$\|f \,:\, \mathcal{A}_{pqr}^s(\mathbb{R}^n)\| \le c\, \|\lambda \,:\, \mathbf{a}_{pqr}(\mathbb{R}^n)\|.$$

Here the constant c does not depend on $\{a_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ nor λ.

2. *Conversely any $f \in \mathcal{A}^s_{pqr}(\mathbb{R}^n)$ admits the following decomposition:*

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}.$$

The sum converges in $\mathcal{S}'(\mathbb{R}^n)$. Here $\{a_{\nu m}\}_{\nu \in \mathbb{N}_0,\, m \in \mathbb{Z}^n} \in \mathbf{Atom}_0$ and the coefficient $\lambda = \{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0,\, m \in \mathbb{Z}^n} \in \mathbf{a}_{pqr}(\mathbb{R}^n)$ fulfills the norm estimate

$$\|\lambda \,:\, \mathbf{a}_{pqr}(\mathbb{R}^n)\| \le c\,\|f \,:\, \mathcal{A}^s_{pqr}(\mathbb{R}^n)\|.$$

In view of our actual construction, unfortunately the coefficient λ does not depend linearly on f. We refer to [4, 22, 29] for more details.

Quarkonial decomposition

As we have noted, the atomic decomposition fails to enjoy linear dependency for the constants. In [29, 30] H. Triebel proposed the quarkonial decomposition. The counterpart for $\mathcal{A}^s_{pqr}(\mathbb{R}^n)$ was obtained in [22, Section 5]. Following [30, Chapter 1, Section 1] and [22, Section 5], we describe the quarkonial decomposition for $\mathcal{A}^s_{pqr}(\mathbb{R}^n)$.

Notation . 1. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a compactly supported function that generates the partition of the unit:

$$\sum_{m \in \mathbb{Z}^n} \psi(x-m) \equiv 1$$

with $\mathrm{supp}(\psi) \subset B(2^{\tilde{d}})$.

2. For $\beta \in \mathbb{N}_0$ define $\psi^\beta(x) := x^\beta \psi(x)$.

3. For $\beta \in \mathbb{N}_0$, $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, define $(\beta qu)_{\nu m}(x) := 2^{-\nu\left(s-\frac{n}{p}\right)} \psi^\beta(2^\nu x - m)$.

4. Let $\rho > \tilde{d}$, where $\tilde{d}$ is a fixed constant appearing just above. Given a triply indexed sequence $\lambda = \{\lambda^\beta_{\nu m}\}_{\beta \in \mathbb{N}_0{}^n,\, \nu \in \mathbb{N}_0,\, m \in \mathbb{Z}^n}$, we define

$$\begin{aligned}
\|\lambda \,:\, \mathbf{n}_{pqr}(\mathbb{R}^n)\|_\rho &:= \sup_{\beta \in \mathbb{N}_0{}^n} 2^{\rho|\beta|} \|\lambda^\beta \,:\, \mathbf{n}_{pqr}(\mathbb{R}^n)\|,\ 0 < q \le p \le \infty,\ 0 < r \le \infty \\
\|\lambda \,:\, \mathbf{e}_{pqr}(\mathbb{R}^n)\|_\rho, &:= \sup_{\beta \in \mathbb{N}_0{}^n} 2^{\rho|\beta|} \|\lambda^\beta \,:\, \mathbf{e}_{pqr}(\mathbb{R}^n)\|,\ 0 < q \le p \le \infty,\ 0 < r \le \infty,
\end{aligned}$$

where $\lambda^\beta := \{\lambda^\beta_{\nu m}\}_{\nu \in \mathbb{N}_0,\, m \in \mathbb{Z}^n}$.

5. We denote by $\|\lambda \,:\, \mathbf{a}_{pqr}(\mathbb{R}^n)\|_\rho$ to denote either $\|\lambda \,:\, \mathbf{n}_{pqr}(\mathbb{R}^n)\|_\rho$ or $\|\lambda \,:\, \mathbf{e}_{pqr}(\mathbb{R}^n)\|_\rho$ as usual. $\mathbf{e}_{\infty qr}(\mathbb{R}^n)$ with $r < \infty$ will not appear for later considerations.

Theorem 3.3 (Quarkonial decomposition for the regular case). *Suppose that the parameters p, q, r, s, ρ satisfy*

$$0 < q \le p < \infty,\ 0 < r \le \infty,\ s > \sigma_q,\ \rho > \tilde{d}$$

for the $\mathcal{N}$-scale

$$0 < q \le p < \infty,\ 0 < r \le \infty,\ s > \sigma_{qr},\ \rho > \tilde{d}$$

for the $\mathcal{E}$-scale. Then any $f \in \mathcal{A}^s_{pqr}(\mathbb{R}^n)$ admits the following decomposition:

$$f = \sum_{\beta \in \mathbb{N}_0{}^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m} (\beta qu)_{\nu m}.$$

Here the coefficient $\lambda = \{\lambda^\beta_{\nu m}\}_{\beta \in \mathbb{N}_0{}^n,\, \nu \in \mathbb{N}_0,\, m \in \mathbb{Z}^n}$ *depends linearly and continuously on* f *and it satisfies*

$$\|\lambda \,:\, \mathbf{a}_{pqr}(\mathbb{R}^n)\|_\rho \le c\, \|f \,:\, \mathcal{A}^s_{pqr}(\mathbb{R}^n)\|.$$

If we set

$$f^\beta := \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m} (\beta qu)_{\nu m}$$

for $\beta \in \mathbb{N}_0{}^n$, *then for all* $\delta \in (0, \rho - \tilde{d})$, *there exists* $c_\delta > 0$ *such that*

$$\|f^\beta \,:\, \mathcal{A}^s_{pqr}(\mathbb{R}^n)\| \le c_\delta\, 2^{-\delta|\beta|} \|f \,:\, \mathcal{A}^s_{pqr}(\mathbb{R}^n)\|.$$

Conversely if $\lambda = \{\lambda^\beta_{\nu m}\}_{\beta \in \mathbb{N}_0{}^n,\, \nu \in \mathbb{N}_0,\, m \in \mathbb{Z}^n}$ *satisfies* $\|\lambda \,:\, \mathbf{a}_{pqr}\|_\rho < \infty$, *then*

$$f := \sum_{\beta \in \mathbb{N}_0{}^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m} (\beta qu)_{\nu m}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ *and satisfies*

$$\|f \,:\, \mathcal{A}^s_{pqr}(\mathbb{R}^n)\| \le c\, \|\lambda \,:\, \mathbf{a}_{pqr}(\mathbb{R}^n)\|_\rho.$$

Theorem 3.4 (Quarkonial decomposition for the general case). *Suppose that an odd integer* L *and the parameters* p, q, r, s, σ, ρ *satisfy*

$$0 < q \le p < \infty,\ 0 < r \le \infty,\ s \in \mathbb{R},\ \sigma > \max(\sigma_q, s),\ \rho > \tilde{d},\ L \ge \max(-1, [\sigma_q - s])$$

for the $\mathcal{N}$*-scale and*

$$0 < q \le p < \infty,\ 0 < r \le \infty,\ s \in \mathbb{R},\ \sigma > \max(\sigma_{qr}, s),\ \rho > \tilde{d},\ L \ge \max(-1, [\sigma_{qr} - s])$$

for the $\mathcal{E}$*-scale. Let* $f \in \mathcal{S}'(\mathbb{R}^n)$. *Set*

$$\begin{aligned}(\beta qu)_{\nu m}(x) &:= 2^{-\nu\left(\sigma - \frac{n}{p}\right)} \psi^\beta(2^\nu x - m)\\ (\beta qu)^{(L)}_{\nu m}(x) &:= 2^{-\nu\left(s - \frac{n}{p}\right)} \left((-\Delta)^{\frac{L+1}{2}} \psi^\beta\right)(2^\nu x - m).\end{aligned}$$

Then $f \in \mathcal{A}^s_{pqr}(\mathbb{R}^n)$ *if and only if there exist triply indexed sequences*

$$\eta = \{\eta^\beta_{\nu m}\}_{\beta \in \mathbb{N}_0{}^n,\ \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n} \text{ and } \lambda = \{\lambda^\beta_{\nu m}\}_{\beta \in \mathbb{N}_0{}^n,\ \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n} \in \mathbf{a}_{pqr}(\mathbb{R}^n)$$

such that f *can be expressed as*

$$f = \sum_{\beta \in \mathbb{N}_0{}^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \eta^\beta_{\nu m} (\beta qu)_{\nu m} + \sum_{\beta \in \mathbb{N}_0{}^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m} (\beta qu)^{(L)}_{\nu m}$$

with

$$\|\eta \,:\, \mathbf{a}_{pqr}(\mathbb{R}^n)\|_\rho + \|\lambda \,:\, \mathbf{a}_{pqr}(\mathbb{R}^n)\|_\rho < \infty. \tag{4}$$

If this is the case, then λ *can be taken so that it depends linearly and continuously on* f *and the following norm equivalence holds*:

$$\|\eta \,:\, \mathbf{a}_{pqr}(\mathbb{R}^n)\|_\rho + \|\lambda \,:\, \mathbf{a}_{pqr}(\mathbb{R}^n)\|_\rho \simeq \|f \,:\, \mathcal{A}^s_{pqr}(\mathbb{R}^n)\|. \tag{5}$$

As applications we obtained the trace results, the boundedness of the pseudo-differential operators, the diffeomorphic property of the function spaces and so on. For details we refer to [20].

Part II

Function spaces coming with A_∞-weights

Now we shall begin with function spaces coming with weights somehow nicer than $A_\infty^{\rm loc}$. Namely, we are going to consider the weighted Morrey spaces whose norm is given by

$$\|f\,:\,\mathcal{M}^p_q(w)\| := \sup_{Q:cube} w(Q)^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q |f|^q\,w\right)^{\frac{1}{q}},$$

where $w \in A_\infty$. For the sake of convenience for readers we define the class A_p with $1 \le p < \infty$.

A brief introduction of the weight-class A_p By "weight " we mean a positive function w which is locally integrable. For a cube Q and a weight w, we write $w(Q) := \int_Q w$ and $m_Q(w) := \frac{w(Q)}{|Q|}$. We also denote $w\{\cdots\} = \int_{\{\cdots\}} w$ for the sake of brevity.

Let $1 \le p < \infty$. We define

$$A_p := \{w\,:\,w \text{ is a weight satisfying } A_p(w) < \infty\},$$

where

$$\begin{aligned} A_1(w) &:= \sup_{Q:cube} m_Q(w)\cdot\|w^{-1}\,:\,L^\infty(Q)\| \\ A_p(w) &:= \sup_{Q:cube} m_Q(w)\cdot m_Q\left(w^{-\frac{1}{p-1}}\right)^{p-1} \quad \text{if} \quad 1 < p < \infty. \end{aligned}$$

We also define $A_\infty := \bigcup_{1\le p<\infty} A_p$. The following facts are well-known.

Theorem 3.5. Let M be the Hardy-Littlewood maximal operator and w a weight.

1. $A_{p_1} \subset A_{p_2}$ *for all* $1 \le p_1 \le p_2 < \infty$.
2. *Let* $w \in A_\infty$. *Then*
$$\{p \in (1,\infty)\,:\,w \in A_p\}$$
is an open interval in $(1,\infty)$.
3. *If* $w \in A_\infty$, *then there exists a constant* $c > 0$ *such that* $w(2Q) \le c\,w(Q)$ *for all cubes* Q.
4. $w \in A_1$ *is equivalent to the following weak-type inequality*:
$$w\{Mf > \lambda\} \le \frac{c}{\lambda}\int |f|w$$
for all measurable functions f.
5. *Let* $1 < p < \infty$. *Then the following assertions are equivalent.*
   (a) $w \in A_p$.

(b) *The following weak-type maximal inequality holds:*

$$w\{Mf > \lambda\} \le \frac{c}{\lambda^p} \int |f|^p \, w$$

for all measurable functions f.

(c) *The following strong-type maximal inequality holds:*

$$\int_{\mathbb{R}^n} Mf^p \, w \le c \int_{\mathbb{R}^n} |f|^p \, w$$

for all measurable functions f.

For details we refer to [2, 6] for example.

Given a weight w, we also use the following vector-valued norm.

$$\|\{f_j\}_{j\in\mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r, w)\| := \sup_{Q:cube} w(Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \left(\sum_{j\in\mathbb{N}_0} |f_j|^r \right)^{\frac{q}{r}} w \right)^{\frac{1}{q}}. \tag{6}$$

4 Function spaces coming with A_∞-weights

Having set down some elementary properties, let us consider the function spaces with A_∞-weights.

Let $w \in A_\infty$. Below to be more precise let $w \in A_u$ with $1 \le u < \infty$.

For $0 < \eta < \infty$ we define the powered Hardy-Littlewood maximal operator by

$$M^{(\eta)} f(x) := \sup_{x\in Q\,:\,\mathrm{cube}} \left(\frac{1}{|Q|} \int_Q |f|^\eta \right)^{\frac{1}{\eta}}$$

for a measurable function f.

Theorem 4.1. *Let $0 < q \le p < \infty$ and $0 < r \le \infty$. Suppose that $w \in A_u$. Assume in addition that $0 < \eta < \min\left(\frac{1}{q}, \frac{1}{u}\right)$. Then there exists $c > 0$ such that*

$$\|\{M^{(\eta)} f_j\}_{j\in\mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r, w)\| \le c \, \|\{f_j\}_{j\in\mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r, w)\|$$

for all sequences of measurable functions $\{f_j\}_{j\in\mathbb{N}_0}$.

