On the global existence of semirelativistic Hartree equations

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Abstract

In this paper we brief some recent results on the global well-posedness of semirelativistic Hartree type equations. Then we improve the theories for radial solutions developed in [6] and extend them to nonradial cases.

1 Problems and results

We consider the following Cauchy problem:

$$\begin{cases} i\partial_t u = \sqrt{1-\Delta}\, u + F(u) \ \text{ in } \ \mathbb{R}^n \times \mathbb{R},\ n \geq 1, \\ u(x,0) = \varphi(x) \ \text{ in } \ \mathbb{R}^n. \end{cases} \tag{1.1}$$

Here $F(u)$ is nonlinear functional of Hartree type such that $F(u) = (V_\gamma * |u|^2)u$, where $V_\gamma(x) = \lambda|x|^{-\gamma}, 0 < \gamma < n, \lambda \in \mathbb{R} \setminus \{0\}$ and $*$ denotes the convolution in $\mathbb{R}^n$. This equation is a generalization of the model case $\gamma = 1$ which is derived rigorously via the mean field theory for the quantum many body system of boson

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particles (boson star) with Coulomb type or gravitational interaction. Hence one may think u is the wave function of one particle. For the details see [8, 9] and see also [12] for stationary problems.

If the solution u of (1.1) has sufficient decay at infinity and smoothness, it satisfies two conservation laws:

$$\begin{aligned} \|u(t)\|_{L^2} &= \|\varphi\|_{L^2} \ \ (L^2 \ \text{conservation}),\\ E(u(t)) \equiv K(u) + V(u) &= E(u(0)) = E(\varphi) \ \ (\text{energy conservation}), \end{aligned} \tag{1.2}$$

where $K(u) = \frac{1}{2}\langle\sqrt{1-\Delta}\, u, u\rangle$, $V(u) = \frac{1}{4}\langle F(u), u\rangle$ and $\langle , \rangle$ is the complex inner product in L^2. As to be seen, the Sobolev space $H^{\frac{1}{2}}$ is the energy space. For the proof of (1.2) a regularizing method is simply applicable as in [13] in the case of $0 < \gamma \le 1$. For local solutions constructed by a contraction argument based on the Strichartz estimates stated below, the case of $1 < \gamma \le 2$ is treated by exactly the same method as in [17] without using approximate or regularizing approach.

The problem for local and global well-posedness was treated first by Lenzmann [13] for the case that $\gamma = 1$ and $n = 3$. By the scaling argument on the massless equation $i\partial_t u = \sqrt{-\Delta} u + F(u)$, the potential V_1 is known to be L^2-critical (actually, L^2-critical for $n \ge 2$). Hence the L^2 and energy conservation laws were used efficiently for the global well-posedness together with the Hardy inequalities such that for $0 < \gamma \le 1, s \ge 0$

$$\sup_{x\in\mathbb{R}^n} |V_\gamma * |u|^2(x)| \lesssim \|u\|_{\dot H^{\frac{\gamma}{2}}}^2 \ \text{ and } \ \||x|^{-\gamma} * |u|^2\|_{H^s_{2n}} \lesssim \|u\|_{H^s}\|u\|_{H^{\frac{\gamma}{2}}}. \tag{1.3}$$

In [13] Lenzmann also showed the global well-posedness for negative λ by assuming the smallness of $L^2(\mathbb{R}^3)$ norm of initial data that $\|\varphi\|_{L^2} < \|Q\|_{L^2}$, where $Q \in H^{\frac{1}{2}}(\mathbb{R}^3)$ is a strictly positive solution of

$$\sqrt{-\Delta} Q + (\frac{\lambda}{|x|} * |Q|^2)Q = -Q$$

and satisfies that $\|Q\|_{L^2} > 4/\pi|\lambda|$. By the same argument, Lenzmann's results were extended to the case when $0 < \gamma \le 1$ and $n \ge 1$ in [2, 4]. Since the case $\gamma < 1$ is L^2 subcritical, the global well-posedness can be easily shown by the energy conservation. See [4]. For the quantum system of fermions (white dwarf), we refer the readers to [11] in which the global well-posedness has been established recently by the similar argument.

The finite time blowup in the energy space is expected for large data as is the case for other dispersive equations, when $\lambda < 0$. But this is not a simple matter. It has been known only for $\gamma = 1$, $n \ge 3$ and for the radial symmetric case. It is necessary to estimate a variance type inequality of which the crucial estimtaes are

$$|V_1 * |u|^2(x)| \le \|\varphi\|_{L^2}^2/|x| \ \text{ and } \ |\nabla(V_1 * |u|^2)(x)| \le \|\varphi\|_{L^2}^2/|x|^2.$$

The variance estimate is possible because the solution u is radial and $\gamma = 1$. For the details see [10] and Remark 5 of [6]. If u is radial and $1 < \gamma \le 2$, then by Lemma 4.7 below there holds the estimate

$$V_\gamma * |u|^2(x) \lesssim \||\cdot|^{-(\gamma-1)/2}u\|_{L^2}/|x|.$$

But we do not know whether the right-hand side can be controlled uniformly in time or not because it may blow up in finite time. It is still open to show the finite time blowup of nonradial solutions for $\gamma = 1$ or of the solution for the nonlinearity with $\gamma > 1$.

If we consider the case $\gamma > 1$ (this case corresponds to the L^2-supercritical one), then we cannot use the Hardy inequality (1.3) any more because the righthand side is energy-supercritical. To circumvent this difficulty, in [2, 4] the authors used a Strichartz estimate for the evolution group $U(t) = e^{-it\sqrt{1-\Delta}}$ and the refined Hardy inequalities such that for any $1 < \gamma < n$, $0 < \varepsilon < n - \gamma$ and $s \ge 0$

$$\begin{aligned} \sup_{x\in\mathbb{R}^n} |V_\gamma * |u|^2(x)| &\lesssim \|u\|_{L^{\frac{2n}{n-(\gamma+\varepsilon)}}} \|u\|_{L^{\frac{2n}{n-(\gamma-\varepsilon)}}}, \\ \||x|^{-\gamma} * |u|^2\|_{H^s_{\frac{2n}{\gamma+\varepsilon}}} &\lesssim \|u\|_{L^{\frac{2n}{n-(\gamma-\varepsilon)}}} \|u\|_{H^s}. \end{aligned} \tag{1.4}$$

The Strichartz estimate is the following (see [15, 16]):

$$\begin{aligned} \|U(t)\varphi\|_{L_T^{q_0} H_{r_0}^{s_0-\alpha_0}} &\lesssim \|\varphi\|_{H^{s_0}}, \\ \left\| \int_0^t U(t-t')F(t')\,dt' \right\|_{L_T^{q_1} H_{r_1}^{s_1-\alpha_1}} &\lesssim \|F\|_{L_T^1 H^{s_1}}, \end{aligned} \tag{1.5}$$

where $(q_i, r_i), i = 0, 1$, satisfy that for some $\theta \in [0, 1]$

$$\begin{aligned} \frac{2}{q_i} = (n-1+\theta)\left(\frac{1}{2} - \frac{1}{r_i}\right), \;\; 2\alpha_i = (n+1+\theta)\left(\frac{1}{2} - \frac{1}{r_i}\right), \\ 2 \le q_i, r_i \le \infty, \;\; (q_i, r_i) \ne (2, \infty). \end{aligned} \tag{1.6}$$

We call the pair (q, r) satisfying (1.6) *admissible pair.* If $\theta = 0$, it is called wave admissible and if $\theta = 1$, then Schrödinger admissible.

