RIMS Kôkyûroku Bessatsu
B22 (2010), 137–144

On the Uniqueness Property of Various Maximal Operators

Lasha Ephremidze

Razmadze Mathematical Institute, Tbilisi, Georgia

1. Introduction. Let $\mathcal{K}$ be any operator which maps one function space into another. The property of $\mathcal{K}$ that it is one-to-one, i.e. the injectivity property of $\mathcal{K}$, is frequently used in studying some problems related to this operator. In many situations such uniqueness theorem for $\mathcal{K}$ is an easy consequence of more profound results concerning $\mathcal{K}$. For example, when $\mathcal{K}$ is the Fourier transform, then we have the inversion formula both for discrete and continuous cases and for the conjugate operator or the Hilbert transform $\mathcal{K}^2 f = -f$. These imply the uniqueness of $\mathcal{K}$. On the other hand, there exist integral operators for which the uniqueness theorem fails to hold.

The situation is different for the classical Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{x\in I} \frac{1}{|I|}\int_I |f(t)|\,dt. \tag{1}$$

Here the uniqueness property cannot be deduced from other known results and it is difficult to handle it in general since M is nonlinear operator and no technique of Banach spaces can be used.

In spite of the fact that this operator plays very important role in Harmonic and Real Analysis, its uniqueness property apparently suffers from the luck of applications. However, we see sufficient theoretical interest to investigate this problem. Obviously, whenever one studies whether operator M is one-to-one, either it should be restricted on the class of positive functions or the modulus sign should be omitted in the definition (1). Both approaches are acceptable and we do so whenever we formulate uniqueness results for different types of maximal operators. We emphasize that most of these results are obtained for one-sided maximal functions. In some sense, we propose a method to reconstruct f from corresponding maximal function. The uniqueness problem for two-sided (classical) Hardy-Littlewood maximal operator is still open even on the real line, though the author believes that the fact itself is correct. The problem seems even more difficult for different type of maximal operators in the high dimensional Euclidian spaces where no positive result is obtained so far.

It should be mentioned that the continuous character of the system of intervals with respect to which the supremum is taken in (1) plays an important role in the validity of obtained uniqueness theorems. Otherwise, if we take the maximal function with respect to a discrete system of partitions, the uniqueness property fails to hold in general (see [8]). We provide a simple counterexample for the dyadic maximal function.

2. The dyadic maximal function. Let $\mathcal{I}$ be the set of dyadic intervals in $[0,1)$,

$$\mathcal{I} = \left\{ \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) : n = 0,1,\dots; k = 0,1,\dots,2^n-1 \right\},$$

and let M_d be the dyadic maximal operator,

$$M_d f(x) = \sup_{x\in I\in\mathcal{I}} \frac{1}{|I|}\int_I |f|\,dm, \qquad f \in L[0,1),$$

where m stands for the Lebesgue measure on $\mathbb{R}$. The following counterexample shows that M_d is not one-to-one on the set of positive integrable functions.

EXAMPLE 1. Let non-negative functions f and g satisfy the following conditions

$$f(x)=\begin{cases}1 \text{ for } x\in[0,1/4),\\ 0 \text{ for } x\in[1/4,1/2),\end{cases}\qquad g(x)=\begin{cases}0 \text{ for } x\in[0,1/4),\\ 1 \text{ for } x\in[1/4,1/2),\end{cases}$$

$f(x)=g(x)$ for $x\in[1/2,1)$ and

$$\int_{1/2}^{1} f\,dm=\int_{1/2}^{1} g\,dm=1.$$

Obviously such functions exist. We have

$$M_df(x)=M_dg(x)=1.25 \quad \text{for } x\in[0,1/2)$$

and

$$M_df(x)=M_dg(x)\geq\frac{1}{1-0.5}\int_{1/2}^{1} f\,dm=2 \quad \text{for } x\in[1/2,1)\,.$$

Thus, $M_df=M_dg$ while f and g are different.