Proof. The proof here is a mixture of [10, 21]. Note that

$$\|\{M^{(\eta)} f_j\}_{j\in\mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r, w)\| = \left\| \{M[|f_j|^\eta]\}_{j\in\mathbb{N}_0} \,:\, \mathcal{M}^{\frac{p}{\eta}}_{\frac{q}{\eta}}(l_r, w) \right\|^{\frac{1}{\eta}}$$

Therefore, we can assume that $\eta = 1$ and $q > u$. Note that since $w \in A_u$, we have

$$\left(\frac{1}{|Q|} \int_Q w \right) \cdot \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{u-1}} \right)^{u-1} \le A_u(w) < \infty$$

for all cubes Q. We let M_w be the maximal operator with respect to w:

$$M_w^{(u)} f(x) := \sup_{\substack{Q:cube \\ x \in Q}} \left(\frac{1}{w(Q)} \int_Q |f|^u \, w \right)^{\frac{1}{u}}, \; M_w f(x) := M_w^{(1)} f(x).$$

In order to use the assumption that $w \in A_u$, we rewrite $Mf(x)$ as follows:

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f| w^{\frac{1}{u}} \cdot w^{-\frac{1}{u}}$$

By the Hölder inequality we obtain

$$Mf(x) \le \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q |f|^u \, w \right)^{\frac{1}{u}} \cdot \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{u-1}} \right)^{\frac{1}{u'}}.$$

From the definition of A_u, we obtain

$$Mf(x) \le A_u(w)^{\frac{1}{u}} \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q |f|^u \, w \right)^{\frac{1}{u}} \cdot \left(\frac{1}{|Q|} \int_Q w \right)^{-\frac{1}{u}} \le A_u(w)^{\frac{1}{u}} \, M_w^{(u)} f(x). \tag{7}$$

Furthermore, w is doubling. Therefore, we are in the position of using the vector-valued maximal inequality on the Morrey spaces coming with general Radon measures, in particular with doubling Radon measures, (see [21]) to obtain

$$\begin{aligned} \|\{M^{(\eta)} f_j\}_{j \in \mathbb{N}_0} : \mathcal{M}^p_q(l_r, w)\| &\le c \, \|\{M_w^{(u)} f_j\}_{j \in \mathbb{N}_0} : \mathcal{M}^p_q(l_r, w)\| \\ &= c \, \left\| \{M_w[|f_j|^u]\}_{j \in \mathbb{N}_0} : \mathcal{M}^{\frac{p}{u}}_{\frac{q}{u}}(l_r, w) \right\|^{\frac{1}{u}} \\ &\le c \, \left\| \{|f_j|^u\}_{j \in \mathbb{N}_0} : \mathcal{M}^{\frac{p}{u}}_{\frac{q}{u}}(l_r, \mu) \right\|^{\frac{1}{u}} \\ &= c \, \|\{f_j\}_{j \in \mathbb{N}_0} : \mathcal{M}^p_q(l_r, w)\|. \end{aligned}$$

This is the desired result. ∎

5 Function spaces $\mathcal{A}^s_{pqr}(w)$ with $w \in A_\infty$

The goal of this section it to define the weighted function spaces $\mathcal{A}^s_{pqr}(w)$ with $w \in A_\infty$ as a model case to Part III.

5.1 Definition

Definition 5.1. Suppose that the parameters p, q, r, s and the weight w satisfy

$$0 < q \le p < \infty, \; 0 < r \le \infty, \; s \in \mathbb{R}, \; w \in A_\infty.$$

Let $\phi_0 \in \mathcal{S}$ be taken so that $\chi_{B(1)} \le \phi_0 \le \chi_{B(2)}$. Define $\phi_j(x) := \phi(2^{-j}x) - \phi(2^{-j+1}x)$ for $j \in \mathbb{N}$. Then ones sets

$$\|f : \mathcal{N}^s_{pqr}(w)\| := \|\{2^{js}\phi_j(D)f\}_{j \in \mathbb{N}_0} : l_r(\mathcal{M}^p_q(w))\|$$
$$\|f : \mathcal{E}^s_{pqr}(w)\| := \|\{2^{js}\phi_j(D)f\}_{j \in \mathbb{N}_0} : \mathcal{M}^p_q(l_r, w)\|$$

for $f \in \mathcal{S}'$. Below by $\mathcal{A}^s_{pqr}(w)$ we denote either $\mathcal{N}^s_{pqr}(w)$ or $\mathcal{E}^s_{pqr}(w)$. $\mathcal{A}^s_{pqr}(w)$ is the set of all tempered distributions $f \in \mathcal{S}'$ for which the quasi-norm $\|f : \mathcal{A}^s_{pqr}(w)\|$ is finite.

Theorem 5.2. *Suppose that the parameters p, q, r, s and the weight w satisfy*

$$0 < q \le p < \infty,\ 0 < r \le \infty,\ s \in \mathbb{R},\ w \in A_\infty.$$

Then the definition of the set $\mathcal{A}^s_{pqr}(w)_{\mathrm{loc}}$ does not depend on $\{\phi_j\}_{j\in\mathbb{N}_0}$ above.

The proof is based on the Plancherel-Polya-Nikolskij inequality, which can be formulated as follows:

Theorem 5.3. [27, Theorem 1.3.1, Section 1.4.1] *Let $f \in \mathcal{M}^p_q$ with $0 < q \le p < \infty$ and* $\mathrm{supp}(\mathcal{F}f) \subset B(1)$. *Then for all $\eta > 0$, there exists $c > 0$ such that there holds*

$$\sup_{y\in\mathbf{R}^n} \frac{|f(x-y)|}{1+|y|^{\frac{n}{\eta}}} \le c\, M^{(\eta)} f(x)$$

for every $x \in \mathbf{R}^n$.

The following proposition is a corollary of the above theorem.

Proposition 5.4. *Suppose that the parameters p, q, r and the weight w satisfy*

$$0 < q \le p < \infty,\ \ 0 < r \le \infty,\ w \in A_\infty.$$

1. *Let $\sigma \gg 1$. Suppose $H \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. Assume further that* $\mathrm{supp}(\mathcal{F}f) \subset B(R)$ *with $0 < R < \infty$. Then we have*

   $$\|H(D)f : \mathcal{M}^p_q(w)\| \le c\, \|H(R\cdot) : H^\sigma_2(\mathbb{R}^n)\| \cdot \|f : \mathcal{M}^p_q(w)\|,$$

   where c is independent of H, f and R.

2. *Let $p < \infty$ and $\sigma \gg 1$. Suppose that we are given a sequence $\{f_j\}_{j\in\mathbb{N}_0}$ in $\mathcal{S}'(\mathbb{R}^n)$ and a positive sequence $\{R_j\}_{j=0}^\infty$ such that* $\mathrm{supp}(\mathcal{F}f_j) \subset B(R_j)$ *for every $j \in \mathbb{N}_0$. Assume further that $\{H_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$. Then we have*

   $$\|\{H_j(D)f_j\}_{j\in\mathbb{N}_0} : \mathcal{M}^p_q(l_r, w)\| \le c \left(\sup_{k\in\mathbb{N}_0} \|H_k(R_k\cdot) : H^\sigma_2(\mathbb{R}^n)\|\right) \cdot \|\{f_j\}_{j\in\mathbb{N}_0} : \mathcal{M}^p_q(l_r, w)\|,$$

   where c is independent of $\{H_j\}_{j\in\mathbb{N}_0}$, $\{f_j\}_{j\in\mathbb{N}_0}$ and $\{R_j\}_{j\in\mathbb{N}_0}$.

Completeness (quasi-Banach property)

Lemma 5.5. *Let $w \in A_\infty$ and $0 < q \le p < \infty$. Then there exist $c, \alpha_0, \alpha_1 > 0$ such that*

$$|f(x)| \le c\, R^{\alpha_0} \langle x\rangle^{\alpha_1} \|f : \mathcal{M}^p_q(w)\|$$

for all $f \in \mathcal{M}^p_q(w)$ with $\mathrm{supp}(\mathcal{F}f) \subset B(R)$.

Proof. Let $0 < \eta < q$. We use the Plancherel-Polya inequality (see Theorem 5.3 above). Note that

$$\sup_{x\in Q} |f(x)| \le c\, R^{\frac{n}{\eta}} \inf_{x\in Q} M^{(\eta)} f(x) \le c_0\, w(Q)^{-\frac{1}{p}} \|M^{(\eta)} f : \mathcal{M}^p_q(w)\|$$

for every cube Q with side-length 1. Recall that there exists a constant $D > 1$ such that $w([-2^k, 2^k]) \ge D^k\, w([-1, 1])$ for all k. Therefore, if we choose $\alpha_1 = \max(\log_2 D, c_0, \log n/\eta)$, then we obtain

$$|f(x)| \le c\, R^{\alpha_1 + \frac{n}{\eta}} \langle x\rangle^{\alpha_1} \|f : \mathcal{M}^p_q(w)\|$$

by virtue of the boundedness of the maximal operator. The proof is now complete. ∎

Corollary 5.6. *$\mathcal{A}^s_{pqr}(w) \subset \mathcal{S}'$ and $\mathcal{A}^s_{pqr}(w)$ is complete.*

Proof. By the Fatou lemma for the Morrey spaces the matter is essentially reduced to showing $\mathcal{A}^s_{pqr}(w) \subset \mathcal{S}'$.

Let α_0 be a constant from Lemma 5.5. Choose an integer L so that $L > \alpha_0 - s$. It is not so hard to show that $(1-\Delta)^{-L} : \mathcal{A}^s_{pqr}(w) \to \mathcal{A}^{s+2L}_{pqr}(w)$ is an isomorphism by virtue of Proposition 5.4. We also remark that $\mathcal{A}^{s+2L}_{pqr}(w) \subset \mathcal{N}^{s+L}_{pq1}(w)$ is a continuous embedding. Pick a test function $\zeta \in \mathcal{S}$. Then by Lemma 5.5, we obtain

$$\int |\phi_j(D)f| \cdot |\zeta| \le c\, 2^{j\alpha_0} \|\phi_j(D)f \,:\, \mathcal{M}^p_q\| . \int |\zeta(x)| \langle x \rangle^{\alpha_0}\, dx.$$

This inequality is summable over $j \in \mathbb{N}_0$ to a quantity less than $c\,\|f \,:\, \mathcal{N}^{s+L}_{pq1}(w)\|$. As a result we conclude $\mathcal{A}^s_{pqr}(w) \subset \mathcal{S}'$, proving completeness of $\mathcal{A}^s_{pqr}(w)$. ∎

5.2 Atomic decomposition

Now we describe the atomic decomposition for the function space $\mathcal{A}^s_{pqr}(w)$.

Definition 5.7. Suppose that the parameters p, q, r and the weight w satisfy

$$0 < q \le p < \infty,\ 0 < r \le \infty,\ w \in A_\infty.$$

1. Let $\lambda = \{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0,\, m \in \mathbb{Z}^n}$ be a doubly indexed sequence. Then define

$$\|\lambda \,:\, \mathbf{n}_{pqr}(w)\| := \left\| \left\{ \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi^{(p)}_{\nu m} \right\}_{\nu \in \mathbb{N}_0} \,:\, l_r(\mathcal{M}^p_q(w)) \right\|$$

$$\|\lambda \,:\, \mathbf{e}_{pqr}(w)\| := \left\| \left\{ \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi^{(p)}_{\nu m} \right\}_{\nu \in \mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r, w) \right\|.$$

2. As before, to unify our formulation we use $\mathbf{a}_{pqr}(w)$ to denote either $\mathbf{n}_{pqr}(w)$ or $\mathbf{e}_{pqr}(w)$ according as $\mathcal{A}^s_{pqr}(w)$ denotes $\mathcal{N}^s_{pqr}(w)$ or $\mathcal{E}^s_{pqr}(w)$.

Theorem 5.8. *Let $w \in A_u$ with $1 \le u < \infty$. Suppose that the parameters p, q, r, s and the integers K, L satisfy*

$$0 < q \le p < \infty,\ 0 < r \le \infty,\ s \in \mathbb{R},\ K \ge (1 + [s])_+,\ L \ge \max(-1, \sigma_q + n(u-1)).$$

Assume in addition that $L \ge [\sigma_{\frac{q}{u}} - s]$ for the $\mathcal{N}$-case and that $L \ge \left[\sigma_{\frac{q}{u}\frac{r}{u}} - s\right]$ for the $\mathcal{E}$-case. Then there exists a constant $c > 0$ such that the following assertion holds.

1. *Let $f \in \mathcal{A}^s_{pqr}(w)$ be taken arbitrarily. Then there exist $\lambda \in \mathbf{a}_{pqr}(w)$ and a family of atoms $A = \{a_{\nu m}\}_{\nu \in \mathbb{N}_0{}^n,\, m \in \mathbb{Z}^n}$ such that*

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m},\ \|\lambda \,:\, \mathbf{a}_{pqr}(w)\| \le c\,\|f \,:\, \mathcal{A}^s_{pqr}(w)\|.$$

2. *Conversely suppose that* $\lambda \in \mathbf{a}_{pqr}(w)$ *and that* $A = \{a_{\nu m}\}_{\nu \in \mathbb{N}_0{}^n,\, m \in \mathbb{Z}^n}$ *is a family of atoms. Then* $f := \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$ *converges in the topology of* $\mathcal{S}'$ *and satisfies the norm estimate*

$$\|f \,:\, \mathcal{A}^s_{pqr}(w)\| \le c \,\|\lambda \,:\, \mathbf{a}_{pqr}(w)\|.$$

Proof. The proof is just a minor modification of the unweighted case. We omit the detail. ∎

Part III

Function spaces coming with $A^{\rm loc}_\infty$ weights

This part contains new results which we shall publish elsewhere. The key tool was obtained by Rychkov [18]. For the sake of convenience for readers, we include the proofs.

6 Topological vector space $\mathcal{S}'_e$ and the maximal inequality

6.1 Topological vector space $\mathcal{S}'_e$

The Besov spaces and the Triebel-Lizorkin spaces are subsets of $\mathcal{S}'$. However, in the weighted framework, it is not enough to consider the function spaces as a subset of $\mathcal{S}'$. Indeed, if we do so, then the completeness of the function spaces will fail. Instead, we enlarge the underlying function spaces.

Definition 6.1. [18] Let $N \in \mathbb{N}_0$. Then define $q_N(f) := \sup_{x \in \mathbb{R}^n} \sup_{|\alpha| \le N} e^{N|x|}|\partial^\alpha f(x)|$ for $f \in C^\infty$. $\mathcal{S}_e$ is a set of all C^∞-functions for which $q_N(f)$ is finite for all $N \in \mathbb{N}$. Topologize $\mathcal{S}_e$ with q_N.

From the very definition of the topology, the topology of $\mathcal{S}_e$ is the weakest topology such that $f \in \mathcal{S}_e \mapsto q_N(f) \in \mathbb{R}$ is continuous for all $N \in \mathbb{N}_0$.

Denote by $\mathcal{S}'_e$ the topological dual of $\mathcal{S}_e$. The following is an elementary fact from topological vector space theory. However, for the sake of convenience for readers, we include its proof.

Proposition 6.2. *A linear functional* $\ell : \mathcal{S}_e \to \mathbb{C}$*, which is not assumed continuous, belongs to* $\mathcal{S}'_e$ *if and only if there exists* N_0 *such that* $|\ell(\varphi)| \le N_0\, q_{N_0}(\varphi)$ *for all* $\varphi \in \mathcal{S}_e$.