Using the Strichartz estimate (1.5) with the wave admissible pairs and Hardy inequality (1.4), in [2, 4] the global well-posedness was shown for $1 < \gamma < \frac{2n}{n+1}$ in $C_b(H^{\frac{1}{2}}) \cap L^q_{loc}(H_r^{\frac{1}{2}-\alpha})$, where $q = \frac{4n}{(n-1)\beta}$, $r = \frac{2n}{n-\beta}$ and $\alpha = \frac{(n+1)\beta}{4n}$ for some $\beta < \frac{2n}{n+1}$ but sufficiently close to $\frac{2n}{n+1}$. Since the case $\gamma > 2, n \ge 3$ is energy supercritical, one cannot use the energy conservation argument. The endpoint Schrödinger admissible pair is useful for the small data global well-posedness and scattering of this energy-supercritical case. More precisely, if $2 < \gamma < n$, $n \ge 3$, $s > \frac{\gamma}{2} - \frac{n-2}{2n}$ and if $\varphi \in H^s$ and $\|\varphi\|_{H^s}$ is sufficiently small, then (1.1) has a unique solution $u \in C_b(H^s) \cap L^2(H^{s-\frac{n+2}{2n}}_{\frac{2n}{n-2}})$ and there is $\varphi^\pm \in H^s$ such that

$$\|u(t) - U(t)\varphi^\pm\|_{H^s} \to 0 \;\text{ as }\; t \to \pm\infty.$$

One of the most important properties of wave functions is that it has finite propagation speed. Here, we say that a wave has the finite propagation speed if its group velocity, the gradient of phase function in frequency space, is bounded. Since $|\nabla\sqrt{1+|\xi|^2}| = |\xi|/\sqrt{1+|\xi|^2} \le 1$, the propagation speed does not exceed the value 1. From this we can show that

$$\lim_{t\to\infty}\int_{|y|\le At} |U(t)\varphi^+(y)|^2\,dy = \|\varphi^+\|_{L^2}^2 \tag{1.7}$$

for any $A > 1$[1]. Hence we deduce that like the Klein-Gordon and wave equations, the nonexistence of scattering is expected for long range case (small γ). In fact, if $0 < \gamma \le 1$ for $n \ge 3$ and $0 < \gamma < \frac{n}{2}$ for $n = 1, 2$, then it can be shown from (1.7) together with the dispersive estimate[2] $\|U(t)\varphi^+\|_{L^\infty} \lesssim t^{-n/2}$ that the functional $H(t) = sgn(\lambda)\mathrm{Re}\langle u(t), U(t)\varphi^+\rangle \gtrsim \log t$ for sufficiently large t, which contradicts the uniform boundedness of H.

Now we consider the case $\frac{2n}{n+1} \le \gamma \le 2$ for the global well-posedness. This case is energy (sub)critical. Hence the global well-posedness is strongly expected by analogy with the Schrödinger and Klein-Gordon equations. But contrary to the Klein-Gordon equation, our equation (1.1) has a different type of inhomogeneous term when we rewrite the solution according to the Duhamel's principle. There is a regularity gain $(1-\Delta)^{-1}$ in the case of Klein-Gordon equation $(-\int_0^t (1-\Delta)^{-1}\sin((t-s)\sqrt{1-\Delta})\,F(u)\,ds)$. The solution u of (1.1) can be written as the following integral equation

$$u(t) = U(t)\varphi - i\int_0^t U(t-t')F(u)(t')dt', \tag{1.8}$$

where

$$(U(t)\varphi)(x) = (e^{-it\sqrt{1-\Delta}}\varphi)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi - t\sqrt{1+|\xi|^2})}\widehat{\varphi}(\xi)\,d\xi.$$

Here $\widehat{\varphi}$ denotes the Fourier transform of φ such that $\widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\,dx$. In this sense, the semirelativistic equation is said to have a regularity preserving nonlinearity like Dirac equation which makes a trouble in using the Strichartz estimates (1.5) since a regularity loss is already in the estimates. Up to now, we still do not know how to treat this regularity preserving property when we consider general initial data. Recently, there have been some improvements under the radial symmetry assumption. In this paper we are going to survey the results for solutions of radial symmetry in case that $\frac{2n}{n+1} \le \gamma \le 2$.

In the next four sections, we introduce two types of Strichartz estimates and generalized Hardy inequalities for functions with angular regularity, and how to show the global well-posedness of the radial solution of the equation (1.1). In Section 6 we extend the global result of radial solution for $\gamma = 2, n \ge 4$ to the

[1] (1.7) is stronger than the one in [2]. But it can be easily verified by a slight modification of the original proof of [2]

[2] For this we need smoothness and decay for φ^+.

nonradial ones with the help of generalized Hardy inequalities. We show the global well-posedness in $C_b(\mathbb{R}; \dot H^{\frac12} H_\sigma^{\frac32}) \cap L^2(\mathbb{R}; |x| L_r^2 H_\sigma^2)$ for $\gamma = 2$ and $n \geq 4$. See Section 2 for the definition of function spaces. The final section is devoted to show some properties of Besov and Sobolev spaces on the polar coordinates.

2 Function spaces

To proceed let us introduce several function spaces to be used in this paper. Let $\mathcal{P}$ denote the totality of polynomials and $\mathcal{S}_0 = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \forall\alpha\ \partial^\alpha \mathcal{F}\varphi(0) = 0\}$. For any $f \in \mathcal{S}'/\mathcal{P} = \mathcal{S}_0'$, the homogeneous Sobolev space $\dot H_p^s$ is defined by $\dot H_p^s = \{f \in \mathcal{S}_0' : \||\nabla|^s f\|_{L^p} < \infty\}$, $s \in \mathbb{R}, 1 \leq p \leq \infty$, where $|\nabla| = \sqrt{-\Delta}$. $H_p^s = (1-\Delta)^{-s/2} L^p$ is the inhomogeneous Sobolev space. We denote $\dot H_2^s$ and H_2^s by $\dot H^s$ and H^s, respectively . We denote the space $L^q(-T, T; B)$ by $L_T^q(B)$ and its norm by $\|\cdot\|_{L_T^q B}$ for some Banach space B, and also $L^q(\mathbb{R}; B)$ by $L^q(B)$ with norm $\|\cdot\|_{L^q B}$, $1 \leq q \leq \infty$. $C_b(B) = C_b(\mathbb{R}; B)$ is the space of bounded and continuous B-valued functions on $\mathbb{R}$.

For the Strichartz estimates of radial functions the following hybrid Sobolev space is useful:

Definition 1. For $s, s' \in \mathbb{R}$ and $1 < p < \infty$

$$\widetilde H_p^{s,s'} = \{f \in \mathcal{S}_0' : \|v\|_{\widetilde H_p^{s,s'}} \equiv \|D^{s,s'} v\|_{L^p} < \infty\}.$$

where $D^{s,s'} = |\nabla|^s (1-\Delta)^{\frac{s'-s}{2}}$.

Next we introduce the Besov space with the mixed norms on the polar coordinates.

Definition 2. For any $1 \leq p, q, \rho \leq \infty$ and $s \in \mathbb{R}$, we define the seminorm $\|\cdot\|_{\dot B^s_{p,q,\rho}}$ for any $f \in \mathcal{S}'(\mathbb{R}^n)$:

$$\|f\|_{\dot B^s_{p,q,\rho}} = \|2^{sj} \varphi_j * f\|_{\ell_j^\rho(\mathbb{Z}; L_r^p L_\sigma^q)},$$

where $\widehat{\varphi_j * f}(\xi) = \chi(\xi/2^j)\widehat f(\xi)$ for some radial Littlewood- Paley function χ such that $\chi(\xi) = 1$ for $|\xi| \leq 1$ and $\chi(\xi) = 0$ for $|\xi| \geq 2$. Here we used

$$\|h\|_{L_r^p L_\sigma^q} = \| \|h(r\sigma)\|_{L_\sigma^q} \|_{L^p(r^{n-1}dr)}, \quad 1 \leq p, q \leq \infty.$$

We also define the seminormed space $\dot B^s_{p,q,\rho}$ by

$$\dot B^s_{p,q,\rho} = \{\sum_{j\in\mathbb{Z}} \varphi_j * f \in \mathcal{S}_0'(\mathbb{R}^n) : \|f\|_{\dot B^s_{p,q,\rho}} < \infty\}.$$

This definition makes sense as a space embedded in $\mathcal{S}'(\mathbb{R}^n)$ if the series converges whenever the semi-norm is finite, which is the case if $s < n/p$ or $s = n/p$ and $\rho = 1$. We will deal with those cases only. Obviously, $\dot B^s_{p,p,\rho} = \dot B^s_{p,\rho}$ is the standard (isotropic) homogeneous Besov space.

We can also define the Sobolev or Bessel-potential space on the polar coordinates:

Definition 3. For any $s, \alpha \in \mathbb{R}$ and $1 < p, q < \infty$, we define the seminormed space $\dot{H}^s_p H^\alpha_{q,\sigma}$ and the norm $\dot{H}^s_p H^\alpha_{q,\sigma}$ for any $f \in \mathcal{S}'_0$ as follows:

$$\dot{H}^s_p H^\alpha_{q,\sigma} = \{f \in \mathcal{S}'_0(\mathbb{R}^n) : \|f\|_{\dot{H}^s_p H^\alpha_{q,\sigma}} < \infty\}, \tag{2.1}$$

and

$$\|f\|_{\dot{H}^s_p H^\alpha_{q,\sigma}} = \||\nabla|^s D^\alpha_\sigma f\|_{L^p_r L^q_\sigma}, \tag{2.2}$$

where $D_\sigma = \sqrt{1-\Delta_\sigma}$.