3. The one-sided Hardy-littlewood maximal function. For a locally integrable function $f\in L_{\rm loc}(\mathbb{R})$, let

$$M_+f(x)=\sup_{y>x}\frac{1}{y-x}\int_x^y f\,dm,\qquad x\in\mathbb{R}.$$

Let $L(\mathbb{T})\subset L_{\rm loc}(\mathbb{R})$ be the class of 2π-periodic integrable (on $\mathbb{T}=[0,2\pi)$) functions.

The main positive result concerning to the uniqueness of maximal functions is the following theorem obtained in [1].

THEOREM 1. *Let $f,g\in L(\mathbb{T})$ and*

$$M_+f(x)=M_+g(x)\quad \textit{for each } x\in\mathbb{R}.$$

Then

$$f(x)=g(x)\quad \textit{for a.a. } x\in\mathbb{R}.$$

We present a little bit different proof than the one given in [1], emphasizing that actually the function f can be reconstructed in unique way from the values of M_+f. It consists of several lemmas.

LEMMA 1. *For $f\in L(\mathbb{T})$, we have*

$$\inf_{x\in\mathbb{T}} M_+f(x)=M_+f(x_0)=\frac{1}{2\pi}\int_{\mathbb{T}} f\,dm=:\lambda_f \tag{2}$$

for some $x_0\in\mathbb{T}$.

Proof. Since f is 2π-periodic,

$$M_+f(x)\geq\frac{1}{2\pi}\int_x^{x+2\pi} f\,dm=\lambda_f \text{ for each } x\in\mathbb{R}. \tag{3}$$

The function

$$h(x)=\int_0^x f\,dm-\lambda_f x$$

is continuous and 2π-periodic, and $h(0) = h(2\pi) = 0$. Thus, if $x_0 \in [0, 2\pi)$ is maximum point of h,

$$h(x_0) = \sup_{x\in\mathbb{R}} h(x),$$

then, for each $x > x_0$, we have

$$0 \geq h(x) - h(x_0) = \int_{x_0}^{x} f\,dm - \lambda_f(x - x_0).$$

So that $M_+f(x_0) \leq \lambda_f$, and taking into account (3), (2) holds. □

In the sequel we use the following lemma due to Riesz

"Rising Sun" Lemma. *For $f \in L_{\text{loc}}(\mathbb{R})$ and $\lambda \in \mathbb{R}$, if (α, β) is a finite connected component of $\{x \in \mathbb{R} : M_+f(x) > \lambda\} =: \{M_+f > \lambda\}$, then*

$$\int_{\alpha}^{\beta} f\,dm = \lambda(\beta - \alpha). \tag{4}$$

Since $M_+f(\alpha) \leq \lambda$, we have $\int_\alpha^y f\,dm \leq \lambda(y - \alpha)$ and subtracting it from (4), we get

$$\int_{y}^{\beta} f\,dm \geq \lambda(\beta - y) \text{ for each } y \in (\alpha, \beta). \tag{5}$$

We use this relation in the sequel as well.

LEMMA 2. *For a.a. $x \in \{M_+f = \lambda_f\}$, we have*

$$f(x) = \lambda_f.$$

Proof. It follows from Lemma 1 that

$$\mathbb{R}\backslash\{M_+f > \lambda_f\} = \{M_+f = \lambda_f\} \tag{6}$$

and 2π-periodic set $\{M_+f > \lambda_f\}$ does not coincide with $\mathbb{R}$.

By the Lebesgue differentiation theorem

$$f(x) \leq \lambda_f \text{ for a.a. } x \in \mathbb{R}\backslash\{M_+f > \lambda_f\} = \{M_+f = \lambda_f\}. \tag{7}$$

By virtue of the "rising sun" lemma

$$\int_{\{M_+f>\lambda_f\}\cap\mathbb{T}} f\,dm = \lambda_f m\big(\{M_+f > \lambda_f\} \cap \mathbb{T}\big)$$

and, subtracting this equation from $\int_\mathbb{T} f\,dm = 2\pi\lambda_f$ (see (2)), we get

$$\int_{(\mathbb{R}\backslash\{M_+f>\lambda_f\})\cap\mathbb{T}} f\,dm = \lambda_f m\big((\mathbb{R}\backslash\{M_+f > \lambda_f\}) \cap \mathbb{T}\big).$$