Proof. The “if ” part is obvious. Let us prove the “only if” part. Note that, assuming that ℓ is continuous $\{\varphi \in \mathcal{S}_e \,:\, |\ell(\varphi)| < 1\}$ is an open set containing 0. Therefore, from the definition of the topology in particular that of the open basis, there exists $r_1, r_2, \ldots, r_M$ and $N_1, N_2, \ldots, N_M$ such that

$$\{\varphi \in \mathcal{S}_e \,:\, q_{N_j}(\varphi) < r_j \text{ for all } j = 1, 2, \ldots, M\} \subset \{\varphi \in \mathcal{S}_e \,:\, |\ell(\varphi)| < 1\}.$$

Since q_j is monotone, it suffices to set

$$N_0 := \max\left(N_1, N_2, \ldots, N_M, [r_1+1], [r_2+1], \ldots, [r_M+1]\right).$$

Indeed, if we consider $\psi_\epsilon := \dfrac{\varphi}{(1+\epsilon)N_0\, q_{N_0}(\varphi)}$ for $\epsilon > 0$, then

$$\psi_\epsilon \in \{\varphi \in \mathcal{S}_e \,:\, q_{N_j}(\varphi) < r_j \text{ for all } j = 1, 2, \ldots, M\} \subset \{\varphi \in \mathcal{S}_e \,:\, |\ell(\varphi)| < 1\}.$$

As a result, we obtain $|\ell(\psi_\epsilon)| \le 1$, which gives us the desired result. ∎

From the definition it is easy to see

Proposition 6.3. *$\mathcal{S}_e \subset \mathcal{S}$ and $\mathcal{S}' \subset \mathcal{S}'_e$ in the sense of continuous embedding.*

6.2 Local reproducing formula

Now we recall the local reproducing formula defined by Rychkov.

Proposition 6.4 ([18, Theorem 1.6]). *Denote by X either $\mathcal{S}$, $\mathcal{S}_e$ or $\mathcal{D}$. Suppose that $\varphi \in \mathcal{D}$ and that it satisfies $\displaystyle\int \varphi \neq 0$. Define $\varphi_j, j \in \mathbb{N}_0$ by*

$$\begin{aligned}\varphi_0(x) &:= \varphi(x)\\ \varphi_j(x) &:= 2^{jn}\varphi(2^j x) - 2^{(j-1)n}\varphi(2^{j-1}x) \text{ for } j \in \mathbb{N}.\end{aligned}$$

Then given an integer L, there exists a function $\psi \in \mathcal{D}$ such that

$$\delta = \sum_{j \in \mathbb{N}_0} \psi_j * \varphi_j$$

in the topology of X' and that ψ_1 has vanishing moment up to order L, where we have defined

$$\begin{aligned}\psi_0(x) &:= \psi(x)\\ \psi_j(x) &:= 2^{jn}\psi(2^j x) - 2^{(j-1)n}\psi(2^{j-1}x) \text{ for } j \in \mathbb{N}.\end{aligned}$$

Proof. Without loss of generality we can assume $\displaystyle\int \varphi = 1$. Define

$$\begin{aligned}g_0(x) &:= \varphi * \varphi(x)\\ g_j(x) &:= 2^{jn}\varphi * \varphi(2^j x) - 2^{(j-1)n}\varphi * \varphi(2^{j-1}x) \text{ for } j \in \mathbb{N}.\end{aligned}$$

Then we have $\delta = \displaystyle\sum_{j \in \mathbb{N}} g_j$ in the topology of X', no matter what X may be. From this we deduce

$$\delta = \left(\sum_{j \in \mathbb{N}} g_j\right) * \left(\sum_{j \in \mathbb{N}} g_j\right) * \ldots * \left(\sum_{j \in \mathbb{N}} g_j\right) \ (L+2 \text{ times}).$$

Let us write g^{*L} as the L-fold convolution of g. A repeated application of the binomial expansion gives us

$$\delta = \sum_{j \in \mathbb{N}_0} \sum_{m=1}^{L+2} \binom{L+2}{m} (g_j)^{*m} \left(\sum_{k=j+1}^{\infty} g_j\right)^{*L+2-m}.$$

We define

$$G_j := \sum_{m=1}^{L+2} \binom{L+2}{m} (g_j)^{*m} \left(\sum_{k=j+1}^{\infty} g_j \right)^{*L+2-m}.$$

Since, for every $j \in \mathbb{N}$, G_j never contains g_0 and $g_j = 2^{(j-1)n} g_1(2^{j-1}\cdot)$, we have $G_j = 2^{(j-1)n} G_1(2^{j-1}\cdot)$ for $j \in \mathbb{N}$. Note that we can factor out φ_j from G_j for each $j \in \mathbb{N}_0$. Therefore, if we set

$$G_j(x) = \varphi_j * \psi_j(x),$$

then we have $\psi_j, j \in \mathbb{N}_0$ is the desired family. ∎

The following is also an important observation on smooth functions made by Schott [23].

Lemma 6.5. *Let $L \in \mathbb{N}$. Then there exists $\Phi_L, \Psi_L \in \mathcal{D}$ such that*

$$\Phi_L(x) - 2^{-n}\Phi_L(2^{-1}x) = \Delta^L \Psi_L(x)$$

and that $\displaystyle\int \Phi_L(x) = 1$.

Proof. The proof is a reproduction of [23, Proposition 4.1]. For the sake of convenience for readers we supply the proof. First, we choose $\phi_1 \in \mathcal{D}(\mathbb{R})$ so that

$$\int_0^\infty \phi_1(r) r^{n-1}\, dr = 1,\ \mathrm{supp}(\phi_1) \subset (0,1).$$

We define $\phi_2, \phi_3, \ldots$ as follows: Suppose that we have defined ϕ_L. Then define $\phi_{L+1}(r)$

$$\phi_{L+1}(r) = \mu_L \phi_L(r) + \lambda_L \phi_L(2r),$$

where μ_L and λ_L are given by the following simultaneous equations.

$$\mu_L + 2^{-n}\lambda_L = 1,\ \mu_L + 2^{-n-2L}\lambda_L = 0.$$

Note that

$$\mathrm{supp}(\phi_L) \subset \mathrm{supp}(\phi_{L-1}) \cup \frac{1}{2}\mathrm{supp}(\phi_{L-1}) \subset \ldots \subset (0,1). \tag{8}$$

Furthermore,

$$\begin{aligned}
\int_0^\infty \phi_L(r) r^{n-1}\, dr &= \mu_{L-1} \int_0^\infty \phi_{L-1}(r) r^{n-1}\, dr + \lambda_{L-1} \int_0^\infty \phi_{L-1}(2r) r^{n-1}\, dr \\
&= \mu_{L-1} \int_0^\infty \phi_{L-1}(r) r^{n-1}\, dr + 2^{-n}\lambda_{L-1} \int_0^\infty \phi_{L-1}(r) r^{n-1}\, dr \\
&= (\mu_{L-1} + 2^{-n}\lambda_{L-1}) \int_0^\infty \phi_{L-1}(r) r^{n-1}\, dr \\
&= \int_0^\infty \phi_{L-1}(r) r^{n-1}\, dr = \ldots = 1.
\end{aligned}$$

We define $\eta_L(r) := \phi_L(r) - 2^{-n}\phi_L(r/2)$ for $r \in \mathbb{R}$. Then we have

$$\begin{aligned}
\mu_L\, \eta_L(r) + \lambda_L\, \eta_L(2r) &= \mu_L\, \phi_L(r) - 2^{-n}\mu_L \phi_L(r/2) + \lambda_L \phi_L(2r) - 2^{-n}\lambda_L \phi_L(r) \\
&= \phi_{L+1}(r) - 2^{-n}\phi_{L+1}(r/2) \\
&= \eta_{L+1}(r).
\end{aligned}$$

As a result we have $\eta_{L+1}(r) = \mu_L\, \eta_L(r) + \lambda_L\, \eta(2r)$.

We define a linear operator $T : C[0,\infty) \to C[0,\infty)$ by

$$(T\phi)(r) = \int_0^r \left(\int_0^t \left(\frac{s}{t}\right)^{n-1} \phi(s)\, ds \right) dt.$$

Note that a series of changing variables gives us

$$\begin{aligned}(T\phi)(2r) &= \int_0^{2r} \left(\int_0^t \left(\frac{s}{t}\right)^{n-1} \phi(s)\, ds \right) dt = \int_0^r \left(\int_0^{2u} \left(\frac{s}{2u}\right)^{n-1} \phi(s)\, ds \right) 2\, du \\ &= 2^{-n+2} \int_0^r \left(\int_0^{2u} \left(\frac{s}{u}\right)^{n-1} \phi(s)\, ds \right) du = 4 \int_0^r \left(\int_0^u \left(\frac{v}{u}\right)^{n-1} \phi(2v)\, dv \right) du \\ &= 4 \cdot T[\phi(2\cdot)](r).\end{aligned}$$

Denote by T^L the L-fold composition of T. As can be verified by induction on L and (8), $T^L \eta_L$ vanishes in a neighborhood of $0 \in [0,\infty)$.

We shall establish by induction that $T^L \eta_L$ agrees with an even polynomial of degree $2L-2$ on $[2,\infty)$. This assertion is true for $L = 0$.

Let $L \geq 0$. Then we have

$$\begin{aligned}T^{L+1}\eta_{L+1}(r) - T^{L+1}\eta_{L+1}(2) &= T[T^L \eta_{L+1}](r) - T[T^L \eta_{L+1}](2) \\ &= \int_2^r \left(\int_0^t \left(\frac{s}{t}\right)^{n-1} T^L \eta_{L+1}(s)\, ds \right) dt \\ &= \int_2^r \left(\int_0^t \left(\frac{s}{t}\right)^{n-1} (\mu_L\, T^L \eta_L(s) + \lambda_L\, T^L[\eta_L(2\cdot)](s))\, ds \right) dt \\ &= \int_2^r \left(\int_0^t \left(\frac{s}{t}\right)^{n-1} (\mu_L\, T^L \eta_L(s) + \lambda_L\, 2^{-2L} T^L \eta_L(2s))\, ds \right) dt,\end{aligned}$$

where for the last equality we used $(T\phi)(2r) = 4 \cdot T[\phi(2\cdot)](r)$. Now we use $\mu_L * 2^{-n-2L}\lambda_L = 1$.

$$\begin{aligned}&T^{L+1}\eta_{L+1}(r) - T^{L+1}\eta_{L+1}(2) \\ &= \int_2^r \left(\int_0^t \mu_L \left(\frac{s}{t}\right)^{n-1} T^L \eta_L(s)\, ds + \int_0^{2t} \lambda_L \left(\frac{s}{t}\right)^{n-1} 2^{-2L-n} T^L \eta_L(s))\, ds \right) dt \\ &= \int_2^r \left(\int_0^t \left(\frac{s}{t}\right)^{n-1} (\mu_L + 2^{-2L-n}\lambda_L)\, T^L \eta_L(s)\, ds + \int_t^{2t} \lambda_L \left(\frac{s}{t}\right)^{n-1} 2^{-2L-n} T^L \eta_L(s))\, ds \right) dt \\ &= \lambda_L\, 2^{-2L-n} \int_2^r \left(\int_t^{2t} \left(\frac{s}{t}\right)^{n-1} T^L \eta_L(s)\, ds \right) dt.\end{aligned}$$

Using this recurrence formula and $\displaystyle\int_2^r \left(\int_t^{2t} \left(\frac{s}{t}\right)^{n-1} s^{2k}\, ds \right) = \frac{2^{n+k}-1}{n+k}(r^{2k+2} - 2^{2k+2})$ we see that our claim holds.

Let us define

$$\psi_L(r) := r(T^L \eta_L)(r) - \sum_{l=0}^{L-1} a_l\, r^{2l},\ 0 \leq r < \infty.$$

The ψ_L agrees with an even polynomial of degree $2L-2$ near a neighborhood of $0 \in [0,\infty)$ and vanishes outside $B(2)$, so that

$$\begin{aligned}\Phi_L(x) &:= \mathrm{vol}(S^1)^{-1} \phi_L(|x|) \\ \Psi_L(x) &:= \mathrm{vol}(S^1)^{-1} \psi_L(|x|)\end{aligned}$$

are smooth function in $\mathcal{D}$. Furthermore, it is easy to see that Φ_L has integral 1.

Finally we observe that

$$\Delta T\rho(|x|) = \left[r^{1-n}\left(r^{n-1}(T\rho)'(r)\right)'\right]_{r=|x|} = \rho(|x|),$$

whenever $\rho : [0, \infty) \to [0, \infty)$ is a smooth function vanishing near 0. Using this observation, we see that

$$\begin{aligned}\Delta^L \psi_L(x) &= \mathrm{vol}(S^1)^{-1}\Delta^L \left[(T^L\eta_L)(r) - \sum_{l=0}^{L-1} a_l\, r^{2l}\right] \\ &= \mathrm{vol}(S^1)^{-1}\Delta^L(T^L\eta_L)(|x|) \\ &= \mathrm{vol}(S^1)^{-1}\eta_L(|x|) \\ &= \Phi_L(x) - 2^{-n}\Phi_L(x/2).\end{aligned}$$

This is the desired result. ∎

Remark 6.6 ([18, Remark 1.8]). Let $M \in \mathbb{N}$. Then there exist $\varphi_0, \Phi \in \mathcal{D}$ such that

$$\Delta^M\Phi(x) = \varphi_0(x) - 2^{-n}\varphi_0(2^{-1}x), \ \int \varphi_0 = 1. \tag{9}$$

Let us set $\varphi = \Delta^M\Phi$. Define

$$\psi_j(x) := 2^{jn}[\Delta^m(\varphi_0 + 2^{-n}\varphi_0(\cdot/2)](2^j x),\ \tilde{\varphi}_j := 2^{jn}[\Delta^{M-m}\Phi](2^j x)$$

for $j \in \mathbb{N}$. Then we have

$$\delta = \varphi_0 * \varphi_0 + \sum_{j\in\mathbb{N}} \psi_j * \tilde{\varphi}_j$$

in the topology of X. This is another candidate of the Littlewood-Paley patch.

6.3 Maximal inequality

Now we take up the maximal operator. Throughout this section we assume that $0 < \eta \le 1$ and $\varphi_0,\, \varphi \in \mathcal{D}$. Define

$$\varphi_j(x) := 2^{jn}\varphi(2^j x) - 2^{(j-1)n}\varphi(2^{j-1}x)$$

for $j \in \mathbb{N}$. We assume that φ_1 has vanishing moment up to order L.

Let $A, B > 0$ and $j \in \mathbb{N}_0$. Then we define $m_{j,A,B}(y) := (1 + 2^j|y|)^A 2^{B|y|}$ and $\varphi^*_{j,A,B}f(x) := \sup\limits_{y\in\mathbb{R}^n} \dfrac{|\varphi_j * f(x-y)|}{m_{j,A,B}(y)}$.

Lemma 6.7 ([18, Lemma 2.9]). *For every $A, B \ge 0$ there exists a constant $c > 0$ depending only on A, B, n, φ_0 such that*

$$|\varphi_j * f(x)| \le c \sum_{k=j}^{\infty} 2^{(j-k)A} 2^{jn} \int_{\mathbb{R}^n} \frac{|\varphi_k * f(x-y)|}{m_{j,A,B}(y)}\, dy,\ x \in \mathbb{R}^n$$

for every $f \in \mathcal{S}'_e$ and $j \in \mathbb{N}_0$.