If $\alpha = 0$, we then denote $\dot{H}^s_p H^0_{q,\sigma}$ by $\dot{H}^s_{p,q}$. $\dot{H}^s_{p,p} = \dot{H}^s_p$ is the standard (isotropic) homogeneous Sobolev space. It is also clear that $\dot{H}^0_{p,q} = L^p_r L^q_\sigma$. Similarly, the inhomogeneous version is defined as follows: for $s, \alpha \in \mathbb{R}$ and $1 \le p, q \le \infty$

$$\|f\|_{H^s_p H^\alpha_{q,\sigma}} = \|(1-\Delta)^{\frac{s}{2}} D^\alpha_\sigma f\|_{L^p_r L^q_\sigma}. \tag{2.3}$$

We denote $H^s_2 H^\alpha_{2,\sigma}$ and $H^0 H^\alpha_\sigma$ by $H^s H^\alpha_\sigma$ and $L^2_r H^\alpha_\sigma$, respectively.

Finally, we consider the weighted hybrid Sobolev space with angular regularity:

Definition 4. $\widetilde{H}^{s,s'}_{a,2} H^\alpha_\sigma = \{f \in \mathcal{S}'_0 : \|f\|_{\widetilde{H}^{s,s'}_{a,2} H^\alpha_\sigma} \equiv \||\cdot|^{-a} D^{s,s'} D^\alpha_\sigma v\|_{L^2} < \infty\}$, where $0 < a < n$, $s, s' \in \mathbb{R}$.

Since Δ and $|x|$ commute with Δ_σ, $\|v\|_{\widetilde{H}^{s,s'}_{a,2} H^\alpha_\sigma} = \|v\|_{H^\alpha_\sigma \widetilde{H}^{s,s'}_{a,2}}$.

3 Improved Strichartz estimates

In this section we introduce two type of Strichartz estimates. The first one is an improvement of $L^q_T L^p$ type Strichartz estimate for radially symmetric functions. If φ and F are radially symmetric, then by the well-known decay property of the Fourier transform of measure on the unit sphere that $|\widehat{d\sigma}(\xi)| \lesssim (1+|\xi|)^{-\frac{n-1}{2}}$, the estimate (1.5) can be extended as follows:

$$\begin{aligned} \|U(t)\varphi\|_{L^p_T \widetilde{H}^{\frac{1}{2}, s-\alpha}_p} &\lesssim \|\varphi\|_{H^s}, \\ \left\|\int_0^t U(t-t')F(t')\,dt'\right\|_{L^p_T \widetilde{H}^{\frac{1}{2}, s-\alpha}_p} &\lesssim \|F\|_{L^1_T H^s}, \end{aligned} \tag{3.1}$$

where $s \in \mathbb{R}$ and $\frac{2n}{n-1} < p < \infty, \alpha = \frac{n}{2} - \frac{n+1}{p}$. Let us observe that the embedding $H^{\frac{1}{2}} \cap \widetilde{H}^{\frac{1}{2}, \frac{1}{2}-\alpha}_{\frac{2n}{n-1}} \hookrightarrow L^r$ holds for any $\frac{2n}{n-1} \le r \le \frac{2n}{n-1-2(\frac{1}{2}-\alpha)} < \frac{2n}{n-2}$. This plays a key role in the proof of global well-posedness. The second inequality follows immediately from the generalized Minkowski inequality. See [3] for details.

Interpolating (1.5) and (3.1), we get a Strichartz estimate with wider range of pairs (q, r). For the global well-posedness of radial solutions, we need only the pairs $(q, \frac{2n}{n-1})$ with q slightly larger than $\frac{2n}{n-1}$. Actually, given $\varepsilon > 0$ we can find q and α such that $\frac{2n}{n-1} < q < \frac{2n}{n-1} + \varepsilon$, $\frac{1}{2n} < \alpha < \frac{1}{2n} + \varepsilon$ and

$$\begin{aligned} \|U(t)\varphi\|_{L_T^q \widetilde{H}^{\frac{1}{2},\frac{1}{2}-\alpha}_{\frac{2n}{n-1}}} &\lesssim \|\varphi\|_{H^{\frac{1}{2}}}, \\ \left\| \int_0^t U(t-t')F(t')\,dt' \right\|_{L_T^q \widetilde{H}^{\frac{1}{2},\frac{1}{2}-\alpha}_{\frac{2n}{n-1}}} &\lesssim \|F\|_{L_T^1 H^{\frac{1}{2}}}, \end{aligned} \tag{3.2}$$

With these pairs we can choose the value of α close to $\frac{1}{2n}$ in such a way that $\gamma < 2 - 2\alpha$, provided γ is close to $\frac{2n-1}{n}$.

The other Strichartz estimate is a L^2-weighted version. Let $\frac{1}{2} < a < \frac{n}{2}$ and $n \geq 2$. Then for any $\varphi \in H^s$, $s \geq 0$, we have

$$\|U(t)\varphi\|_{L_T^2(\widetilde{H}^{1-a,s+1/2-a}_{a,2} H_\sigma^{a-\frac{1}{2}})} \lesssim \|\varphi\|_{H^s}. \tag{3.3}$$

The constants in the estimates can be chosen independently of T. For the proof of (3.3), one can use the spherical harmonic expansion. From the duality and interpolation argument it follows that for $1 \leq q \leq 2$

$$\| \int_0^T U(t-t')F(t')\,dt' \|_{L_T^2 \widetilde{H}^{s',s''}_{a,2} H_\sigma^\alpha} \lesssim \|F\|_{L_T^q \widetilde{H}^{0,1/2}_{-2a(1-1/q),2}}, \tag{3.4}$$

where $s' = (1-a)(3-2/q)$, $s'' = s' - (1-1/q)$ and $\alpha = (a-1/2)(3-2/q)$. For details see Section 2 of [6].

For the nonhomogeneous Strichartz estimate, one can use the low-diagonal operator estimate in [1, 7, 18] as follows:

Lemma 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach spaces. Let K be an operator defined by kernel k mapping $\mathcal{B}$ to $\mathcal{A}$ such that $KG(t) = \int_0^T k(t-t')G(t')\,dt'$ and satisfy that $\|KG\|_{L_T^q(\mathcal{A})} \leq C\|G\|_{L_T^p(\mathcal{B})}$ for some $1 \leq p, q \leq \infty$. Here C does not depend on T. If $p < q$, then the low-diagonal operator $\widetilde{K}$ defined by $\widetilde{K}G = \int_0^t k(t-t')G(t')\,dt'$ satisfies that $\|\widetilde{K}G\|_{L_T^q(\mathcal{A})} \leq \widetilde{C}\|G\|_{L_T^p(\mathcal{B})}$, where $\widetilde{C}$ does not depend on T.*

Let $\mathcal{A} = \widetilde{H}^{s',s''}_{a,2} H_\sigma^\alpha$ for s', s'', α as in (3.4), $\mathcal{B} = \widetilde{H}^{0,1/2}_{-2a(1-1/q),2}$ and $k(t)$ be $U(t)$. Then if $1 \leq q < 2$, we get the nonhomogeneous Strichartz estimate as follows.

$$\| \int_0^t U(t-t')F(t')\,dt' \|_{L_T^2 \widetilde{H}^{s',s''}_{a,2} H_\sigma^\alpha} \lesssim \|F\|_{L_T^q \widetilde{H}^{0,1/2}_{-2a(1-1/q),2}}, \tag{3.5}$$

where $s' = (1-a)(3-2/q)$, $s'' = s' - (1-1/q)$ and $\alpha = (a-1/2)(3-2/q)$.

Remark 1. In view of the weighted nonhomogeneous Strichartz estimate (3.5), we can get a better angular regularity gain, if $a, q > 1$. However, instead, we lose the spatial regularity because $s', s'' < 0$ and we need a spatial decay for F as $|x| \to \infty$.

For the global well-posedness we need only the case that $a = 1, s = \frac{1}{2}$ and $q = 1$. In that case we can obtain a simple L^2-weighted Strichartz estimate such that

$$\begin{aligned} \||x|^{-1}U(t)\varphi\|_{L^2_T(L^2_r H^{\frac{1}{2}}_\sigma)} &\lesssim \|\varphi\|_{\dot H^{\frac{1}{2}}}, \\ \left\||x|^{-1}\int_0^t U(t-t')F(t')\,dt'\right\|_{L^2_T(L^2_r H^{\frac{1}{2}}_\sigma)} &\lesssim \|F\|_{L^1_T \dot H^{\frac{1}{2}}}. \end{aligned} \tag{3.6}$$

Here by a simple argument the inhomogeneous Sobolv norm of $H^{\frac{1}{2}}$ was replaced by the homogeneous one of $\dot H^{\frac{1}{2}}$ Actually, without using Lemma 3.1 the second inequality of (3.6) can be easily shown by the generalized Minkowski inequality as of (3.1).