Thus, taking into account (6),

$$\int_{\{M_+f=\lambda_f\}\cap\mathbb{T}} f\,dm = \lambda_f m\big(\{M_+f = \lambda_f\} \cap \mathbb{T}\big).$$

and, by virtue of (7), the lemma follows. □

It follows from Lemmas 1 and 2 that the function M_+f uniquely determines f on $\{M_+f = \lambda_f\}$. The following lemma completes the proof of Theorem 1 by determining f on the remaining set.

LEMMA 3. *Let $f \in L(\mathbb{T})$, and let (α,β) be a finite connected component of $\{M_+ f > \lambda_f\}$. For each $x \in (\alpha,\beta)$, we have that*

$$\int_x^\beta f\,dm \tag{8}$$

is uniquely determined by the values of the function $M_+ f$ on (α,β).

Proof. Assume $x \in (\alpha,\beta)$ fixed and let

$$\lambda_x = M_+ f(x).$$

For each $\lambda \in [\lambda_f, \lambda_x)$, let $(\alpha_\lambda, \beta_\lambda)$ be the connected component of $\{M_+ f > \lambda\}$ which contains x. Note that $(\alpha_{\lambda_f}, \beta_{\lambda_f}) = (\alpha,\beta)$, by the hypothesis.

The intervals $(\alpha_\lambda, \beta_\lambda)$, $\lambda \in [\lambda_f, \lambda_x)$, are imbedded,

$$(\alpha_\lambda, \beta_\lambda) \supset (\alpha_\tau, \beta_\tau), \quad \lambda_f \le \lambda \le \tau < \lambda_x,$$

and

$$(\alpha_\lambda, \beta_\lambda) = \cup_{\tau > \lambda} (\alpha_\tau, \beta_\tau).$$

Thus, introducing the function $\Psi : [\lambda_f, \lambda_x] \longrightarrow [x, \beta]$ defined by

$$\Psi(\lambda) = \begin{cases} \beta_\lambda, & \text{if } \lambda_f \le \lambda < \lambda_x, \\ x \text{ if } \lambda = \lambda_x, \end{cases}$$

one can see that Ψ is non-increasing and continuous from the right. Observe that Ψ is uniquely determined by $M_+ f$.

Let D be the set of discontinuity points of Ψ,

$$D = \big\{\lambda \in (\lambda_0, \lambda_x] : \Psi(\lambda -) \ne \Psi(\lambda)\big\},$$

and C be the set where Ψ does not decrease strictly,

$$C = \big\{\lambda \in [\lambda_0, \lambda_x] : \Psi(\lambda') = \Psi(\lambda) \text{ for some } \lambda' \ne \lambda\big\}$$

Note that D is at most countable and C consists of at most countable union of disjoint intervals.

Let

$$E = (\lambda_0, \lambda_x) \backslash (D \cup C).$$

Ψ is strictly decreasing on E, so that Ψ is one-to-one on E, and Ψ^{-1} exists on $\Psi(E) = \{y \in [x,\beta] : \exists \lambda \in E \text{ such that } \Psi(\lambda) = y\}$.

Since Ψ is monotonic function, we have a disjoint representation of image of Ψ:

$$(x,\beta) = \Psi(E) \bigcup \cup_{\lambda \in D} (\Psi(\lambda), \Psi(\lambda -)),$$

where the equation holds up to a set of measure 0. Thus

$$\int_x^\beta f\,dm = \int_{\Psi(E)} f\,dm + \sum_{\lambda \in D} \int_{\Psi(\lambda)}^{\Psi(\lambda-)} f\,dm. \tag{9}$$

We will deduce from (9) that

$$\int_x^\beta f\,dm = \int_{\Psi(E)} \Psi^{-1} dm + \sum_{\lambda \in D} \lambda\big(\Psi(\lambda -) - \Psi(\lambda)\big). \tag{10}$$

Thus (8) will be expressed in terms of Ψ, and the lemma will be proved.