Proof. By the local reproducing formula, there exists $\psi_0, \psi \in \mathcal{D}$ such that $\int_{\mathbb{R}^n} x^\beta \psi(x)\, dx = 0$ for all $\beta \in {\mathbb{N}_0}^n$ with $|\beta| \le L$ and that

$$\sum_{j\in\mathbb{N}_0} \psi_j * \varphi_j = \delta, \text{ in } \mathcal{S}'_e$$

where we have set $\psi_j(x) := 2^{jn}\varphi(2^j x) - 2^{(j-1)n}\varphi(2^{j-1}x)$ for $j \in \mathbb{N}$. Define

$$(\varphi_0)_j(x) := 2^{jn}\varphi_0(2^j x),\ (\psi_0)_j(x) := 2^{jn}\psi_0(2^j x)$$

for $j \in \mathbb{N}_0$. We remark that $(\varphi_0)_j$ and $(\psi_0)_j$ are different from φ_j and ψ_j respectively. Along this decomposition of δ, we expand $\varphi_j * f$:

$$\varphi_j * f = (\psi_0)_j * (\varphi_0)_j * \varphi_j * f + \sum_{k=j+1}^{\infty} \varphi_j * \psi_k * \varphi_k * f.$$

Observe that $\mathrm{supp}(\varphi_j * \psi_k) \subset \mathrm{supp}(\varphi_j) + \mathrm{supp}(\psi_k) \subset B(c\, 2^{-j})$ for $k \ge j+1$. Furthermore, $|\mathcal{F}(\psi_1)(\xi)| \le c\, |\xi|^{L+1}$ by virtue of the moment condition of ψ_1. Therefore,

$$|\mathcal{F}(\varphi_j)(\xi)\mathcal{F}(\psi_k)(\xi)| \le c\, |2^{1-k}\xi|^{L+1} |\mathcal{F}(\varphi_1)(2^{1-j}\xi)| = c\, 2^{(j-k)(L+1)} |2^{1-j}\xi|^{L+1} \cdot |\mathcal{F}(\varphi_1)(2^{1-j}\xi)|.$$

A similar estimate holds for the partial derivatives. Along with this estimate, $\| \mathcal{F}f \|_\infty \le c \| f \|_1$ gives us that

$$|\varphi_j * \psi_k(x)| \le c\, 2^{(j-k)(L+1)+jn}$$

for all $x \in \mathbb{R}^n$. Combining the above observations, we conclude

$$|\varphi_j * \psi_k(x)| \le c\, 2^{(j-k)(L+1)+jn} \chi_{B(c\, 2^{-j})}(x)$$

for all $x \in \mathbb{R}^n$. Since

$$m_{j,A,B}(y) \le c$$

for $y \in B(c\, 2^{-j})$. Therefore, it follows that

$$|\varphi_j * f(x)| \le c \sum_{k=j}^{\infty} 2^{(j-k)A} 2^{jn} \int_{\mathbb{R}^n} \frac{|\varphi_k * f(x-y)|}{m_{j,A,B}(y)}\, dy,$$

which concludes the proof. ∎

Set $M_{A,B} f(x,j) := \sup_{k \ge j} 2^{(j-k)A} \dfrac{|\varphi_k * f(x-y)|}{m_{j,A,B}(y)}$.

Lemma 6.8 ([18, Lemma 2.9]). *Let $0 < \eta < 1$. For every $A, B \ge 0$ there exists a constant $c > 0$ depending only on A, B, n, η, φ_0 such that*

$$|\varphi_j * f(x)|^\eta \le c \sum_{k=j}^{\infty} 2^{(j-k)A\eta} 2^{jn} \int_{\mathbb{R}^n} \frac{|\varphi_k * f(x-y)|^\eta}{m_{j,A\eta,B\eta}(y)}\, dy$$

for every $f \in \mathcal{S}'_e$ and each $x \in \mathbb{R}^n$.

Proof. Note that we have proved in Lemma 6.7

$$|\varphi_k * f(x-y)| \le c \sum_{l=k}^{\infty} h 2^{(k-l)A} 2^{ln} \int_{\mathbb{R}^n} \frac{|\varphi_l * f(x-y-z)|}{m_{k,A,B}(z)}\, dz,$$

which is the case when $\eta = 1$. With this inequality, we pass to the general case. First, this inequality gives us

$$\begin{aligned} 2^{(j-k)A} \frac{|\varphi_k * f(x-y)|}{m_{j,A,B}(y)} &\le c \sum_{l=k}^{\infty} 2^{(j-l)A} 2^{ln} \int_{\mathbb{R}^n} \frac{|\varphi_l * f(x-y-z)|}{m_{j,A,B}(y) m_{k,A,B}(z)} \, dz \\ &\le c \sum_{l=k}^{\infty} 2^{(j-l)A} 2^{ln} \int_{\mathbb{R}^n} \frac{|\varphi_l * f(x-y-z)|}{m_{j,A,B}(y) m_{j,A,B}(z)} \, dy. \end{aligned}$$

Now we make use of the Peetre inequality $\langle a+b \rangle \le \sqrt{2} \langle a \rangle \cdot \langle b \rangle$:

$$\begin{aligned} \frac{|\varphi_l * f(x-y-z)|}{m_{j,A,B}(y) m_{j,A,B}(z)} &\le c \left(\frac{|\varphi_l * f(x-y-z)|}{m_{j,A,B}(y+z)} \right)^{1-\eta} \cdot \left(\frac{|\varphi_l * f(x-y-z)|}{m_{j,A,B}(y+z)} \right)^{\eta} \\ &\le c \, 2^{(l-j)(1-\eta)A} M_{A,B} f(x,j)^{(1-\eta)} \cdot \left(\frac{|\varphi_l * f(x-y-z)|}{m_{j,A,B}(y+z)} \right)^{\eta}. \end{aligned}$$

Inserting this estimate, we obtain

$$2^{(j-k)A} \frac{|\varphi_k * f(x-y)|}{m_{j,A,B}(y)} \le c \, M_{A,B} f(x,j)^{1-\eta} \sum_{l=k}^{\infty} 2^{(j-l)A\eta} 2^{ln} \int_{\mathbb{R}^n} \frac{|\varphi_l * f(x-z)|^{\eta}}{m_{j,A\eta,B\eta}(z)} \, dz.$$

Taking supremum of both side over $k \in \mathbb{N}$ with $k \ge j$, we obtain

$$\begin{aligned} M_{A,B} f(x,j) &= \sup_{\substack{k \ge j \\ y \in \mathbb{R}^n}} 2^{(j-k)A} \frac{|\varphi_k * f(x-y)|}{m_{j,A,B}(y)} \\ &\le c \, M_{A,B} f(x,j)^{1-\eta} \sup_{\substack{k \ge j \\ y \in \mathbb{R}^n}} \sum_{l=k}^{\infty} 2^{(j-l)A\eta} 2^{ln} \int_{\mathbb{R}^n} \frac{|\varphi_l * f(x-z)|^{\eta}}{m_{j,A\eta,B\eta}(z)} \, dz \\ &\le c \, M_{A,B} f(x,j)^{1-\eta} \sum_{l=j}^{\infty} 2^{(j-l)A\eta} 2^{ln} \int_{\mathbb{R}^n} \frac{|\varphi_l * f(x-z)|^{\eta}}{m_{j,A\eta,B\eta}(z)} \, dz. \end{aligned}$$

In summary we have obtained

$$M_{A,B} f(x,j) \le c \, M_{A,B} f(x,j)^{1-\eta} \sum_{l=j}^{\infty} 2^{(j-l)A\eta} 2^{ln} \int_{\mathbb{R}^n} \frac{|\varphi_l * f(x-z)|^{\eta}}{m_{j,A\eta,B\eta}(z)} \, dz. \tag{10}$$

Therefore, the assetion in the theorem is obtained, once we justify we can divide $M_{A,B} f(x,j)^{\eta}$ in the above inequality. Since $f \in \mathcal{S}'_e$, we have

$$|\varphi_l * f(x-z)| \le c \, q_N(\varphi_l(x-z-*))$$

for some $N \in \mathbb{N}$ with N depending on f. Note that

$$\begin{aligned} q_N(\varphi_l(x-z-*)) &= \sup_{|\alpha| \le N, \, w \in \mathbb{R}^n} e^{N|w|} |\partial^{\alpha} \varphi_l(x-z-w)| \\ &\le c \, 2^{lN} q_N(\varphi) \sup_{w \in \mathbb{R}^n} e^{N|w| - 2^l \, N|x-z-w|} \\ &\le c \, 2^{lN} q_N(\varphi) \sup_{w \in \mathbb{R}^n} e^{N|x-z-w| - 2^l \, N|w|} \\ &\le c \, 2^{lN} \exp(N|x-y|). \end{aligned}$$

Therefore, there exists A and B such that $M_{A,B}f(x,j) < \infty$, whenever $A \geq A_f$ and $B \geq B_f$. Dividing both sides by $M_{A,B}f(x,j)^{1-\eta}$, we obtain

$$M_{A,B}f(x,j)^\eta \leq c \sum_{l=j}^\infty 2^{(j-l)A\eta} 2^{ln} \int_{\mathbb{R}^n} \frac{|\varphi_l * f(x-z)|^\eta}{m_{j,A\eta,B\eta}(z)}\,dz,$$

whenever $A \geq A_f$ and $B \geq B_f$. In particular

$$|\varphi_j * f(x)|^\eta \leq c \sum_{l=j}^\infty 2^{(j-l)A\eta} 2^{ln} \int_{\mathbb{R}^n} \frac{|\varphi_l * f(x-z)|^\eta}{m_{j,A\eta,B\eta}(z)}\,dz,$$

provided $A \geq A_f$ and $B \geq B_f$. Note that the constant c in the above estimate does depend on A and B. Therefore, we see that there exists $c = c_f$ depending on f such that

$$|\varphi_j * f(x)|^\eta \leq c_f \sum_{l=j}^\infty 2^{(j-l)A\eta} 2^{ln} \int_{\mathbb{R}^n} \frac{|\varphi_l * f(x-z)|^\eta}{m_{j,A\eta,B\eta}(z)}\,dz.$$

Since the right-hand side is decreasing with respect to A and B, we see that the formula is valid for all $A, B \geq 0$. As a consequence, we obtain

$$|\varphi_j * f(x)|^\eta \leq c_f \sum_{l=j}^\infty 2^{(j-l)A\eta} 2^{ln} \int_{\mathbb{R}^n} \frac{|\varphi_l * f(x-z)|^\eta}{m_{j,A\eta,B\eta}(z)}\,dz$$

for all $A, B > 0$. Another application of the Peetre inequality gives us

$$\begin{aligned} M_{A,B}f(x,j)^\eta &= \sup_{\substack{k \geq j \\ y \in \mathbb{R}^n}} \left(2^{(j-k)A} \frac{|\varphi_k * f(x-y)|}{m_{j,A,B}(y)} \right)^\eta \\ &\leq c_f \sum_{l=k}^\infty 2^{(j-l)A\eta} 2^{ln} \int_{\mathbb{R}^n} \frac{|\varphi_l * f(x-y-z)|^\eta}{m_{j,A\eta,B\eta}(y) m_{j,A\eta,B\eta}(z)}\,dz \\ &\leq c_f \sup_{\substack{k \geq j \\ y \in \mathbb{R}^n}} \sum_{l=k}^\infty 2^{(j-l)A\eta} 2^{ln} \int_{\mathbb{R}^n} \frac{|\varphi_l * f(x-y-z)|^\eta}{m_{j,A\eta,B\eta}(y+z)}\,dz \end{aligned}$$

Notice that the most right-hand side is increasing with k. Therefore, after a change of variables we obtain

$$M_{A,B}f(x,j)^\eta \leq c_f \sup_{z \in \mathbb{R}^n} \sum_{l=j}^\infty 2^{(j-l)A\eta} 2^{ln} \int_{\mathbb{R}^n} \frac{|\varphi_l * f(x-z)|^\eta}{m_{j,A\eta,B\eta}(z)}\,dz < \infty.$$

As a result we conclude $M_{A,B}f(x,j) < \infty$, whenever x satisfies

$$\sum_{l=j}^\infty 2^{(j-l)A\eta} 2^{ln} \int_{\mathbb{R}^n} \frac{|\varphi_l * f(x-z)|^\eta}{m_{j,A\eta,B\eta}(z)}\,dz < \infty.$$

We now return to (10) to conclude

$$M_{A,B}f(x,j) \leq c \sum_{l=j}^\infty 2^{(j-l)A\eta} 2^{ln} \int_{\mathbb{R}^n} \frac{|\varphi_l * f(x-z)|^\eta}{m_{j,A\eta,B\eta}(z)}\,dz.$$

This gives us the desired result in view of the definition of $M_{A,B}f(x,j)$. ∎

Now we return to the maximal operator given by

$$\varphi^*_{j,A,B} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\varphi_j * f(x-y)|}{m_{j,A,B}(y)}, \tag{11}$$

which we shall use in this paper.

Definition 6.9. Let us set

$$\begin{aligned}
M_{\mathrm{loc},\leq \mathrm{r}} f(x) &:= \sup_{x \in Q,\, \ell(Q) \leq r} \frac{1}{|Q|} \int_Q |f| \, dx \\
M^{(\eta)}_{\mathrm{loc},\leq \mathrm{r}} f(x) &:= \sup_{x \in Q,\, \ell(Q) \leq r} \left(\frac{1}{|Q|} \int_Q |f|^\eta \, dx \right)^{\frac{1}{\eta}} \\
M_{\mathrm{loc}} f(x) &:= M_{\mathrm{loc},\leq \mathrm{L}} f(x) \\
M^{(\eta)}_{\mathrm{loc}} f(x) &:= M^{(\eta)}_{\mathrm{loc},\leq \mathrm{L}} f(x)
\end{aligned}$$

for $0 < \eta, r < \infty$ and a measurable function f.

To formulate our result, we define an operator K_B.