4 Weighted Sobolev inequalities on the polar coordinates

The Hardy inequalities (1.3) and (1.4) can be improved for radial functions. More generally, we can extend the radial improvements, Lemmas 2 and 3 of [6], to the non-radial case in this section. By using different amount of regularity in the radial and the angular directions, we can get more general Sobolev type inequalities with weights.

Lemma 4.1. *Let $p, q, q_1 \in [1, \infty]$, $0 \le \gamma \le (n-1)/p$ and*

$$\frac{1}{q_1} \ge \frac{1}{q} - \frac{1}{p} + \frac{\gamma}{n-1}.$$

Then we have

$$\||x|^\gamma f\|_{L^\infty_r L^{q_1}_\sigma} \lesssim \|f\|_{\dot B^{n/p-\gamma}_{p,q,1}}. \tag{4.1}$$

Moreover, if $0 < \gamma/(n-1) < 1/p$ and $\gamma/(n-1) \ne 1/p - 1/q$, then we have

$$\||x|^\gamma f\|_{L^\infty_r L^{q_1,\infty}_\sigma} \lesssim \|f\|_{\dot B^{n/p-\gamma}_{p,q,\infty}} \lesssim \|f\|_{\dot H^{n/p-\gamma}_{p,q}}, \tag{4.2}$$

where $L^{q_1,\infty}_\sigma$ is the weak L^{q_1} space on the unit sphere.

Proof. We start with (4.1). If $\gamma = 0$, it is the special case of (I) in Proposition 7.2:

$$\dot B^{n/p}_{p,q,1} \subset \dot B^0_{\infty,q_1,1} \subset L^\infty_r L^{q_1}_\sigma.$$

Next we consider the other endpoint $\gamma = (n-1)/p$. By integration in the radial direction, we have for any $r > 0$ and $\sigma \in S^{n-1}$,

$$|f(r\sigma)| \le \int_r^\infty |\partial_r f(s\sigma)|ds \le r^{1-n}\int_0^\infty |\partial_r f(s\sigma)|s^{n-1}ds.$$

Integrating for σ, we get

$$\||x|^{n-1}f\|_{L^\infty_r L^q_\sigma} \le \|\partial_r f\|_{L^q_\sigma L^1_r} \le \|\partial_r f\|_{L^1_r L^q_\sigma} \lesssim \|f\|_{\dot B^1_{1,q,1}}, \tag{4.3}$$

where we used the Minkowski inequality and (II) of Proposition 7.2. Then by the complex interpolation with the trivial embedding $\dot B^0_{\infty,q,1} \subset L^\infty_r L^q_\sigma$, we obtain (4.1) for $\gamma = (n-1)/p$.

For the first inequality of (4.2), if $\gamma/(n-1) \in (0, 1/p - 1/q)$, then we can apply the real interpolation to (4.1) with p, q fixed but γ, q_1 changing slightly. Then by (IV) of Proposition 7.2 and also the interpolation property of the Lorentz spaces, we get (4.2). If $1/p - 1/q < \gamma/(n-1) < 1/p$, then we can apply the real interpolation in the same way but with $q_1 = \infty$ fixed, and we get the desired estimate. The second inequality of (4.2) follows from (7.25). ☐

Remark 2. From (4.2) and the Sobolev embedding on S^{n-1} we have for any $1 \le p < \infty$, any $0 < \gamma < (n-1)/p$ and any $\alpha > \gamma$,

$$\||x|^\gamma h\|_{L^\infty} \lesssim \|h\|_{\dot H^{n/p-\gamma}_p H^\alpha_{p,\sigma}}. \tag{4.4}$$

In fact, from the hypothesis $\alpha > \gamma$ we can take a number $q < (n-1)/\gamma$ such that $H^\alpha_{q,\sigma} \hookrightarrow L^\infty_\sigma$. Since $L^{(n-1)/\gamma,\infty}_\sigma \hookrightarrow L^q_\sigma$, we have

$$\||x|^\gamma h\|_{L^\infty_r L^\infty_\sigma} \lesssim \||x|^\gamma h\|_{L^\infty_r H^\alpha_{q,\sigma}} \lesssim \|(1-\Delta_\sigma)^{\alpha/2}|x|^\gamma h\|_{L^\infty_r L^{(n-1)/\gamma,\infty}}.$$

Then by direct application of (4.2), we can deduce (4.4).

It should be noted that the inequality (4.2) improves the recent result of [5] for $p = 2$, in which the following inequality was proved

$$\||x|^\gamma h\|_{L^\infty} \lesssim \|h\|_{\dot H^{n/2-\gamma} H^\alpha_\sigma}$$

for $0 < \gamma < (n-1)/2, \alpha > n/2 - 1 + \gamma$. The authors of [5] used the spherical harmonic expansion and the boundedness of the spherical harmonic functions w.r.t. the supremum norm, which cause the regularity loss for α up to $n/2 - 1$.

By duality we immediately get

Lemma 4.2. *Let $p, q, q_1 \in [1, \infty]$, $0 < \gamma < (n-1)/p'$,*

$$\frac{1}{q_1} \ge \frac{1}{q} - \frac{1}{p'} + \frac{\gamma}{n-1}, \quad \frac{\gamma}{n-1} \ne \frac{1}{q_1} - \frac{1}{p}. \tag{4.5}$$

Then we have

$$\||x|^{-n/p-\gamma} * f\|_{L^p_r L^{q_1}_\sigma} \lesssim \|f\|_{\dot B^{-n/p'+\gamma}_{p,q_1,1}} \lesssim \||x|^{-\gamma} f\|_{L^1_r L^{q,1}_\sigma}. \tag{4.6}$$

We can further add a weight on the left.

Lemma 4.3. *Let $p, q, q_1 \in [1, \infty]$, $0 \le \delta < \gamma < (n-1)/p'$,*

$$\frac{1}{q_1} \ge \frac{1}{q} - \frac{1}{p'} + \frac{\gamma}{n-1}, \quad \frac{\gamma}{n-1} \ne \frac{1}{q_1} - \frac{1}{p}.$$

Then we have

$$\||x|^{\delta}(|x|^{-n/p-\gamma} * f)\|_{L_r^p L_\sigma^{q_1}} \lesssim \||x|^{-(\gamma-\delta)} f\|_{L_r^1 L_\sigma^{q,1}}. \tag{4.7}$$

If $p = \infty$, then $\delta = \gamma$ is also allowed.

Proof. The function in the norm on the left is bounded by

$$\begin{aligned}&\left(\int_{|x| \lesssim |y|} + \int_{|y| \ll |x| \sim |x-y|}\right) |x|^{\delta} |x-y|^{-n/p-\gamma} |f(y)| dy \\ &\lesssim |x|^{-n/p-\gamma} * (|x|^{\delta}|f|) + |x|^{-n/p-(\gamma-\delta)} * |f|,\end{aligned}$$

hence it is bounded in the norm by the above lemma

$$\lesssim \||x|^{-\gamma}(|x|^{\delta}|f|)\|_{L_r^1 L_\sigma^{q,1}} + \||x|^{-(\gamma-\delta)}|f|\|_{L_r^1 L_\sigma^{q,1}} \lesssim \||x|^{-(\gamma-\delta)} f\|_{L_r^1 L_\sigma^{q,1}}.$$

where $\gamma = \delta$ is excluded because then $\gamma - \delta = 0$ in the second term, but the estimate is trivially correct if $p = \infty$. Note that we get some room for q_1 when we use the above lemma with $\gamma - \delta$, so we need not worry about the case $(\gamma - \delta)/(n-1) = 1/q_1 - 1/p$. ☐

Remark 3. Using Hölder and Sobolev embedding on the unit sphere, the right hand side of (4.6) with f replaced by fg is bounded by

$$\|f\|_{L_r^2 L_\sigma^{2(n-1)/(n-1-\gamma),2}} \||x|^{-\gamma} g\|_{L_r^2 L_\sigma^{2(n-1)/(n-1-\gamma),2}} \lesssim \|f\|_{L_r^2 H_\sigma^{\frac{\gamma}{2}}} \||x|^{-\gamma} g\|_{L_r^2 H_\sigma^{\frac{\gamma}{2}}}.$$