For each $\lambda \in E$, we have

$$f(\Psi(\lambda)) = \lambda \tag{11}$$

whenever $\Psi(\lambda)$ is the Lebesgue point of f. Indeed, since $\beta_\lambda = \Psi(\lambda) \notin \{M_+f = \lambda\}$, $M_+f(\Psi(\lambda)) \leq \lambda$, so that

$$f(\Psi(\lambda)) \leq \lambda. \tag{12}$$

On the other hand, by virtue of (5),

$$\frac{1}{\beta_\lambda - y}\int_y^{\beta_\lambda} f\,dm \geq \lambda$$

for each $y \in (\alpha_\lambda, \beta_\lambda)$, so that letting $y \to \beta_\lambda$, we get

$$f(\beta_\lambda) = f(\Psi(\lambda)) \geq \lambda \tag{13}$$

whenever $\beta_\lambda = \Psi(\lambda)$ is the Lebesgue point of f. (13) and (12) imply (11), so that $f = \Psi^{-1}$ almost everywhere on $\Psi(E)$ and we have

$$\int_{\Psi(E)} f\,dm = \int_{\Psi(E)} \Psi^{-1} dm. \tag{14}$$

For each $\lambda \in D$,

$$\int_{\Psi(\lambda)}^{\Psi(\lambda-)} f\,dm = \lambda\big(\Psi(\lambda-) - \Psi(\lambda)\big). \tag{15}$$

Indeed, since $M_+f(\Psi(\lambda)) \leq \lambda$, we have

$$\int_{\Psi(\lambda)}^{\Psi(\lambda-)} f\,dm \leq \lambda\big(\Psi(\lambda-) - \Psi(\lambda)\big). \tag{16}$$

For each $\lambda' < \lambda$, we have $\Psi(\lambda) \in (a_{\lambda'}, b_{\lambda'})$. So that, by virtue of (5),

$$\int_{\Psi(\lambda)}^{\Psi(\lambda')} f\,dm \geq \lambda'\big(\Psi(\lambda') - \Psi(\lambda)\big). \tag{17}$$

Letting $\lambda' \to \lambda$ from below, we have $\Psi(\lambda') \to \Psi(\lambda-)$. Thus it follows from (17) that

$$\int_{\Psi(\lambda)}^{\Psi(\lambda-)} f\,dm \geq \lambda\big(\Psi(\lambda-) - \Psi(\lambda)\big). \tag{18}$$

(15) follows from (16) and (18), and the relations (9), (14), and (15) imply (10). □

Exactly in the same way one can prove

COROLLARY 1. *Let $f \in L(\mathbb{R})$, and let (α, β) be a finite connected component of $\{M_+f > \lambda\}$. For each $x \in (\alpha, \beta)$, we have that*

$$\int_x^\beta f\,dm$$

is uniquely determined by the values of the function M_+f on (α, β).

The values of maximal function M_+f on a set of complete measure S, $m(\mathbb{R}\backslash S) = 0$, uniquely determines M_+f everywhere on $\mathbb{R}$. This follows from the equation

$$M_+f(x) = \lim_{\delta\to 0+} \operatorname*{ess\,inf}_{y\in(x,x+\delta)} M_+f(y) \tag{19}$$

proved in [2, Lemma 3]. Thus we can slightly strengthen Theorem 1.

THEOREM 2. *Let $f, g \in L(\mathbb{T})$ and*

$$M_+f(x) = M_+g(x) \quad \textit{for a.a.} \quad x \in \mathbb{R}.$$

Then

$$f(x) = g(x) \quad \textit{for a.a.} \ x \in \mathbb{R}.$$

For non-periodic integrable functions on the real line, Theorem 1 is not valid in general since $M_+f \equiv 0$ for every negative $f \in L(\mathbb{R})$. But M_+ remains one-to-one on the set of nonnegative functions.