Definition 6.10. Let $B > 0$. Then define

$$K_B f(x) := \int_{\mathbb{R}^n} |f(y)| 2^{-B|x-y|} \, dy.$$

Proposition 6.11. [18] *Let $\eta > 0$ and $A > n/\eta$. Then we have*

$$\varphi^*_{j,A,B} f(x)^\eta \leq c \sum_{k=j}^{\infty} 2^{(j-k)(A\eta - n)} \left(M_{\mathrm{loc}}[|\varphi_k * f|^\eta](x) + K_{B\eta}[|\varphi_k * f|^\eta](x) \right)$$

Proof. Again by the Peetre inequality we obtain

$$\begin{aligned}
\frac{|\varphi_j * f(x)|^\eta}{m_{j,A\eta,B\eta}(y)} &\leq c \sum_{k=j}^{\infty} 2^{(j-k)A\eta} 2^{jk} \int_{\mathbb{R}^n} \frac{|\varphi_k * f(x-z)|^\eta}{m_{j,A\eta,B\eta}(y) m_{j,A\eta,B\eta}(z-y)} \, dz \\
&\leq c \sum_{k=j}^{\infty} 2^{(j-k)A\eta} 2^{jk} \int_{\mathbb{R}^n} \frac{|\varphi_k * f(x-z)|^\eta}{m_{j,A\eta,B\eta}(z)} \, dz.
\end{aligned}$$

We decompose the inner integral of the most right-hand side according to the unit ball.

$$\int_{\mathbb{R}^n} \frac{|\varphi_k * f(x-z)|^\eta}{m_{j,A\eta,B\eta}(z)} \, dz = 2^{jn} \int_{|z| \leq 1} \frac{|\varphi_k * f(x-z)|^\eta}{(1 + 2^j |y|)^\eta} \, dz + 2^{j(n - A\eta)} \int_{|z| > 1} \frac{|\varphi_k * f(x-z)|^\eta}{2^{B\eta|z|}} \, dz.$$

Since $j \geq 0$, the first term is bounded by the local maximal operator :

$$2^{jn} \int_{|z| \leq 1} \frac{|\varphi_k * f(x-z)|^\eta}{(1 + 2^j |y|)^\eta} \, dz \leq c \, M_{\mathrm{loc}}[|\varphi_k * f|^\eta](x).$$

This is a well-known technique that can be found in [24], for example. As for the second term a crude estimate suffices :

$$2^{j(n - A\eta)} \int_{|z| > 1} \frac{|\varphi_k * f(x-z)|^\eta}{2^{B\eta|z|}} \, dz \leq c \, 2^{j(n - A\eta)} K_{B\eta}[|\varphi_k * f|^\eta](x)$$

Therefore the proof is now complete. ∎

Lemma 6.12. [18] *Let* $s \in \mathbb{R}$ *and* $w \in A_u^{\mathrm{loc}}$ *for some* $1 \le u < \infty$. *Pick* $\varphi_0, \psi_0 \in \mathcal{D}$ *so that* $\displaystyle\int \varphi_0 \ne 0$. *Define*

$$\varphi_j(x) := 2^{jn}\varphi_0(2^j x) - 2^{(j-1)n}\varphi_0(2^{j-1}x),\ \psi_j(x) := 2^{jn}\psi_0(2^j x) - 2^{(j-1)n}\psi_0(2^{j-1}x)$$

for $j \in \mathbb{N}$. *Assume in addition that* φ_1 *has vanishing moment up to order* $\max(-1,[s])$. *Then we have*

$$2^{js}\psi^*_{j,A,B}f(x) \le c \sum_{k \in \mathbb{N}_0} 2^{-\epsilon|j-k|}2^{ks}\varphi^*_{k,A,B}f(x)$$

for some constant $c > 0$ *and* $\epsilon > 0$.

Here that φ_1 has vanishing moment up to order -1 means no condition.

Proof. Note that there exist $\eta_0, \eta_1, \ldots \in \mathcal{D}$ such that η_1 has vanishing moment up to order sufficiently large, say M, and that

$$\sum_{j \in \mathbb{N}_0} \eta_j * \varphi_j = \delta$$

in $\mathcal{S}_e'$ and $\eta_j(x) = 2^{(j-1)n}\eta_1(2^{j-1}x)$ for $j \ge 2$. Therefore,

$$\begin{aligned}
|\psi_j * f(x-y)| &\le \sum_{k \in \mathbb{N}_0} |\psi_j * \eta_k * \varphi_k * f(x-y)| \\
&\le \sum_{k \in \mathbb{N}_0} \int_{\mathbb{R}^n} |\psi_j * \eta_k(z)| \cdot |\varphi_j * f(x-y-z)|\,dz \\
&\le \sup_{\overline{z} \in \mathbb{R}^n} \left(\frac{|\varphi_k * f(x-y-\overline{z})|}{m_{j,A,B}(\overline{z})} \right) \int_{\mathbb{R}^n} |\psi_j * \eta_k(z)| m_{j,A,B}(z)\,dz.
\end{aligned}$$

Going through the same argument as before, we see

$$|\psi_j * \eta_k(z)| \le c\, 2^{-(j-k)_+(1+[s])_+ -(k-j)_+(M+1)+n\min(j,k)} \chi_{B(2^{-\min(j,k)})}(z).$$

Therefore, it follows that

$$\frac{|\varphi_k * f(x-y-z)|}{m_{j,A,B}(z)} \le c\, 2^{-(j-k)_+(1+[s])_+ -(k-j)_+(M+1-A)+n\min(j,k)} \chi_{B(2^{-\min(j,k)})}(z).$$

Choosing M sufficiently large, we obtain $\epsilon > 0$ such that

$$\int_{\mathbb{R}^n} |\psi_j * \eta_k(z)| m_{j,A,B}(z)\,dz \le c\, 2^{-\epsilon|j-k|}.$$

Therefore, the desired result follows. ∎

7 Weight class A_p^{loc}

We mean by "weight" a non-negative measurable function which is locally integrable on $\mathbb{R}^n$.

Rychkov defined the class of the weights as follows:

$$A_p^{\mathrm{loc}} := \{w \,:\, w \text{ is a weight with } A_p^{\mathrm{loc}}(w) < \infty\},\ 1 \le p < \infty,$$

where

$$A_1^{\mathrm{loc}}(w) := \sup_{\ell(Q)\leq 1} m_Q(w)\cdot \|w^{-1} \, : \, L^\infty(Q)\|$$
$$A_p^{\mathrm{loc}}(w) := \sup_{\ell(Q)\leq 1} m_Q(w)\cdot m_Q(w^{-\frac{1}{p-1}})^{p-1}, \; 1<p<\infty.$$

We also define

$$A_\infty^{\mathrm{loc}} := \bigcup_{1\leq p<\infty} A_p^{\mathrm{loc}}$$

as a set. This class of weights enjoys properties analogous to A_p such as the openness property and the (local) reverse Hölder inequality.

The class A_p^{loc} ($1 \leq p < \infty$) is independent of the upper bound for the length of cube used in its definition. We define

$$A_{1,r}^{\mathrm{loc}}(w) := \sup_{\ell(Q)\leq r} m_Q(w)\cdot \|w^{-1} \, : \, L^\infty(Q)\| = \mathrm{esssup}_{x\in\mathbb{R}^n} \frac{M_{\mathrm{loc},\leq \mathrm{r}}w(x)}{w(x)}$$
$$A_{p,r}^{\mathrm{loc}}(w) := \sup_{\ell(Q)\leq r} m_Q(w)\cdot m_Q(w^{-\frac{1}{p-1}})^{p-1}, \; 1<p<\infty$$

for a weight w and $r > 0$.

The class of A_p^{loc} weights is independent of the upper bound for $\ell(Q)$ used in its definition. Namely, we can replace $\ell(Q) \leq 1$ by $\ell(Q) \leq r$ for any $0 < r < \infty$. If $0 < r \leq 1$, then it is obvious that

$$A_{p,r}^{\mathrm{loc}}(w) \leq A_p^{\mathrm{loc}}(w).$$

Our purpose is to clarify the indepence for $r \geq 1$. Lemarié [12] first showed the independence in the case of $1 \leq p < \infty$. Later, Rychkov [18] gave a more precise estimate for $1 < p < \infty$. Now we reexamine their result.

Now motivated by argument of Rychkov, we prove the following, whose global counterpart is well-known.

Proposition 7.1. *Let $p \geq 1$.*

1. *Let $r > 0$. Then*

$$w\{M_{\mathrm{loc},\leq \mathrm{r}}f > \lambda\} \leq \frac{3^p\, A_{p,3r}^{\mathrm{loc}}(w)}{\lambda^p}\int |f|^p\, w \tag{12}$$

   for every $f \in L^p(w)$.

2. *Conversely*

$$w\{M_{\mathrm{loc},\leq \mathrm{r}}f > \lambda\} \leq \frac{M}{\lambda^p}\int |f|^p\, w \tag{13}$$

   implies $A_{p,r}^{\mathrm{loc}}(w) \leq M$.

Proof. Note that

$$M_{\mathrm{loc},\leq \mathrm{r}}f(x) := \sup_{x\in Q,\, \ell(Q)<r} m_Q(|f|).$$

Therefore, a standard argument shows that for every compact set $K \subset \{M_{\mathrm{loc},\le \mathrm{r}} f > \lambda\}$, we can find a finite collection of cubes $Q_1, Q_2, \ldots, Q_L$ with side length less than r such that

$$\chi_K \le \sum_{j=1}^{L} \chi_{3Q_j}, \sum_{j=1}^{L} \chi_{Q_j} \le 1$$

and that $m_{Q_j}(|f|) > \lambda$. As a result we obtain

$$\begin{aligned} w(K) &\le \sum_{j=1}^{L} w(3Q_j) \le A^{\mathrm{loc}}_{p,3r}(w) \sum_{j=1}^{L} |3Q_j|^p \left(\int_{3Q_j} w^{-\frac{1}{p-1}} \right)^{-\frac{1}{p-1}} \\ &\le \frac{3^p\, A^{\mathrm{loc}}_{p,3r}(w)}{\lambda^p} \sum_{j=1}^{L} \left(\int_{Q_j} |f| \right)^p \left(\int_{3Q_j} w^{-\frac{1}{p-1}} \right)^{-\frac{1}{p-1}} \\ &\le \frac{3^p\, A^{\mathrm{loc}}_{p,3r}(w)}{\lambda^p} \int |f|^p\, w. \end{aligned}$$

Therefore *1* is established. To prove *2*, it suffices to test the inequality with $f = \chi_Q \cdot w^{-\frac{1}{p-1}}$ and $\lambda = m_Q(f)$, as we did for the classical case. ∎

The following is a corollary of the above observation. This fact seems somehow well-known. However, for the sake of convenience for readers we supply the proof.

Corollary 7.2. *If $w \in A^{\mathrm{loc}}_{\infty,r}$, then there exists c_w such that*

$$w(Q + [-r, r]^n) \le c_w\, w(Q)$$

for all cubes. Here we have set

$$Q + [-r, r]^n := \{a + b \,:\, a \in Q,\, b \in [-r, r]^n\}.$$

Proof. If we choose $f = \chi_{Q(r)}$, then we obtain

$$w(Q + [-r/2, r/2]^n) \le w\{M_{\mathrm{loc},\le \mathrm{M}} f > 2^{-n}\} \le c_{w,0}\, w(Q).$$

Hence the constant c_w can be taken as ${c_{w,0}}^2$. ∎

It is obvious that

$$A^{\mathrm{loc}}_{p,r}(w) \le A^{\mathrm{loc}}_{p}(w) \quad \text{if} \quad 0 < r \le 1.$$

In the case of $r \ge 1$, the following proposition shows the independence.

Proposition 7.3.

1. (See [12, Lemma 7].[1]) *Let $w \in A_1^{\mathrm{loc,r}}$ and $0 < r < \infty$. Then we have*

$$A^{\mathrm{loc}}_{1,2r}(w) \le 4^n A^{\mathrm{loc}}_{1,r}(w)^3.$$

2. *Let $1 < p < \infty$, $w \in A^{\mathrm{loc}}_p$ and $r \ge 1$. Then we have*

$$A^{\mathrm{loc}}_{p,r}(w) \le \exp(c_w\, r), \tag{14}$$

where $c_w > 0$ is a constant independent of r.

[1] [12] is written in French. So in the actual article one should refer to Lemme 7.

Proof. Lemarié-Rieusset [12] proved *1* for $n = 1$. The proof we give here is the generalization for the multi-dimensional case. On the other hand, Rychkov [18, Subsection 1.1,p148–p151] proved *2*. We follow his proof and investigate how the estimate depends on r.

Let $\chi_r(x) = r^{-n}\chi_{Q(r)}$. Then we have

$$M_{\mathrm{loc},\le \mathrm{r}} f(x) = \sup_{0<R\le r} \sup_{a\in Q(R)} |\chi_R(\cdot - a) * f(x)|.$$

In view of this remark, we have

$$\begin{aligned} M_{\mathrm{loc},\le 2\mathrm{r}} w(x) &\le 2^n\, M_{\mathrm{loc},\le \mathrm{r}} M_{\mathrm{loc},\le \frac{3}{2}\mathrm{r}} w(x) \\ &\le 4^n\, M_{\mathrm{loc},\le \mathrm{r}} M_{\mathrm{loc},\le \mathrm{r}} M_{\mathrm{loc},\le \mathrm{r}} w(x) \\ &\le 4^n\, A^{\mathrm{loc}}_{p,r}(w)^3 w(x). \end{aligned}$$

This is the desired result for $p = 1$.

With the estimate

$$w(Q') \le c_w\, w(Q)$$

for all cubes Q, Q' such that $\ell(Q) \le \ell(Q') \le \ell(Q) + 1$, $Q \subset Q'$. It is not so hard to see

$$A_{p,r+1}(w) \le c_w\, A_{p,r}(w).$$

This proves the assertion *2*. ∎

The following proposition is a tool that allows us to reduce the matter to the theory of A_∞-weights.

Proposition 7.4. [18] *Let $r > 0$, Q be a cube with $\ell(Q) = r$ and $w \in A^{\mathrm{loc}}_p$ with $1 < p < \infty$. Then there exists $\overline{w} \in A_p$ such that*

$$A_p(\overline{w}) \le c\, A^{\mathrm{loc}}_p(w) \text{ and } \overline{w} = w \text{ on } Q.$$

Proof. We content ourselves with constructing $\overline{w}$. It is straightforward to check that $\overline{w}$ is the desired weight. Rychkov [18] proved above for $r = 1$. The passage to the general case is essentially the same.

Let us write $Q = \prod_{i=1}^{n} [q_i, q_i + r]$. For $v \in \mathbb{R}$ and $u \in [v, v + 2r)$, define

$$\tau_v(u) := \begin{cases} u & \text{if} \quad u \in [v, v + r), \\ 2(v + r) - u & \text{if} \quad u \in [v + r, v + 2r), \end{cases}$$

and the $2r\mathbb{Z}^n$-periodic function $\overline{w}$ by

$$\overline{w}(x) = w(\tau_{q_1}(x_1), \ldots, \tau_{q_n}(x_n)) \quad \text{if} \quad x \in \prod_{i=1}^{n} [q_i, q_i + 2r).$$

Then $\overline{w}$ is the desired weight. ∎

8 Weighted function space $\mathcal{A}^s_{pqr}(w)_{\mathrm{loc}}$

8.1 Weighted local-Morrey maximal inequality

Throughout this part we assume that $w \in A_u^{\mathrm{loc}}$ with $1 \le u < \infty$ for definiteness.

Proposition 8.1. *Let $0 < q \le p < \infty$ and $0 < r \le \infty$. Suppose that $w \in A_u^{\mathrm{loc}}$ and that $0 < \eta < \min(q^{-1}, u^{-1})$. Then*

$$\left\| \{M_{\mathrm{loc}}^{(\eta)} f_j\}_{j \in \mathbb{N}_0} \, : \, \mathcal{M}_q^p(l_r, w)_{\mathrm{loc}} \right\| \le c \, \left\| \{f_j\}_{j \in \mathbb{N}_0} \, : \, \mathcal{M}_q^p(l_r, w)_{\mathrm{loc}} \right\|$$

for some constant $c > 0$.