Also the right hand side of (4.7) with fg is bounded by

$$\|f\|_{L_r^2 H_\sigma^{\frac{\gamma}{2}}} \||x|^{-(\gamma-\delta)} g\|_{L_r^2 H_\sigma^{\frac{\gamma}{2}}}.$$

Remark 4. The Lemma 4.2 is necessary for the energy estimate of nonlinear term via Remark 3. In fact, for radial function u

$$\begin{aligned}\|(V_\gamma * |u|^2)u\|_{H^{\frac12}} &\lesssim \|(V_\gamma * |u|^2)u\|_{L^2} + \|(V_\gamma * |u|^2)u\|_{\dot H^{\frac12}} \\ &\lesssim \|V_\gamma * |u|^2\|_{L^\infty} \|u\|_{H^{\frac12}} + \|V_{\gamma+\frac12} * |u|^2\|_{L^{2n}} \|u\|_{L^{\frac{2n}{n-1}}} \\ &= I + II\end{aligned}$$

The first part I is estimated by Lemma 4.2 with $p = \infty$ as follows: if $0 < \gamma < n-1$

$$\begin{aligned}I &\lesssim \||x|^{-\gamma}|u|^2\|_{L^1} \|u\|_{H^{\frac12}} \lesssim \||x|^{-\frac{\gamma}{2}} u\|_{L^2}^2 \|u\|_{H^{\frac12}} \\ &\lesssim \|u\|_{L^2}^{2-\gamma} \|u\|_{H^{\frac12}} \||x|^{-1} u\|_{L^2}^{\gamma}.\end{aligned}$$

For II we have from Lemma 4.2 with $p = 2n$

$$\begin{aligned} II &\lesssim \|V_{\gamma+\frac{1}{2}} * |u|^2\|_{L^{2n}} \|u\|_{H^{\frac{1}{2}}} \lesssim \||x|^{-\gamma}|u|^2\|_{L^1} \|u\|_{H^{\frac{1}{2}}} \\ &\lesssim \|u\|_{L^2}^{2-\gamma} \|u\|_{H^{\frac{1}{2}}} \||x|^{-1}u\|_{L^2}^{\gamma}. \end{aligned}$$

Here to use Lemma 4.2 with $p = 2n$ we should restrict the range of γ to $0 < \gamma < \gamma_0 \equiv (n-1)(2n-1)/2n$. If $n = 3$, $\gamma_0 = \frac{5}{3}$. If $n \geq 4$, then $\gamma_0 > 2$. These estimates are the key parts of the proof of Theorem 5.2 below.

5 Global existence of radial solutions

Now we introduce the improved results for radial solutions. The first one is the following (see Theorem 1 of [3]).

Theorem 5.1. *Let γ satisfy $1 < \gamma < \frac{2n-1}{n}, n \geq 2, s \geq \frac{1}{2}$. If $\lambda > 0$, then for any radially symmetric function $\varphi \in H^s$, (1.8) has a unique radially symmetric solution $u \in C(\mathbb{R}; H^s) \cap L^q_{loc}\widetilde{H}^{\frac{1}{2},\frac{1}{2}-\sigma}_{\frac{2n}{n-1}}$ for $q = \frac{2n}{n-1} + \varepsilon$ and $\sigma = \frac{1}{2n} + \varepsilon'$ with sufficiently small $\varepsilon, \varepsilon' > 0$. For all time the energy and L^2 norm of $u(t)$ are conserved. If $\lambda < 0$, then there exists $\rho > 0$ such that the same conclusion as above holds for φ with $\|\varphi\|_{L^2} \leq \rho$. Moreover, for $s > \frac{1}{2}$*

$$\|u(t)\|_{H^s} \lesssim \|\varphi\|_{H^s} \exp\left(C|t|(1 + \mathcal{E}(\varphi))^{\frac{q}{q-2}}\right), \tag{5.1}$$

where $\mathcal{E}(\varphi) = E(\varphi)$ if $\lambda > 0$ and $\mathcal{E}(\varphi) = (K(\varphi))^{\gamma}$ if $\lambda < 0$.

This theorem shows the global well-posedness for $\frac{2n}{n+1} \leq \gamma < \frac{2n-1}{n}$ and $n \geq 2$. The proof relies heavily on the Strichartz estimate (3.2) and the Hardy inequality (1.4). For example one needs

$$\begin{aligned} &\left\| \int_0^t U(t-t')F(u)(t')\,dt' \right\|_{L^q_T \widetilde{H}^{\frac{1}{2},\frac{1}{2}-\sigma}_{\frac{2n}{n-1}}} \\ &\lesssim \|F(u)\|_{L^1_T H^s} \lesssim \|V_\gamma * |u|^2\|_{L^1_T L^\infty} \|u\|_{L^\infty_T H^s} + \|V_\gamma * |u|^2\|_{L^{q'}_T H^s_{\frac{2n}{\gamma-\varepsilon_0}}} \|u\|_{L^q_T L^{\frac{2n}{n-\gamma+\varepsilon_0}}} \\ &\lesssim T^{1-\frac{2}{q}} \|u\|_{L^q_T L^{\frac{2n}{n-\gamma-\varepsilon_0}}} \|u\|_{L^q_T L^{\frac{2n}{n-\gamma+\varepsilon_0}}} \|u\|_{L^\infty_T H^s} \\ &\qquad + T^{1-\frac{2}{q}} \||u|^2\|_{L^q_T H^s_{\frac{2n}{2n-\gamma-\varepsilon_0}}} \|u\|_{L^q_T L^{\frac{2n}{n-\gamma+\varepsilon_0}}} \\ &\lesssim T^{1-\frac{2}{q}} \left(\|u\|_{L^q_T L^{\frac{2n}{n-\gamma-\varepsilon_0}}} \|u\|_{L^q_T L^{\frac{2n}{n-\gamma+\varepsilon_0}}} \right) \|u\|_{L^\infty_T H^s}, \end{aligned}$$

where $0 < \varepsilon_0 < n - \gamma$.

The second result is on the improvement for $n \geq 4$.

Theorem 5.2. (1) *Let* $1<\gamma<5/3$ *for* $n=3$ *and* $1<\gamma<2$ *for* $n\ge 4$. *Let* $\varphi\in H^{\frac12}$ *be radially symmetric and assume that* $\|\varphi\|_{L^2}\le\rho$ *for some sufficiently small* ρ *if* $\lambda<0$. *Then there exists a unique radial solution* $u\in C_b(H^{\frac12})\cap L^2_{loc}(|x|L^2)$ *of* (1.8) *satisfying the energy and* L^2 *conservations* (1.2).
(2) *Let* $\gamma=2$ *and* $n\ge 4$. *Let* $\varphi\in H^1$ *be radially symmetric. If* $\|\varphi\|_{H^1}$ *is sufficiently small, then there exists a unique radial solution* $u\in C_b(H^1)\cap L^2(|x|L^2)$ *to* (1.8). *Moreover, there exist radial functions* φ^+ *and* φ^- *such that*

$$\|u(t)-U(t)\varphi^{\pm}\|_{H^1}\to 0\quad\text{as}\quad t\to\pm\infty.$$

If $n=3$, then Theorem 5.2 improves the original results of Theorem 1 in [6] via Lemma 4.2 and the argument in Remark 4 up to $\gamma<5/3$. If $n\ge 4$, then it shows that the answer on question for the global well-posedness is positive at least for the radially symmetric case with dimension $n\ge 4$. For (2), it is enough to adopt Lemma 4.2 with f replaced by $|u|^2$, Hölder inequality and (3.6). The bound ρ in Theorems 5.1 and 5.2 can be taken as

$$\rho\le\min\Big(1,(8^{\gamma}C^{\gamma}(1+\|\varphi\|^2_{\dot H^{1/2}})^{\gamma(\gamma-1)})^{-1/(4-\gamma)}\Big).$$

For this, see [3, 4, 6].

6 Extension to nonradial cases

In this section we apply the estimates (3.6) and (4.6) to the nonradial cases. We first show the following lemma.