THEOREM 3. *Let $0 \le f, g \in L(\mathbb{R})$ and*

$$M_+f(x) = M_+g(x) \tag{20}$$

for a.a. $x \in \mathbb{R}$. Then

$$f(x) = g(x) \quad \textit{for a.a.} \ x \in \mathbb{R}.$$

Proof. By virtue of (19), we can assume that (20) holds for every $x \in \mathbb{R}$.

Like in the proof of Theorem 1, we show that M_+f uniquely determines f.

Obviously, if $M_+f(x) = 0$, then $f(y) = 0 = M_+f(y)$ for a.a. $y > x$. Thus $f = 0$ a.e. on $\{M_+f = 0\}$.

For each $\lambda > 0$, $m\{M_+f > \lambda\} < \infty$ since the weak (1,1) type inequality

$$m\{M_+f > \lambda\} \le \frac{1}{\lambda} \int_{\mathbb{R}} f\, dm$$

holds for operator M_+ (see [6]). Thus each connected component of $\{M_+f > \lambda\}$ is finite, and we can use Corollary 1. Consequently f is uniquely determined by M_+f on

$$\{M_+f > 0\} = \cup_{\lambda\searrow 0}\{M_+f > \lambda\}$$

as well. □

4. The ergodic case. Let $(T_t)_{t\in\Gamma}$ be a group of measure-preserving transformations on a probability space $(X, \mathbb{S}, P)$, $P(X) = 1$, where Γ is either $\mathbb{Z}$ (the discrete case) or $\mathbb{R}$ (the continuous case). It means that

a) $T_t : X \longrightarrow X$ is a measurable map for each $t \in \Gamma$;

b) $T_t \circ T_\tau = T_{t+\tau}$, $t.\tau \in \Gamma$ and $T_0 = id_X$;

c) $P(T_tS) = P(S)$ for each $S \in \mathbb{S}$ and $t \in \Gamma$;

In the continuous case it is assumed that the map $(x, t) \mapsto T_tx$ from $X \times \mathbb{R}$ to X is $\mathbb{S}\otimes\mathcal{B}$ measurable, where $\mathcal{B}$ stands for the Borel σ-algebra on $\mathbb{R}$.

Without lose of generality (for current purposes), we assume that $(T_t)_{t\in\Gamma}$ is irreducible or ergodic. It means that $P(S \triangle T_tS) = 0$ for each $t \in \Gamma$, where $A \triangle B = (A\backslash B) \cup (B\backslash A)$, implies that either $P(S) = 0$ or $P(X\backslash S) = 0$.

The ergodic maximal function is defined as

$$f^*(x) = \begin{cases} \sup_{b>0} \frac{1}{b} \int_0^b f(T_tx)dt \text{ in the continuous case,} \\ \sup_{n>0} \frac{1}{n} \sum_{k=0}^{n-1} f(T^kx) \text{ in the discrete case,} \end{cases}$$

and it plays the similar role in Ergodic Theory as Hardy-Littlewood maximal function does in Harmonic Analysis.

The following theorem has been proved for the continuous case in [2] and for the discrete case in [3]. In the latter case another simple proof of the theorem has been also proposed in [7].

THEOREM 4. *Let $f, g \in L(X)$ and*

$$f^*(x) = g^*(x) \text{ for a.a. } x \in X \text{ (with respect to measure } P).$$

Then

$$f(x) = g(x) \quad \text{for a.a. } x \in X.$$

In the discrete case Theorem B has been generalized for the two sided maximal operator $f \mapsto \sup_{n,m \geq 0} \frac{1}{n+m+1} \sum_{k=-n}^{m} f(T^k x)$ in [4], while in the continuous case, the proof of the uniqueness theorem has not yet been found even for the Hardy-Littlewood maximal operator $Mf(x) = \sup_{a<x<b} \frac{1}{b-a} \int_a^b f(t)\,dt$ on the real line, as it was mentioned in the introduction.