Proof. To prove this, from the definition of the norm we have only to prove that

$$w(Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \left(\sum_{j \in \mathbb{N}_0} M_{\mathrm{loc}}^{(\eta)} {f_j}^r \right)^{\frac{q}{r}} w \right)^{\frac{1}{q}} \le c \, \left\| \{f_j\}_{j \in \mathbb{N}_0} \, : \, \mathcal{M}_q^p(l_r, w)_{\mathrm{loc}} \right\|$$

for all cubes Q having sidelength less than 1. Fix such a cube Q. Let R be a cube which has side-length 1 and is concentric to Q.

By Proposition 7.4, there exists a weight $\overline{w} \in A_\infty$ such that

$$w = \overline{w} \text{ on } 3R, \ A_p(\overline{w}) \le c \, A_p^{\mathrm{loc}}(w).$$

Observe that

$$w(Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \left(\sum_{j \in \mathbb{N}_0} M_{\mathrm{loc}}^{(\eta)} {f_j}^r \right)^{\frac{q}{r}} w \right)^{\frac{1}{q}} = \overline{w}(Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \left(\sum_{j \in \mathbb{N}_0} M^{(\eta)} [\chi_{3Q} \cdot f_j]^r \right)^{\frac{q}{r}} \overline{w} \right)^{\frac{1}{q}}.$$

Now let us use the vector-valued maximal inequality (for A_∞-weights) to the right-hand side. Using this inequality, we obtain

$$\begin{aligned} \overline{w}(Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \left(\sum_{j \in \mathbb{N}_0} M^{(\eta)} [\chi_{3Q} \cdot f_j]^r \right)^{\frac{q}{r}} \overline{w} \right)^{\frac{1}{q}} &\le c \, \| \{\chi_{3Q} \cdot f_j\}_{j \in \mathbb{N}_0} \, : \, \mathcal{M}_q^p(l_r, \overline{w}) \| \\ &\le c \, \| \{\chi_{3Q} \cdot f_j\}_{j \in \mathbb{N}_0} \, : \, \mathcal{M}_q^p(l_r, w) \|. \end{aligned}$$

Since we are assuming that Q has sidelength less than 1, it follows that

$$\| \{\chi_{3Q} \cdot f_j\}_{j \in \mathbb{N}_0} \, : \, \mathcal{M}_q^p(l_r, w) \| \simeq \| \{\chi_{3Q} \cdot f_j\}_{j \in \mathbb{N}_0} \, : \, \mathcal{M}_q^p(l_r, w)_{\mathrm{loc}} \|.$$

Therefore, putting our observations all together, we obtain

$$w(Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \left(\sum_{j \in \mathbb{N}_0} M_{\mathrm{loc}}^{(\eta)} {f_j}^r \right)^{\frac{q}{r}} w \right)^{\frac{1}{q}} \le c \, \| \{f_j\}_{j \in \mathbb{N}_0} \, : \, \mathcal{M}_q^p(l_r, w)_{\mathrm{loc}} \|.$$

The proof is now therefore complete. ∎

Proposition 8.2. *Let $1 < q \le p < \infty$ and $1 < r \le \infty$. Assume that $w \in A_q^{\rm loc}$. Then there exist constants $c, B(w) > 0$ such that*

$$\left\| \{K_B f_j\}_{j \in \mathbb{N}_0} \,:\, \mathcal{M}_q^p(l_r, w)_{\rm loc} \right\| \le c \left\| \{f_j\}_{j \in \mathbb{N}_0} \,:\, \mathcal{M}_q^p(l_r, w)_{\rm loc} \right\|$$

for all B with $B > \dfrac{B(w)}{p}$.

Proof. Let Q be a fixed cube with sidelength 1. Since K_B is a linear operator, we can assume that $f_1 = f$, $f_2 = f_3 = \ldots = f_j = \ldots = 0$. For details of this technique, we refer to [19] for example. Note that

$$\begin{aligned} K_B f(x) &\le c \sum_{m \in \mathbb{Z}^n} e^{-B|x-m|} \int_{m+[0,1]^n} |f| \\ &\le c \sum_{m \in \mathbb{Z}^n} e^{-B|x-m|} w(m+[0,1]^n)^{-\frac{1}{q}} \left(\int_{m+[0,1]^n} |f|^q \, w \right)^{\frac{1}{q}} \\ &\le c \sum_{m \in \mathbb{Z}^n} e^{-B|x-m|} w(m+[0,1]^n)^{-\frac{1}{p}} \| f \,:\, \mathcal{M}_q^p(w)_{\rm loc} \|. \end{aligned}$$

As a consequence

$$\begin{aligned} \|K_B \,:\, \mathcal{M}_q^p(w)_{\rm loc}\| &\le c \|f \,:\, \mathcal{M}_q^p(w)_{\rm loc}\| \cdot \sup_{\substack{Q: cube \\ \ell(Q) \le 1}} \sum_{m \in \mathbb{Z}^n} \left(\frac{e^{-B|c(Q)-m|} w(Q)^{\frac{1}{p}}}{w(m+[0,1]^n)^{\frac{1}{p}}} \right) \\ &\le c \|f \,:\, \mathcal{M}_q^p(w)_{\rm loc}\|. \end{aligned}$$

This is the desired result. ∎

8.2 Definition and elementary properties

Now we define the function space keeping the results due to Rychkov in mind.

Definition 8.3. Suppose that the parameters p, q, r, s satisfy

$$0 < q \le p < \infty,\ 0 < r \le \infty,\ s \in \mathbb{R}.$$

Let $\varphi_0 \in \mathcal{D}$. Define

$$\varphi_j(x) = 2^{jn} \varphi_0(2^j x) - 2^{(j-1)n} \varphi_0(2^{j-1} x)$$

for $j \in \mathbb{N}$. Assume that $\displaystyle\int \varphi_0 \ne 0$ and φ_1 has vanishing moment up to order $\max(-1, [s])$. Then define

$$\begin{aligned} \|f \,:\, \mathcal{N}_{pqr}^s(w)_{\rm loc}\| &:= \|\{2^{js} \varphi_j * f\}_{j \in \mathbb{N}_0} \,:\, l_r(\mathcal{M}_q^p(w)_{\rm loc})\| \\ \|f \,:\, \mathcal{E}_{pqr}^s(w)_{\rm loc}\| &:= \|\{2^{js} \varphi_j * f\}_{j \in \mathbb{N}_0} \,:\, \mathcal{M}_q^p(l_r, w)_{\rm loc}\|. \end{aligned}$$

$\mathcal{A}_{pqr}^s(w)_{\rm loc}$ denotes either $\mathcal{N}_{pqr}^s(w)_{\rm loc}$ or $\mathcal{E}_{pqr}^s(w)_{\rm loc}$.

Theorem 8.4. *Suppose that the parameters p, q, r, s and the weight w satisfy*

$$w \in A_\infty^{\rm loc},\ 0 < q \le p < \infty,\ 0 < r \le \infty,\ s \in \mathbb{R}.$$

Then the definition of the set $\mathcal{A}_{pqr}^s(w)_{\rm loc}$ does not depend on $\{\varphi_j\}$ above.

Proof. Let us concentrate on the $\mathcal{E}$-case, the proof for the $\mathcal{N}$-case being similar. Suppose that $\{\widetilde{\varphi_j}\}_{j\in\mathbb{N}_0}$ is a family that satisfies the same condition as $\{\varphi_j\}_{j\in\mathbb{N}_0}$. Then we have

$$2^{js}\widetilde{\varphi}^*_{j,A,B}f(x) \le c \sum_{k\in\mathbb{N}_0} 2^{-\epsilon|j-k|}2^{ks}\varphi^*_{k,A,B}f(x),$$

where $\widetilde{\varphi}^*_{j,A,B}f$ and $\varphi^*_{k,A,B}f$ are the maximal operators given by (11). Therefore, it follows that

$$\left(\sum_{j\in\mathbb{N}_0} 2^{jsr}\widetilde{\varphi}^*_{j,A,B}f(x)^r\right)^{\frac{1}{r}} \le c\left(\sum_{j\in\mathbb{N}_0} 2^{jsr}\varphi^*_{j,A,B}f(x)^r\right)^{\frac{1}{r}}.$$

By the definition of the maximal operator $\widetilde{\varphi}^*_{j,A,B}$, we obtain

$$\|2^{js}\widetilde{\varphi}_j * f \,:\, \mathcal{M}^p_q(l_r,w)_{\rm loc}\| \le \left\|2^{js}\widetilde{\varphi}^*_{j,A,B}f \,:\, \mathcal{M}^p_q(l_r,w)_{\rm loc}\right\| \le c\left\|2^{js}\varphi^*_{j,A,B}f \,:\, \mathcal{M}^p_q(l_r,w)_{\rm loc}\right\|.$$

Now we invoke estimate

$$\varphi^*_{j,A,B}f(x)^\eta \le c\sum_{k=j}^{\infty} 2^{(j-k)(A\,\eta-n)}\left(M_{\rm loc}[|\varphi_k * f|^\eta](x) + K_{B\,\eta}[|\varphi_k * f|^\eta](x)\right),$$

which yields

$$\left(\sum_{j\in\mathbb{N}_0} 2^{jsr}\varphi^*_{j,A,B}f(x)^r\right)^{\frac{1}{r}} \le c\left(\sum_{j\in\mathbb{N}_0}\left(2^{jsr}M^{(\eta)}_{\rm loc}[|\varphi_j * f|^\eta](x)^r + 2^{jsr}K_{B\,\eta}[|\varphi_j * f|^\eta](x)^{\frac{r}{\eta}}\right)\right)^{\frac{1}{r}}.$$

Hence it follows that

$$\begin{aligned}
&\left\|\{2^{js}\varphi^*_{j,A,B}f\}_{j\in\mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r,w)_{\rm loc}\right\| \\
&\le c\left(\left\|\{2^{js}M^{(\eta)}_{\rm loc}[|\varphi_j * f|]\}_{j\in\mathbb{N}_0} + \{2^{js}K_{B\,\eta}[|\varphi_j * f|^\eta]^{\frac{1}{\eta}}\}_{j\in\mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r,w)_{\rm loc}\right\|\right) \\
&\le c\,\|\{2^{js}\varphi_j * f\}_{j\in\mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r,w)_{\rm loc}\|.
\end{aligned}$$

Putting our observations all together, we obtain

$$\|\{2^{js}\widetilde{\varphi}_j * f\}_{j\in\mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r,w)_{\rm loc}\| \le c\,\|\{2^{js}\varphi_j * f\}_{j\in\mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r,w)_{\rm loc}\|,$$

which shows that the function space $\mathcal{E}^s_{pqr}(w)_{\rm loc}$ is defined independently of $\{\varphi_j\}_{j\in\mathbb{N}_0}$. ∎

It is helpful to summarize our observation above in a quantified form.

Lemma 8.5. *Let $0 < q \le p < \infty$, $0 < r \le \infty$, $s \in \mathbb{R}$. There exists a constant $c > 0$ such that*

$$\left\|\sup_{y\in\mathbb{R}^n}\frac{|f*\gamma(*-y)|}{(1+|y|)^A 2^{|y|B}} \,:\, \mathcal{M}^p_q(w)_{\rm loc}\right\| \le c\,\|f \,:\, \mathcal{A}^s_{pqr}(w)_{\rm loc}\| \cdot \sup_{|\alpha|\le L}|D^\alpha\gamma|$$

for all $f \in \mathcal{A}^s_{pqr}(w)_{\rm loc}$ and $\gamma \in \mathcal{D}$.

Proposition 8.6. *The function space $\mathcal{A}^s_{pqr}(w)_{\rm loc}$ is complete.*

Proof. Note that

$$|f*\gamma(0)| \le c\,w(B(1))^{-\frac{1}{p}}\|f \,:\, \mathcal{A}^s_{pqr}(w)_{\rm loc}\| \cdot \sup_{|\alpha|\le L}|D^\alpha\gamma|$$

for all $f \in \mathcal{A}^s_{pqr}(w)_{\rm loc}$ and $\gamma \in \mathcal{D}$. This fact yields that any Cauchy sequence in $\mathcal{A}^s_{pqr}(w)_{\rm loc}$ converges at least in $\mathcal{D}'$. Once we show the convergence in $\mathcal{D}'$, it is the same as before that we conclude the convergence of the sequence in $\mathcal{A}^s_{pqr}(w)_{\rm loc}$ by using the Fatou type inequality. ∎

8.3 Atomic decomposition

Now we describe the atomic decomposition for the function space $\mathcal{A}^s_{pqr}(w)_{\rm loc}$.

Definition 8.7. Suppose that the parameters p, q, r, s satisfy

$$0 < q \le p < \infty,\ 0 < r \le \infty.$$

Let $w \in A^{\rm loc}_\infty$.

1. Let $\lambda = \{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0,\, m \in \mathbb{Z}^n}$ be a doubly indexed sequence. Then define

$$\|\lambda \,:\, \mathbf{n}_{pqr}(w)_{\rm loc}\| := \left\| \left\{ \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi^{(p)}_{\nu m} \right\}_{\nu \in \mathbb{N}_0} \,:\, l_r(\mathcal{M}^p_q(w)_{\rm loc}) \right\|$$

$$\|\lambda \,:\, \mathbf{e}_{pqr}(w)_{\rm loc}\| := \left\| \left\{ \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi^{(p)}_{\nu m} \right\}_{\nu \in \mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r, w)_{\rm loc} \right\|.$$

2. As before, to unify our formulation we use $\mathbf{a}_{pqr}(w)_{\rm loc}$ to denote either $\mathbf{n}_{pqr}(w)_{\rm loc}$ or $\mathbf{e}_{pqr}(w)_{\rm loc}$ according as $\mathcal{A}^s_{pqr}(w)_{\rm loc}$ denotes $\mathcal{N}^s_{pqr}(w)_{\rm loc}$ or $\mathcal{E}^s_{pqr}(w)_{\rm loc}$.

Theorem 8.8. *Let $w \in A^{\rm loc}_u$. Suppose that the parameters p, q, r, s and the integers K, L satisfy*

$$0 < q \le p \le \infty,\ 0 < r \le \infty,\ s \in \mathbb{R},\ K \ge (1 + [s])_+,\ L \ge \max(-1, \sigma_q + n(u-1)).$$

Assume in addition that $L \ge [\sigma_{\frac{q}{u}} - s]$ for the $\mathcal{N}$-case and that $L \ge [\sigma_{\frac{q}{u}\frac{r}{u}} - s]$ for the $\mathcal{E}$-case. Then there exists a constant $c > 0$ such that the following assertion holds.