Lemma 6.1. *For any* $0\le\alpha\le\gamma<(n-1)(2n-1)/(2n)$ *we have*

$$\|(V_\gamma*|u|^2)u\|_{\dot H^{\frac12}H^\alpha_\sigma}\lesssim\||x|^{-\gamma/2}u\|^2_{L^2_rH^\gamma_\sigma}\|u\|_{\dot H^{\frac12}H^\alpha_\sigma}.$$

Proof. Let $v\equiv V_\gamma*|u|^2$. We want to estimate the mixed Sobolev norm $\|vu\|_{\dot H^{1/2}H^\alpha_\sigma}$ for $0<\alpha\le\gamma<(n-1)(2n-1)/(2n)$ by the Leibniz rule. For the x derivative, we can use the (radial) Littlewood-Paley decomposition, which commutes with the angular derivative D_σ. We have

$$\|vu\|_{\dot H^{1/2}H^\alpha_\sigma}\sim\|2^{j/2}\varphi_jD^\alpha_\sigma(vu)\|_{\ell^2_{j\in\mathbb Z}L^2_x}. \tag{6.1}$$

Further applying the decomposition to v and u, we can bound the above by the $\ell^2_{j\in\mathbb Z}$ norm of

$$\begin{aligned}&\sum_{l=-1}^{1}\Big[\|2^{j/2}D^\alpha_\sigma(v_{<j-1}u_{j+l})\|_{L^2_x}+\|2^{j/2}D^\alpha_\sigma(v_{j+l}u_{<j-1})\|_{L^2_x}\Big]\\&\qquad+\sum_{k\ge j-1}\ \sum_{l\ge j-1,\,|k-l|\le 2}\|2^{j/2}D^\alpha_\sigma(v_ku_l)\|_{L^2_x},\end{aligned}\tag{6.2}$$

where we denoted

$$u_j = \varphi_j * u, \quad u_{<j} = u - \sum_{k \geq j} \varphi_k * u, . \tag{6.3}$$

Then for the angular derivative D_σ^α we apply the standard multiplication estimate, (which we can transfer from $\mathbb{R}^{n-1}$ by using local coordinates on S^{n-1})

$$\|vu\|_{H_\sigma^\alpha} \lesssim \begin{cases} \|v\|_{H_{q_1,\sigma}^\alpha} \|u\|_{H_\sigma^\alpha}, \\ \|v\|_{H_{q_2,\sigma}^\alpha} \|u\|_{H_{2n/(n-1),\sigma}^\alpha}, \end{cases} \tag{6.4}$$

where we choose q_1, q_2 such that

$$\begin{cases} q_2 \geq 2, \ \frac{1}{q_2} - \frac{1}{2n} \leq \frac{1}{q_1} < \frac{\alpha}{n-1}, \ \text{ if } \ \alpha > 0, \\ q_1 = \infty, q_2 = 2n, \ \text{ if } \ \alpha = 0. \end{cases} \tag{6.5}$$

Then (6.2) is bounded by

$$\begin{aligned} &\sum_{|l-j| \leq 1} \Big[\|D_\sigma^\alpha v\|_{L_r^\infty L_\sigma^{q_1}} 2^{l/2} \|D_\sigma^\alpha u_l\|_{L_x^2} + 2^{l/2} \|D_\sigma^\alpha v_l\|_{L_r^{2n} L_\sigma^{q_2}} \|D_\sigma^\alpha u\|_{L_x^{2n/(n-1)}} \Big] \\ &+ \sum_{k \geq j-1} \sum_{l \geq j-1, \, |k-l| \leq 2} 2^{(j-l)/2} \|D_\sigma^\alpha v_k\|_{L_r^\infty L_\sigma^{q_1}} 2^{l/2} \|D_\sigma^\alpha u_l\|_{L_x^2}, \end{aligned} \tag{6.6}$$

where we have discarded the low frequency restriction $_{<j-1}$ by using Lemma 7.1. Then the $\ell^2_{j \in \mathbb{Z}}$ norm is bounded by

$$\|D_\sigma^\alpha v\|_{\dot{B}^{1/2}_{2n,q_2,1}} \|D_\sigma^\alpha u\|_{\dot{H}^{1/2}}, \tag{6.7}$$

where we used the embedding $\dot{B}^{1/2}_{2n,q_2,1} \subset \dot{B}^0_{\infty,q_1,1} \subset L_r^\infty L_\sigma^{q_1}$ and the Young inequality $\ell^1 * \ell^2 \subset \ell^2$ for the sum over $k \sim l \gg j$.

For the v norm in (6.7), we apply Lemma 4.2 after using (7.24). Then it is bounded by

$$\|D_\sigma^\alpha |u|^2\|_{\dot{B}^{1/2-n+\gamma}_{2n,q_2,1}} \lesssim \||x|^{-\gamma} D_\sigma^\alpha |u|^2\|_{L_r^1 L_\sigma^q}, \tag{6.8}$$

for any $q \in [1, \infty]$ satisfying

$$\frac{1}{q} < \frac{1}{q_2} + 1 - \frac{1}{2n} - \frac{\gamma}{n-1} \ \text{ for } \ \alpha \geq 0, \tag{6.9}$$

where we were able to replace the Lorentz space $L_\sigma^{q,1}$ with the Lebesgue space by excluding the critical case (the equality). Note also that the second condition in (4.5) is already satisfied because from (6.5) we have

$$\frac{1}{q_2} - \frac{1}{2n} < \frac{\alpha}{n-1} \leq \frac{\gamma}{n-1} \ \text{ for } \ \alpha < 0, \quad 0 < \frac{\gamma}{n-1} \ \text{ for } \ \alpha = 0. \tag{6.10}$$

For (6.8) we use again the product estimate in the Sobolev space on S^{n-1}:

$$\|uv\|_{H^{\alpha}_{q_0,\sigma}} \lesssim \|u\|_{H^{\gamma}_{\sigma}} \|v\|_{H^{\gamma}_{\sigma}}, \tag{6.11}$$

which holds for any $q_0 \in [1, \infty]$ satisfying

$$\frac{1}{q_0} > \max\left[\frac{1}{2} + \frac{\alpha - \gamma}{n-1}, 1 + \frac{\alpha - 2\gamma}{n-1}\right]. \tag{6.12}$$

Under the conditions (6.5) and (6.9), we can choose any q satisfying

$$\frac{1}{q} < \min\left[1 + \frac{\alpha - \gamma}{n-1}, \frac{3}{2} - \frac{1}{2n} - \frac{\gamma}{n-1}\right], \tag{6.13}$$

hence in particular we can make $q_0 \le q$. Then we can bound (6.8) by

$$\||x|^{-\gamma/2} u\|^2_{L^2_r H^{\gamma}_{\sigma}}. \tag{6.14}$$

□

Remark 5. If we further assume that $\gamma \le 2$, then we get from the complex interpolations

$$\||x|^{-\gamma/2} u\|^2_{L^2_r H^{\gamma}_{\sigma}} \lesssim \left\{ \begin{array}{c} \|u\|^{2-\gamma}_{L^2_r H^{\gamma}_{\sigma}} \||x|^{-1} u\|^{\gamma}_{L^2_r H^{\gamma}_{\sigma}} \\ \|u\|^{2-\gamma}_{L^2_x} \||x|^{-1} u\|^{\gamma}_{L^2_r H^2_{\sigma}} \end{array} \right\}. \tag{6.15}$$

By the same argument we can get the following.

Lemma 6.2. *For any* $0 < \alpha \le \gamma < (n-1)(2n-1)/(2n)$, *we have*

$$\begin{aligned} &\|(V_\gamma * |u|^2)u - (V_\gamma * |v|^2)v\|_{\dot{H}^{\frac{1}{2}} H^{\alpha}_{\sigma}} \\ &\quad \lesssim \||x|^{-\gamma/2}(u-v)\|_{L^2_r H^{\gamma}_{\sigma}} (\||x|^{-\gamma/2} u\|_{L^2_r H^{\gamma}_{\sigma}} + \||x|^{-\gamma/2} v\|_{L^2_r H^{\gamma}_{\sigma}}) \|u\|_{\dot{H}^{\frac{1}{2}} H^{\alpha}_{\sigma}} \\ &\qquad + \||x|^{-\gamma/2} v\|^2_{L^2_r H^{\gamma}_{\sigma}} \|u - v\|_{\dot{H}^{\frac{1}{2}} H^{\alpha}_{\sigma}}. \end{aligned}$$

Remark 6. By a slight change of Littlewood-Paley decomposition $\{\varphi'_j\}_{j\ge 0}$ such that $\varphi'_j = \varphi_j$ for $j \ge 1$ and $\varphi'_0 = 1 - \sum_{j\ge 1} \varphi_j$, we can replace the homogeneous $\dot{H}^{\frac{1}{2}}$ norm with the inhomogeneous one $H^{\frac{1}{2}}$.