For σ-finite measure spaces, the ergodic maximal operator is one-to-one on the set of positive functions (see [3, Theorem 2]) as it is in Theorem 3.

5. Maximal functions of Borel measures. Maximal functions can be defined not only for integrable functions, but for measures as well. Let $\mathcal{M}(\mathbb{T})$ denotes the set of (signed) Borel measures ν on the unit circle. For notational convenience we assume that the one-sided maximal functions $M_+\nu$ of such measures are 2π-periodic functions on the real line defined by

$$M_+\nu(x) = \sup_{y \in (x, x+2\pi]} \frac{\nu[x, y)}{y - x},$$

where it is always naturally assumed that $\nu(B) = \nu\{e^{i\theta} : \theta \in B\}$, whenever a Borel measurable set $B \subset [x, x+2\pi)$ for some $x \in \mathbb{R}$. At the same time, without causing any confusion, we assume that $\nu(B) = \nu(B \cap \mathbb{T})$, whenever B is a 2π-periodic subset of $\mathbb{R}$.

As the following counter example shows there exist $\nu, \mu \in \mathcal{M}(\mathbb{T})$ such that that $M_+\mu = M_+\nu$ everywhere except one point, while μ and ν are essentially different.

EXAMPLE 2. Let $\delta_{\{x\}} \in \mathcal{M}(\mathbb{T})$ be the Dirac measure concentrated at e^{ix} and $\mu, \nu \in \mathcal{M}(\mathbb{T})$ be defined by the equalities: $\nu = \delta_{\{0\}} = \delta_{\{2\pi\}}$ and $\mu = g dm - \delta_{\{\pi\}} + \delta_{\{2\pi\}}$, where $g(e^{ix}) = \pi/(2\pi - x)^2$ for $x \in (0, \pi)$ and $g(e^{ix}) = 0$ for $x \in [\pi, 2\pi]$. Then, for each $x \in (0, 2\pi]$, we have

$$M_+\nu(x) = \frac{1}{2\pi - x} \quad \text{and} \quad M_+\mu(x) = \begin{cases} \frac{1}{2\pi - x} & \text{for } x \neq \pi, \\ 1/4\pi & \text{for } x = \pi. \end{cases}$$

Thus Theorem 2 cannot be generalized for Borel measures. Nevertheless, we claim that the following generalization of Theorem 1 is valid.

THEOREM 5. *Let $\nu, \mu \in \mathcal{M}(\mathbb{T})$ and*

$$M_+\nu(x) = M_+\mu(x) \quad \text{for every } x \in \mathbb{R}.$$

Then

$$\mu = \nu.$$

A general idea of proving Theorem 5 is similar to the one used for the proof of Theorem 1, but the details are much more involved (see [5]).

References

[1] L. Ephremidze, On the uniqueness of maximal functions, Georgian Math. J. 3 (1996), 49–52.
[2] L. Ephremidze, On the uniqueness of the maximal operators for ergodic flows, Rev. Mat. Complut. 15 (2002), 1–10.
[3] L. Ephremidze, On the uniqueness of the ergodic maximal fuction, Fund. Math. 174 (2002), 217–228.
[4] L. Ephremidze, On the uniqueness of the two-sided ergodic maximal fuction, Georgian Math. J. 12 (2005), 45–52.
[5] L. Ephremidze and N. Fujii, On the uniqueness of the one-sided maximal functions of Borel measures, J. Math. Soc. Japan (to appear).
[6] M. Guzman, Deferentiation of integrals in $\mathbb{R}^n$, Lecture Notes in Mathematics, 481, Springer-Verlag, 1975.
[7] R. L. Jones, On the uniqueness of the ergodic maximal fuction, Proc. Amer. Math. Soc. 132 (2003), 1087–1090.
[8] P. Sjögren, How to recognize a discrete maximal function, Indiana Univ. Math. J. 37 (1988), 891–898.

A. Razmadze Mathematical Institute
Georgian Academy of Sciences
Tbilisi 0193, Georgia
E-mail: lasha@rmi.acnet.ge