1. *Let $f \in \mathcal{A}^s_{pqr}(w)_{\rm loc}$ be taken arbitrarily. Then there exist $\lambda \in \mathbf{a}_{pqr}(w)_{\rm loc}$ and a family of atoms $A = \{a_{\nu m}\}_{\nu \in \mathbb{N}_0{}^n,\, m \in \mathbb{Z}^n}$ such that*

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m},\ \|\lambda \,:\, \mathbf{a}_{pqr}(w)_{\rm loc}\| \le c\,\|f \,:\, \mathcal{A}^s_{pqr}(w)_{\rm loc}\|.$$

2. *Conversely suppose that $\lambda \in \mathbf{a}_{pqr}(w)_{\rm loc}$ and that $A = \{a_{\nu m}\}_{\nu \in \mathbb{N}_0{}^n,\, m \in \mathbb{Z}^n}$ is a family of atoms. Then $f := \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$ converges in the topology of $\mathcal{S}'_e$ and satisfies the norm estimate*

$$\|f \,:\, \mathcal{A}^s_{pqr}(w)_{\rm loc}\| \le c\,\|\lambda \,:\, \mathbf{a}_{pqr}(w)_{\rm loc}\|.$$

Let us remark that if $w \equiv 1$, then the conditions on K and L are exactly the ones in [29].

The proof of this theorem consists of several steps, which are split into a series of lemmas. Let us first prove

Lemma 8.9. *Suppose that $\lambda \in \mathbf{a}_{pqr}(w)_{\rm loc}$ and that $A = \{a_{\nu m}\}_{\nu \in \mathbb{N}_0{}^n,\, m \in \mathbb{Z}^n}$ is a family of atoms. Then $f := \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$ converges in the topology of $\mathcal{S}'_e$.*

Proof. By virtue of the Minkowski inequality it is trivial that

$$l_{\min(q,r)}(\mathcal{M}^p_q(w)_{\rm loc}) \subset \mathcal{M}^p_q(l_r, w)_{\rm loc} \subset l_\infty(\mathcal{M}^p_q(w)_{\rm loc})$$

in the sense of the continuous embedding. Therefore, let us assume that $a = n$ and that $r = \infty$, that is, we concentrate on the case of Besov-Morrey type.

First, let us check that the sum $\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$ converges in the topology of $\mathcal{S}_e'$ for every fixed $\nu \in \mathbb{N}_0$. Then we obtain

$$w(Q_{\nu m}) \geq c\, \exp(-N_0\, 2^{-\nu}|m|) w(2^{\nu+2}|m| Q_{\nu m}) \geq c\, \exp(-N_0\, 2^{-\nu}|m|) w(Q_{\nu 0}),$$

where N_0 and c are numbers depending on the weight w. Furthermore, we have

$$|\lambda_{\nu m}| \leq c\, w(Q_{\nu m})^{-1/p}.$$

Therefore, the coupling $\left\langle \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}, \varphi \right\rangle = \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} \lambda_{\nu m} a_{\nu m} \varphi$ converges absolutely for all $\varphi \in \mathcal{S}_e$.

In view of this we have only to prove $\lim_{P\to\infty} \sum_{\nu=1}^{P} \left(\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \right)$ converges in $\mathcal{S}_e'$.

Pick a test function $\varphi \in \mathcal{S}_e$ again. Then by virtue of the moment condition we have

$$\left\langle \sum_{\nu=1}^{P} \left(\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \right), \varphi \right\rangle = \sum_{\nu=1}^{P} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} \int_{\mathbb{R}^n} a_{\nu m} \varphi_{\nu m},$$

where $\varphi_{\nu m}$ is a remainder term of the Taylor expansion given by

$$\varphi_{\nu m}(x) := \varphi(x) - \left(\sum_{|\beta|\leq L} \frac{\partial^\beta \varphi(2^{-\nu}m)}{\beta!} (x - 2^{-\nu}m)^\beta \right).$$

By the mean value theorem we have

$$|\varphi_{\nu m}(x)| \leq c\, 2^{-\nu(L+1)} \left(\sup_{\substack{|\gamma|=L+1 \\ y\in d\, Q_{\nu m}}} |\partial^\gamma \varphi(y)| \right)$$

for $x \in d\, Q_{\nu m}$. Thus, the pointwise estimate $|a_{\nu m}(x)| \leq 2^{-\nu(s-\frac{n}{p})}$, $x \in \mathbb{R}^n$ yields

$$\begin{aligned} e^{N|x|} |a_{\nu m}(x) \varphi_{\nu m}(x)| &\leq c\, 2^{-\nu\left(s-\frac{n}{p}+L+1\right)} \left(\sup_{\substack{|\gamma|=L+1 \\ y\in d\, Q_{\nu m}}} e^{N|y|} |\partial^\gamma \varphi(y)| \right) \chi_{c\, Q_{\nu m}}(x) \\ &\leq c\, 2^{-\nu\left(s-\frac{n}{p}+L+1\right)} q_N(\varphi) \chi_{c\, Q_{\nu m}}(x). \end{aligned}$$

Here N is a constant chosen as large as we wish. Below let us assume that N is sufficiently large, say, $N \gg 1$. Adding the estimate above over $m \in \mathbb{Z}^n$, we obtain

$$\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m} a_{\nu m}(x) \varphi_{\nu m}(x)| \leq c\, 2^{-\nu\left(s-\frac{n}{p}+L+1\right)} q_N(\varphi) \cdot e^{-N|x|} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| \chi_{d\, Q_{\nu m}}(x).$$

Let us write $Q(r) := \{x \in \mathbb{R}^n \,:\, \max(|x_1|, |x_2|, \ldots, |x_n|) \leq r\}$. Inserting this pointwise estimate, we have

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|\lambda_{\nu m}a_{\nu m}\varphi_{\nu m}|\\
&\le c\,2^{-\nu\left(s-\frac{n}{p}+L+1\right)}q_N(\varphi)\int_{\mathbb{R}^n}e^{-N|x|}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\chi_{d\,Q_{\nu m}}(x)\,dx\\
&\le c\,2^{-\nu\left(s-\frac{n}{p}+L+1\right)}q_N(\varphi)\sum_{k=0}^{\infty}\exp(-2^{k-1}N)\int_{Q(2^k)}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\chi_{d\,Q_{\nu m}}\\
&\le c\,2^{-\nu\left(s-\frac{n}{p}+n+L+1\right)}q_N(\varphi)\sum_{k=0}^{\infty}\exp(-2^{k-1}N)\int_{Q(2^k)}\left|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(1)}_{\nu m}(x)\right|\,dx.
\end{aligned}
$$

Here the constant $c>0$ depends on N. We define two auxiliary constants $0<\eta<1$ and $1<\mu<\infty$ by

$$
\eta:=\frac{q}{1+(u-1)q}\quad\text{ and }\quad\mu:=\frac{q}{\eta}=1+(u-1)q.
$$

Denote by μ' the conjugate exponent of μ : $\mu'=\dfrac{\mu}{\mu-1}$. Then we have

$$
\eta\mu'=\frac{\eta\mu}{\mu-1}=\frac{1}{u-1}.
$$

Let us assume for the time being that $\eta<1$. Keeping this in mind, we estimate the integral in question :

$$
\begin{aligned}
\int_{Q(2^k)}\left|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(1)}_{\nu m}(x)\right|\,dx &\le \left(\int_{Q(2^k)}\left|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(\eta)}_{\nu m}(x)\right|^{\eta}\,dx\right)^{\frac{1}{\eta}}\\
&=2^{-\frac{n\nu}{p}+\frac{n(\nu+k)}{\eta}}\left(\int_{Q(2^k)}\left|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(p)}_{\nu m}(x)\right|^{\eta}\,dx\right)^{\frac{1}{\eta}}.
\end{aligned}
$$

If $\eta\ge 1$, a similar calculation works and we obtain

$$
\int_{Q(2^k)}\left|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(1)}_{\nu m}(x)\right|\,dx\le 2^{-\frac{n\nu}{p}+\frac{n(\nu+k)}{\eta}}\left(\int_{Q(2^k)}\left|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(p)}_{\nu m}(x)\right|^{\eta}\,dx\right)^{\frac{1}{\eta}}.
$$

Applying the Hölder inequality, we obtain

$$
\begin{aligned}
&\int_{Q(2^k)}\left|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(1)}_{Q_{\nu m}}\right|\\
&\le 2^{-\frac{n\nu}{p}+\frac{n(\nu+k)}{\eta}}\left(\int_{Q(2^k)}w^{-\eta}\left|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(p)}_{\nu m}\right|^{\eta}w^{\eta}\,dx\right)^{\frac{1}{\eta}}\\
&\le 2^{-\frac{n\nu}{p}+\frac{n(\nu+k)}{\eta}}\left(\int_{Q(2^k)}w(x)^{-\eta\mu'}\,dx\right)^{\frac{1}{\mu'}}\left(\int_{Q(2^k)}\left|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(p)}_{Q_{\nu m}}\right|^{q}w\right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $w^{-\eta\mu'}=w^{-\frac{1}{u-1}}\in A^{\mathrm{loc}}_{u'}$, we see

$$
\int_{Q(2^k)}w(x)^{-\eta\mu'}\,dx\le c\,\exp(N_{q,\eta,u,w}\cdot 2^k)\int_{Q(1)}w(x)^{-\frac{1}{u-1}}\,dx\le c\,\exp(N_{q,\eta,u,w}\cdot 2^k),
$$

where c and $N_{q,\eta,u,w}$ depend on q, η, u and the $A_u^{\rm loc}$-constant of w. Recall that N is at our disposal. Let us choose it so that $N > N_{q,\eta,u,w}$. Thus, we finally obtain

$$\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|\lambda_{\nu m}a_{\nu m}(x)\varphi_{\nu m}(x)|\,dx \le c\,2^{-\nu\left(s-\frac{n}{\eta}+n+L+1\right)}q_N(\varphi)\left\|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}\,:\,\mathcal{M}_q^p(w)_{\rm loc}\right\|.$$

Now by the assumption, L is sufficiently large:

$$s-\frac{n}{\eta}+n+L+1 > \sigma_q + u\,n - \frac{n}{\eta} = \sigma_q - n\left(\frac{1}{q}-1\right) \ge 0.$$

Thus, we are in the position of adding these inequalities over $\nu \in \mathbb{N}_0$:

$$\sum_{\nu=0}^{\infty}\left|\int_{\mathbb{R}^n}\left(\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}\right)\varphi\right| \le c\,q_N(\varphi)\,\|\lambda\,:\,\mathbf{n}_{pq\infty}(w)_{\rm loc}\|. \tag{15}$$

This proves $\displaystyle\lim_{P\to\infty}\sum_{\nu=0}^{P}\left(\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}\right)$ exists in $\mathcal{S}_e'$. ∎

Proof of Theorem 8.8 1. Denote by $\{\varphi_\nu\}_{\nu\in\mathbb{N}_0}$ the family described above. Let $\{\psi_\nu\}_{\nu\in\mathbb{N}_0}$ be taken so that

$$\sum_{\nu\in\mathbb{N}_0}\varphi_\nu * \psi_\nu = \delta$$

in $\mathcal{S}_e'$ and that $L_{\psi_1} \ge L$. Here δ means the Dirac delta distribution. Then we have

$$f = \sum_{\nu\in\mathbb{N}_0}\varphi_\nu * \psi_\nu * f = \sum_{\nu\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n}\Phi_{\nu m}(f),$$

where $\Phi_{\nu m}(f)(x) := \displaystyle\int_{Q_{\nu m}}\psi_\nu(x-y)\varphi_\nu * f(y)\,dy$. If $\nu \in \mathbb{N}$, then it enjoys the same moment condition as ψ_ν and the order L of the moment condition can be made as large as we wish. Observe also that $\Phi_{\nu m}$ is supported on $d\,Q_{\nu m}$ for some $d > 1$. Denote

$$\lambda_{\nu m} := \sup_{\substack{\alpha\in{\mathbb{N}_0}^n\\ |\alpha|\le K}} 2^{\nu(s-\frac{n}{p})-\nu|\alpha|}\|\partial^\alpha\Phi_{\nu m}(f)\,:\,L_\infty\|. \tag{16}$$

Define

$$a_{\nu m} := \begin{cases} {\lambda_{\nu m}}^{-1}\cdot\Phi_{\nu m} & \text{if } \lambda_{\nu m}\neq 0\\ 0 & \text{otherwise.}\end{cases} \tag{17}$$

Then we have from the observation above $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,\,m\in\mathbb{Z}^n} \in \mathbf{Atom}$. Furthermore f is decomposed as

$$f = \sum_{\nu\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}.$$

Let us see the size of coefficients. To do this we majorize the coefficient with the maximal operator which is given by $\varphi^*_{\nu,A,B}f(x) = \sup_{y\in\mathbb{R}^n} \dfrac{|\varphi_\nu * f(x-y)|}{m_{\nu,A,B}(y)}$.

$$
\begin{aligned}
\lambda_{\nu m} &\le \sup_{\substack{\alpha\in\mathbb{N}_0{}^n\\ |\alpha|\le K}} \sup_{x\in\mathbb{R}^n} 2^{\nu(s-\frac{n}{p})-\nu|\alpha|} \int_{Q_{\nu m}} |\partial_x^\alpha \left[\psi_\nu(x-y)\right] \varphi_\nu * f(y)|\, dy \\
&\le c\, 2^{\nu(s-\frac{n}{p})} \sup_{y\in Q_{\nu m}+Q(2^{-\nu+1})} |\varphi_\nu * f(y)| \\
&\le c\, 2^{\nu(s-\frac{n}{p})} \sup_{|y|\le c\, 2^{-\nu}} |\varphi_\nu * f(x-y)| \\
&\le c\, 2^{\nu(s-\frac{n}{p})} \varphi^*_{\nu,A,B} f(x)
\end{aligned}
$$

for all $x \in d\, Q_{\nu m}$. As a consequence

$$\sum_{\nu\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi^{(p)}_{\nu m}(x) \le c\, 2^{\nu s}\varphi^*_{\nu,A,B}f(x).$$

In view of the maximal estimate, which can now be formulated as

$$\left\|2^{\nu s}\varphi^*_{\nu,A,B} f \,:\, \mathcal{M}^p_q(l_r,w)_{\rm loc}\right\| \le c\, \|f \,:\, \mathcal{E}^s_{pqr}(w)_{\rm loc}\|$$

under our notation, we obtain

$$\|\lambda \,:\, \mathbf{e}_{pqr}(w)_{\rm loc}\| \le c\, \|f \,:\, \mathcal{E}^s_{pqr}(w)_{\rm loc}\|.$$

Thus, f was decomposed as we wish. ∎

Proof of 2. We deal with the F-scale, the proof for the B-scale being similar. Suppose that we are given

$$\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} \in \textbf{Atom} \text{ and } \lambda = \{\lambda_{\nu m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} \in a^w_{p,q}.$$