In view of the above lemmas and the Strichartz estimate (3.6), it seems possible to get a global existence for each $\alpha \le \gamma < \frac{5}{3}$ for $n = 3$ or $\alpha \le \gamma < 2$ for $n \ge 4$. However, since we do not know the boundedness of $\|u(t)\|_{H^{\frac{1}{2}} H^{\alpha}_{\sigma}}$ with respect to t, for the present we cannot use the time iteration argument as used in the proof of the previous results. The following is the local existence result for these γ.

Proposition 6.3. *Let $\frac{3}{2} \le \gamma < \frac{5}{3}$ for $n = 3$ and $\frac{3}{2} \le \gamma < 2$ for $n \ge 4$. For any $\varphi \in H^{\frac{1}{2}} H_\sigma^{\frac{3}{2}}$, there exists a $T^* = T^*(\varphi) \in (0, \infty]$ such that there exists a unique solution $u \in C([-T, T]; H^{\frac{1}{2}} H_\sigma^{\frac{3}{2}}) \cap L^2(-T, T; |x| L_r^2 H_\sigma^2)$ to (1.8) for all $T < T^*$. Moreover, if $T^* < \infty$, then*

$$\int_0^{T^*} \||x|^{-\gamma/2} u\|_{L_r^2 H_\sigma^\gamma}^2 \, dt = \infty. \tag{6.16}$$

Proof. Let $X_{T,\rho}$ be a complete metric space with the metric $d(u, v) = \|u - v\|_{X_T}$ and of functions such that $\|v\|_{X_T} \le \rho$, where $X_T = C([-T, T]; H^{\frac{1}{2}} H_\sigma^{\frac{3}{2}} \cap L_T^2(|x| L_r^2 H_\sigma^2))$. Then we claim that the map N defined by $N(u) = U(t)\varphi - i \int_0^t U(t - t') F(u) \, dt'$ is a contraction on X_T, provided T is sufficiently small.

From the Strichartz estimate (3.6), Lemma 6.1 with $\alpha = \frac{3}{2}$ and γ as stated and Remark 6, we have for any $u \in X_{T,\rho}$

$$\begin{aligned} &\|N(u)\|_{X_T} \\ &\lesssim \|\varphi\|_{H^{\frac{1}{2}} H_\sigma^{\frac{3}{2}}} + T^{1-\frac{\gamma}{2}} \|u\|_{X_T}^3 \lesssim \|\varphi\|_{H^{\frac{1}{2}} H_\sigma^{\frac{3}{2}}} + T^{1-\frac{\gamma}{2}} \rho^3. \end{aligned} \tag{6.17}$$

Hence choosing ρ satisfying $\rho/2 \ge C\|\varphi\|_{H^{\frac{1}{2}} H_\sigma^{\frac{3}{2}}}$ and T so small that the mapping N maps $X_{T,\rho}$ to itself. We also have from Lemma 6.2 that

$$d(N(u), N(v)) \le C T^{1-\frac{\gamma}{2}} \rho^2 d(u, v).$$

Thus by the choice of ρ and T such that $C T^{1-\frac{\gamma}{2}} \rho^2 \le \frac{1}{2}$, N becomes a contraction mapping.

Now we suppose that the maximal existence time T^* of the local solution constructed as above is finite. Then we have $\lim_{t \to T^*} \|u(t)\|_{H^{\frac{1}{2}} H_\sigma^{\frac{3}{2}} \cap L(-t,t; |x| L_r^2 H_\sigma^2)} \to +\infty$. To show the blowup criterion we observe from (3.6), Lemma 6.1 and (6.15) that for any $0 < T < T^*$

$$\begin{aligned} &\|u\|_{L^\infty(-T,T; H^{\frac{1}{2}} H_\sigma^{\frac{3}{2}}) \cap L^2(-T,T; |x| L_r^2 H_\sigma^2)} \\ &\qquad \lesssim \|\varphi\|_{H^{\frac{1}{2}} H_\sigma^{\frac{3}{2}}} + \int_0^T \|(V_\gamma * |u|^2) u\|_{H^{\frac{1}{2}} H_\sigma^{\frac{3}{2}}}(t) \, dt \\ &\qquad \lesssim \|\varphi\|_{H^{\frac{1}{2}} H_\sigma^{\frac{3}{2}}} + \int_0^T \||x|^{-\gamma/2} u\|_{L_r^2 H_\sigma^\gamma}^2 \|u\|_{L^\infty(-t,t; H^{\frac{1}{2}} H_\sigma^{\frac{3}{2}})} \, dt. \end{aligned} \tag{6.18}$$

By Gronwall's inequality we get

$$\|u\|_{L^\infty(-T,T; H^{\frac{1}{2}} H_\sigma^{\frac{3}{2}})} \lesssim \|\varphi\|_{H^{\frac{1}{2}} H_\sigma^{\frac{3}{2}}} \exp\left[C \int_0^T \||x|^{-\gamma/2} u\|_{L_r^2 H_\sigma^\gamma}^2 \, dt \right]$$

and by inserting this into (6.18) we also get

$$\begin{aligned} &\|u\|_{L^\infty(-T,T; H^{\frac{1}{2}} H_\sigma^{\frac{3}{2}}) \cap L^2(-T,T; |x| L_r^2 H_\sigma^2)} \\ &\qquad \lesssim \|\varphi\|_{H^{\frac{1}{2}} H_\sigma^{\frac{3}{2}}} \exp\left[C \int_0^T \||x|^{-\gamma/2} u\|_{L_r^2 H_\sigma^\gamma}^2 \, dt \right]. \end{aligned}$$

This implies the blowup criterion (6.16).

∎

Remark 7. Consider the following equation with angular derivatives:

$$iu_t - D_{\sigma,\varepsilon}^3\sqrt{1-\Delta}u = F(u) = \lambda(V_\gamma * |u|^2)u, \quad u(0) = \varphi, \tag{6.19}$$

where $D_{\sigma,\varepsilon} = \sqrt{1-\varepsilon\Delta_\sigma}$ and $\varepsilon > 0$. One can easily show that the solution satisfies the conservation laws:

$$\begin{aligned}&\|u(t)\|_{L_x^2} = \|\varphi\|_{L_x^2},\\ &E_{\sigma,\varepsilon}(u(t)) = \frac{1}{2}\langle D_{\sigma,\varepsilon}^3\sqrt{1-\Delta}\,u, u\rangle + \frac{1}{4}\langle F(u), u\rangle = E_{\sigma,\varepsilon}(\varphi).\end{aligned}$$

Since $\|u(t)\|_{H^{\frac12}H_\sigma^{\frac32}} \le C_\varepsilon E_{\sigma,\varepsilon}(\varphi)^{\frac12}$ for $\varepsilon > 0$ and $\lambda > 0$, we can proceed to the global existence with time iteration. In case that $\lambda < 0$ we need a smallness of $\|\varphi\|_{L_x^2}$. Now let u_ε be the global solution to (6.19) for each ε. Then by the compactness argument we can easily show that u_ε converges to a global weak solution of the original equation along some sequence $\varepsilon \to +0$. It will be interesting to see if the convergence holds strongly for the whole sequence $\varepsilon \to +0$. Global weak solutions can be constructed also by mollifying V_γ.

For the case $\gamma = 2$ and $n \ge 4$ we can show the global existence, provided a smallness of initial data is assumed.

Theorem 6.4. *Let $\gamma = 2$ and $n \ge 4$. If $\varphi \in \dot H^{\frac12}H_\sigma^{\frac32}$ and $\|\varphi\|_{\dot H^{\frac12}H_\sigma^{\frac32}}$ is sufficiently small, then there exists a unique solution $u \in C_b(\mathbb{R}; \dot H^{\frac12}H_\sigma^{\frac32}) \cap L^2(\mathbb{R}; |x|L_r^2H_\sigma^2)$ to (1.8). Moreover, there exist two functions φ^+ and φ^- such that*

$$\|u(t) - U(t)\varphi^\pm\|_{\dot H^{\frac12}H_\sigma^{\frac32}} \to 0 \quad \text{as} \quad t \to \pm\infty.$$

Proof. Let Y_ρ be a complete metric space with the metric $d(u,v) = \|u-v\|_Y$ and of functions such that $\|v\|_Y \le \rho$, where $Y = C_b(\mathbb{R}; \dot H^{\frac12}H_\sigma^{\frac32}) \cap L^2(\mathbb{R}; |x|L_r^2H_\sigma^2)$. Then we claim that the map N defined by $N(u) = U(t)\varphi - i\int_0^t U(t-t')F(u)\,dt'$ is a contraction on Y, provided ρ is sufficiently small.