Assume that $\nu > k \ge 0$ or $\nu = k = 0$ for the time being and let us estimate $\varphi_k * a_{\nu m}$. The same argument as the non-weighted case works to obtain

$$|2^{ks}\varphi_k * a_{\nu m}(x)| \le c\, 2^{-(\nu-k)(L+1+s+n)+n\nu/p}\chi_{c_0\, 2^{\nu-k}Q_{\nu m}}(x),$$

for some $c > 0$. For details of this calculation we refer to [28]. Keeping this in mind, let us estimate

$$\sum_{m\in\mathbb{Z}^n} |2^{\nu s}\lambda_{\nu m}\varphi_k * a_{\nu m}(x)|.$$

We adopt the following notations:

$$
\begin{aligned}
&Q(x,\ell) := \{y \in \mathbb{R}^n \,:\, \max(|y_1-x_1|, |y_2-x_2|, \ldots, |y_n-x_n|) \le \ell\} \\
&B(x,\ell) := \{y \in \mathbb{R}^n \,:\, |x-y| < \ell\}.
\end{aligned}
$$

Let $\eta < \min\left(1, \dfrac{q}{u}, \dfrac{r}{u}\right)$. A trivial inequality $(a+b)^\theta \le a^\theta + b^\theta$ for $0 < \theta \le 1$ gives us

$$\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|\chi_{c\,Q(2^{-\nu}m,2^{-k})}(x) \le \sum_{\substack{m\in\mathbb{Z}^n\\ x\in B(2^{-\nu}m,c\,2^{-k})}} |\lambda_{\nu m}| \le \left(\sum_{\substack{m\in\mathbb{Z}^n\\ x\in B(2^{-\nu}m,c\,2^{-k})}} |\lambda_{\nu m}|^\eta\right)^{\frac{1}{\eta}}.$$

Therefore, it follows that

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|\chi_{c\,Q(2^{-\nu}m,2^{-k})} &\le c\,2^{\frac{\nu n}{\eta}}\left(\int_{B(x,c_0\,2^{-k})}\left|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi_{Q_{\nu m}}(y)\right|^\eta dy\right)^{\frac{1}{\eta}} \\ &= c\,2^{\frac{n(\nu-k)}{\eta}} m^{(\eta)}_{B(x,c_0\,2^{-k})}\left(\left|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi_{Q_{\nu m}}\right|\right) \\ &\le c\,2^{\frac{n(\nu-k)}{\eta}} M^{(\eta)}\left[\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi_{Q_{\nu m}}\right](x).\end{aligned}$$

Inserting this estimate, we are led to

$$\sum_{m\in\mathbb{Z}^n}|2^{ks}\lambda_{\nu m}\varphi_k * a_{\nu m}(x)| \le c\,2^{-2\delta_0(\nu-k)}M^{(\eta)}\left[\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(p)}_{\nu m}\right](x) \tag{18}$$

for some $\delta_0>0$ and some η which is slightly less than $\min\left(1,\dfrac{q}{u},\dfrac{r}{u}\right)$, if $\nu>k\ge 0$ or $\nu=k=0$.

Let us turn to the remaining case when $k\ge\nu\ge 0$ with $k\ne 0$, which requires us an elaboration. Let $L\gg 1$. We can assume $\varphi_1=\Delta^L\rho$ for some $L\in\mathbb{N}_0$ sufficiently large and $\rho\in C^\infty_c$. For details, we refer to Lemma 6.5. Once φ_1 is decomposed as above, the same argument as the non-weighted case again works and we obtain

$$|2^{ks}\varphi_k * a_{\nu m}(x)| \le c\,2^{-(k-\nu)(K-s)+\nu\frac{n}{p}}\chi_{c_0\,Q_{\nu m}}(x) \le c\,2^{-2\delta_0(k-\nu)+\nu\frac{n}{p}}\chi_{c_0\,Q_{\nu m}}(x) \tag{19}$$

for some $\delta_0>0$. We remark that the constants δ_0 in (18) and (19) can be assumed identical if we replace them by smaller numbers if necessary. Therefore, going through a similar argument, we obtain

$$\sum_{m\in\mathbb{Z}^n}|2^{ks}\lambda_{\nu m}\varphi_k * a_{\nu m}(x)| \le c\,2^{-2\delta_0(\nu-k)}M^{(\eta)}\left[\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(p)}_{\nu m}\right](x)$$

for $k\ge\nu\ge 0$ with $k\ne 1$.

In view of this estimate, an argument similar to the non-weighted case works to obtain

$$\begin{aligned}\left\|\sum_{\nu\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m} \,:\, \mathcal{E}^s_{pqr}(w)_{\rm loc}\right\| &\le c\left\|M^{(\eta)}\left[\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(p)}_{\nu m}\right] \,:\, \mathcal{M}^p_q(l_r,w)_{\rm loc}\right\| \\ &\le c\,\|\lambda \,:\, \mathbf{e}_{pqr}(w)_{\rm loc}\|.\end{aligned}$$

This is the desired result. ∎

9 Function spaces generated by the weighted global Morrey spaces

Finally in this report we consider the weighted Morrey spaces whose norm is given by

$$\|f \,:\, \mathcal{M}^p_q(w)\| := \sup_{Q:cube} w(2Q)^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q |f|^q\,w\right)^{\frac{1}{q}},$$

where we assume $w \in A_p^{\rm loc}$ again. We also use the following vector-valued norm.

$$\|f_j \,:\, \mathcal{M}^p_q(l_r, w)\| := \sup_{Q:cube} w(2Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \left(\sum_{j \in \mathbb{N}_0} |f_j|^r \right)^{\frac{q}{r}} w \right)^{\frac{1}{q}}.$$

As was shown in [21], the number 2 can be replaced by any number strictly greater than 1. It will give equivalent norm.

9.1 Weighted global maximal inequalities

Theorem 9.1. *Suppose that* $1 < q \le p < \infty$ *and* $1 < r \le \infty$. *If* $w \in A_q^{\rm loc}$, *then*

$$\|\{M_{\rm loc} f_j\}_{j \in \mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r, w)\| \le c \, \|\{f_j\}_{j \in \mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r, w)\|$$

for all $\{f_j\}_{j \in \mathbb{N}_0}$.

Proof. Going through an argument similar to (7), we can deduce

$$M_{\rm loc} f_j(x) \le c \sup_{\substack{Q:cube \\ x \in Q}} \left(\frac{1}{w(2Q)} \int_Q |f_j|^u \, w \right)^{\frac{1}{u}}.$$

As a result we can use the non-doubling theory in [21]. ∎

Theorem 9.2. *Suppose that* $1 < q \le p < \infty$ *and* $1 < r \le \infty$. *If* $w \in A_q^{\rm loc}$, *then*

$$\|\{K_B f_j\}_{j \in \mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r, w)\| \le c \, \|\{f_j\}_{j \in \mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r, w)\|$$

whenever $B > \dfrac{B(w)}{q}$.

Proof. Note that

$$|K_B f(x)| \le c \sum_{j=1}^{\infty} e^{-Bj} \, M_j^{\rm loc} f(x),$$

where

$$M_j^{\rm loc} f(x) := \sup_{0 < \ell \le j} \frac{1}{|B(x,j)|} \int_{B(x,j)} |f(y)| \, dy$$

If we reexamine the proof of [21], we obtain

$$\|M_j^{\rm loc} f \,:\, \mathcal{M}^p_q(\mu)\| \le c \, \exp(D\, j) \|f \,:\, \mathcal{M}^p_q(\mu)\|$$

for some $D > 0$. As a result, taking $B > D$, we have

$$\|K_B f \,:\, \mathcal{M}^p_q(\mu)\| \le c \, \|f \,:\, \mathcal{M}^p_q(\mu)\|,$$

which is the desired result. ∎

9.2 Definition and elementary properties

Keeping to the notation used in this section, we define the function space.

Definition 9.3. Suppose that the parameters p, q, r, s satisfy

$$0 < q \le p < \infty,\ 0 < r \le \infty,\ s \in \mathbb{R}.$$

Let $\varphi_0 \in \mathcal{D}$. Define

$$\varphi_j(x) := 2^{jn}\varphi_0(2^j x) - 2^{(j-1)n}\varphi_0(2^{j-1}x)$$

for $j \in \mathbb{N}$. Assume that $\displaystyle\int \varphi_0 \ne 0$ and that φ_1 has vanishing moment up to order $\max(-1, [s])$. Then define

$$\|f \,:\, \mathcal{N}^s_{pqr}(w)\| := \|\{2^{js}\varphi_j * f\}_{j\in\mathbb{N}_0} \,:\, l_r(\mathcal{M}^p_q(w))\|$$
$$\|f \,:\, \mathcal{E}^s_{pqr}(w)\| := \|\{2^{js}\varphi_j * f\}_{j\in\mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r, w)\|.$$

$\mathcal{A}^s_{pqr}(w)$ denotes either $\mathcal{N}^s_{pqr}(w)$ or $\mathcal{E}^s_{pqr}(w)$.

Theorem 9.4. *Suppose that the parameters p, q, r, s and the weight w satisfy*

$$w \in A^{\rm loc}_\infty,\ 0 < q \le p < \infty,\ 0 < r \le \infty,\ s \in \mathbb{R}.$$

Then the definition of the set $\mathcal{A}^s_{pqr}(w)$ does not depend on $\{\varphi_j\}_{j=0}^\infty$ above.

Proposition 9.5. $\mathcal{A}^s_{pqr}(w) \subset \mathcal{A}^s_{pqr}(w)_{\rm loc}$.

Proof. This is clear from the structure of the underlying function spaces $\mathcal{M}^p_q(w) \subset \mathcal{M}^p_q(w)_{\rm loc}$. ∎

Corollary 9.6. *$\mathcal{A}^s_{pqr}(w)$ is complete.*

9.3 Atomic decomposition

Now we describe the atomic decomposition for the function space $\mathcal{A}^s_{pqr}(w)$.

Definition 9.7. Suppose that the parameters p, q, r, s satisfy

$$0 < q \le p < \infty,\ 0 < r \le \infty.$$

Let $w \in A^{\rm loc}_\infty$.

1. Let $\lambda = \{\lambda_{\nu m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n}$ be a doubly indexed sequence. Then define

$$\|\lambda \,:\, \mathbf{n}_{pqr}(w)\| := \left\| \left\{ \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi^{(p)}_{\nu m} \right\}_{\nu\in\mathbb{N}_0} \,:\, l_r(\mathcal{M}^p_q(w)) \right\|$$
$$\|\lambda \,:\, \mathbf{e}_{pqr}(w)\| := \left\| \left\{ \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi^{(p)}_{\nu m} \right\}_{\nu\in\mathbb{N}_0} \,:\, \mathcal{M}^p_q(l_r, w) \right\|.$$

2. As before, to unify our formulation we use $\mathbf{a}_{pqr}(w)$ to denote either $\mathbf{n}_{pqr}(w)$ or $\mathbf{e}_{pqr}(w)$ according as $\mathcal{A}^s_{pqr}(w)$ denotes $\mathcal{N}^s_{pqr}(w)$ or $\mathcal{E}^s_{pqr}(w)$.

Theorem 9.8. *Let $w \in A_u^{\mathrm{loc}}$. Suppose that the parameters p, q, r, s and the integers K, L satisfy*

$$0 < q \le p < \infty,\ 0 < r \le \infty,\ s \in \mathbb{R},\ K \ge (1 + [s])_+,\ L \ge \max(-1, \sigma_q + n(u-1)).$$

Assume in addition that $L \ge [\sigma_{\frac{q}{u}} - s]$ for the $\mathcal{N}$-case and that $L \ge [\sigma_{\frac{q}{u}\frac{r}{u}} - s]$ for the $\mathcal{E}$-case. Then there exists a constant $c > 0$ such that the following assertion holds.

1. *Let $f \in \mathcal{A}^s_{pqr}(w)$ be taken arbitrarily. Then there exist $\lambda \in \mathbf{a}_{pqr}(w)$ and a family of atoms $A = \{a_{\nu m}\}_{\nu \in \mathbb{N}_0{}^n,\, m \in \mathbb{Z}^n}$ such that*

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m},\ \|\lambda : \mathbf{a}_{pqr}(w)\| \le c \, \|f : \mathcal{A}^s_{pqr}(w)\|.$$

2. *Conversely suppose that $\lambda \in \mathbf{a}_{pqr}(w)$ and that $A = \{a_{\nu m}\}_{\nu \in \mathbb{N}_0{}^n,\, m \in \mathbb{Z}^n}$ is a family of atoms. Then $f := \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$ converges in the topology of $\mathcal{S}'$ and satisfies the norm estimate*

$$\|f : \mathcal{A}^s_{pqr}(w)\| \le c \, \|\lambda : \mathbf{a}_{pqr}(w)\|.$$

The fact that the series converges in $\mathcal{S}'_e$ is clear from Proposition 9.5. The remaining part will be proved elsewhere.

Part IV

Open problems

Finally to conclude this report we present some problems.

10 Weighted Morrey maximal inequality

First we consider the Morrey counterpart of the A_p-class.

Problem 10.1. *Let $1 < q \le p < \infty$. Characterize the weight w satisfying*

$$\|Mf : \mathcal{M}^p_q(w)\| \le c \, \|f : \mathcal{M}^p_q(w)\|.$$

We note that if we replace $\mathcal{M}^p_q(w)$ to $\mathcal{M}^p_q(w)_{\mathrm{loc}}$, then the problem was solved recently.

As for this problem Komori and Shirai [10] gave a partial answer. They showed, as we have seen in this paper, that $w \in A_q$ is a sufficient condition.

11 Characterization of predual of the Morrey spaces

It is not easy to characterize the dual of the Morrey spaces. However, we are still able to construct the predual of the Morrey spaces. The second question is whether we can obtain a characterization of the predual space.

Definition 11.1. Let $1 \le p \le q < \infty$. An L^q function a is said to be a block, if it is supported on a cube Q and $\|a\|_q \le |Q|^{\frac{1}{q}-\frac{1}{p}}$.

Definition 11.2. Let $1 \le p \le q < \infty$. $\mathcal{H}^p_q$ is a set of all functions f such that it can be written as $f = \sum_{j=1}^{\infty} \lambda_j \, a_j$ where each a_j is a block and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. One defines $\|f \,:\, \mathcal{H}^p_q\| := \inf \sum_{j=1}^{\infty} |\lambda_j|$, where the sequence in inf runs over all admissible expression above.

The following are due to Zorko for example.

Theorem 11.3. [1, 31] *Let $1 < q \le p < \infty$. Then $\mathcal{H}^{q'}_{p'}$ is a Banach space and the dual is $\mathcal{M}^p_q$.*

Problem 11.4. *Let $1 < q \le p < \infty$. Let f be a measurable function such that*

$$\int |fg| \le c \, \|g \,:\, \mathcal{M}^p_q\|$$

for all $g \in \mathcal{M}^p_q$. Then can we conclude that $f \in \mathcal{H}^p_q$?

It is a well-known fact that the answer is yes in the case $p = q$. It is also easy to see that the converse of the above conjuncture holds. However, given a function above, can we really prove that f admits the decomposition described above ?

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