From the Strichartz estimate (3.6) and Lemma 6.1 with $\gamma = 2$ and $\alpha = \frac32$, we have for any $u \in Y_\rho$

$$\|N(u)\|_Y \lesssim \|\varphi\|_{\dot H^{\frac12}H_\sigma^{\frac32}} + \|u\|_Y^3 \lesssim \|\varphi\|_{\dot H^{\frac12}H_\sigma^{\frac32}} + \rho^3.$$

Hence choosing ρ so small that $C\|\varphi\|_{\dot H^{\frac12}H_\sigma^{\frac32}} \le \frac{\rho}{2}$ and $C\rho^3 \le \frac{\rho}{2}$, the mapping N maps Y_ρ to itself. We also have from Lemma 6.2 that

$$d(N(u), N(v)) \le C\rho^2 d(u,v).$$

Thus by the choice of ρ such that $C\rho^2 \le \frac12$, N becomes a contraction.

As for the scattering, let us define functions $\varphi_\pm$ by

$$\varphi_\pm = \varphi - i\int_0^{\pm\infty} U(-s)F(u)(s)\,ds.$$

Then clearly $\varphi_\pm \in \dot{H}^{\frac{1}{2}} H_\sigma^{\frac{3}{2}}$ and one can show that

$$\begin{aligned}\|u(t) - U(t)\varphi_\pm\|_{\dot{H}^{\frac{1}{2}} H_\sigma^{\frac{3}{2}}} &\lesssim \||x|^{-1}u\|^2_{L^2(I_t^\pm; L_r^2 H_\sigma^2)} \|u\|_{L^\infty(\mathbb{R}; \dot{H}^{\frac{1}{2}} H_\sigma^{\frac{3}{2}})} \\ &\to 0 \quad \text{as} \quad t \to \pm\infty,\end{aligned}$$

where $I_t^+ = (t, \infty)$ and $I_t^- = (-\infty, t)$. This proves the Theorem 6.4. □

7 Appendix

In this section we investigate the basic properties of mixed normed Besov and Sobolev spaces, such as the Sobolev embedding, the interpolation property, lifting property by derivative, etc. The proofs are the same as the the case of Besov or Sobolev space. The only extra ingredients are the commutativity of the radial convolution and the mixed norms on the polar coordinates:

Lemma 7.1. *If $\psi(x)$ is radially symmetric, then*

$$\|\psi * f\|_{L_r^p L_\sigma^q} \le \|f\|_{L_r^{p_1} L_\sigma^{q_1}} \|\psi\|_{L_x^{p_2}}, \tag{7.1}$$

for all $p_1, p_2, p, q, q_1 \in [1, \infty]$ satisfying

$$\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{p}, \quad \frac{1}{q_1} + \frac{1}{p_2} - 1 \le \frac{1}{q}. \tag{7.2}$$

This does not hold in general if ψ is not symmetric.

Proof. The case $p_2 = \infty$ is trivial:

$$\|\psi * f\|_{L_r^\infty L_\sigma^\infty} \le \|f\|_{L_r^1 L_\sigma^1} \|\psi\|_{L^\infty}. \tag{7.3}$$

Hence by the complex interpolation it suffices to prove the estimate for the case $p_2 = 1$. We use the pointwise estimate: for any $r > 0$ and $\theta \in S^{n-1}$, we have

$$\int_{S^{n-1}} |\psi * f|(r\sigma)d\sigma \le (|\psi| * F)(r\theta), \tag{7.4}$$

where F is the radially symmetric function defined by

$$F(r\theta) = \int_{S^{n-1}} |f(r\sigma)|d\sigma, \tag{7.5}$$

and the measure $d\sigma$ on S^{n-1} is normalized.

(7.4) is proved as follows. First we can replace ψ and f with $|\psi|$ and $|f|$ respectively. Since ψ is radially symmetric, the left hand side is invariant with respect to rotation of f:

$$\int_{S^{n-1}} |\psi| * |f|(r\sigma)d\sigma = \int_{S^{n-1}} (|\psi| * |f(Ax)|)(r\sigma)d\sigma, \tag{7.6}$$

for any $A \in SO(n)$. Integrating it over all A with the normalized Haar measure, and applying Fubini, we get

$$\begin{aligned} = \int_{S^{n-1}} |\psi| * \left[\int_{SO(n)} |f(Ax)|dA\right](r\sigma)d\sigma &= \int_{S^{n-1}} (|\psi| * F)(r\sigma)d\sigma \\ &= (|\psi| * F)(r\theta), \end{aligned} \tag{7.7}$$

where we used the radial symmetry of $|\psi| * F$ as well.

Taking the L^p_r norm of (7.4), and applying the Minkowski inequality, we get

$$\|\psi * f\|_{L^p_r L^1_\sigma} \le \||\psi| * F\|_{L^p_x} \le \|\psi\|_{L^1}\|F\|_{L^p} = \|\psi\|_{L^1}\|f\|_{L^p_r L^1_\sigma}, \tag{7.8}$$

which is the desired inequality for $p_2 = 1$ and $q = 1$. Then by duality we get

$$\begin{aligned} |\langle \psi * f|g\rangle| = |\langle f|\psi^\dagger * g\rangle| &\le \|f\|_{L^p_r L^\infty_\sigma} \|\psi^\dagger * g\|_{L^{p'}_r L^1_\sigma} \\ &\le \|f\|_{L^p_r L^\infty_\sigma} \|\psi^\dagger\|_{L^1} \|g\|_{L^{p'}_r L^1_\sigma}, \end{aligned} \tag{7.9}$$

where $\psi^\dagger(x) := \overline{\psi}(-x)$, which implies the desired inequality for $p_2 = 1$ and $q = \infty$. Then we get for all $q \in [1, \infty]$ by the complex interpolation. $\square$

Then we get the following in the same way as for the isotropic Besov spaces:

Proposition 7.2. (I) *Sobolev embedding: If $s_1 \le s_2$, $\rho_1 \ge \rho_2$, and*

$$\frac{s_1 - s_2}{n} = \frac{1}{p_1} - \frac{1}{p_2} \le \frac{1}{q_1} - \frac{1}{q_2}, \tag{7.10}$$

then we have

$$\|f\|_{\dot{B}^{s_1}_{p_1,q_1,\rho_1}} \lesssim \|f\|_{\dot{B}^{s_2}_{p_2,q_2,\rho_2}}. \tag{7.11}$$

(II) *Lifting:*

$$\|\nabla f\|_{\dot{B}^{s}_{p,q,\rho}} \sim \|f\|_{\dot{B}^{s+1}_{p,q,\rho}}. \tag{7.12}$$

(III) *Complex interpolation: Let $\theta \in [0, 1]$, and*

$$s = (1-\theta)s_1 + \theta s_2, \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2},$$
$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad \frac{1}{\rho} = \frac{1-\theta}{\rho_1} + \frac{\theta}{\rho_2}.$$

Then we have

$$[\dot B^{s_1}_{p_1,q_1,\rho_1}, \dot B^{s_2}_{p_2,q_2,\rho_2}]_\theta = \dot B^{s}_{p,q,\rho},$$

where $[\cdot,\cdot]_\theta$ *denotes the complex interpolation functor.*
(IV) *Real interpolation: Let* $\theta \in (0,1)$*,* $s = (1-\theta)s_1 + \theta s_2$ *and* $s_1 \neq s_2$*. Then we have*

$$(\dot B^{s_1}_{p,q,\rho_1}, \dot B^{s_2}_{p,q,\rho_2})_{\theta,\rho} = \dot B^{s}_{p,q,\rho},$$

where $(\cdot,\cdot)_{\theta,\rho}$ *denotes the real interpolation functor.*

Proof. (I) and (II) are achieved by writing those operation in terms of convolution with radial Schwartz functions with the scaling property. Let $\widehat{\varphi}_j := \varphi_{j-1} + \varphi_j + \varphi_{j+1}$. Then the Fourier support property implies that $\varphi_j = \widehat{\varphi}_j * \varphi_j$. Hence by the above lemma

$$\|\varphi_j * f\|_{L^{p_1}_r L^{q_1}_\sigma} \le \|\widehat{\varphi}_j\|_{L^{p_0}} \|\varphi_j * f\|_{L^{p_2}_r L^{q_2}_\sigma}, \tag{7.13}$$

where $p_0 \in [1,\infty]$ is chosen such that $1/p_1 = 1/p_0 + 1/p_2 - 1$, i.e. $n/p_0' = s_2 - s_1$. The scaling property implies that

$$\|\widehat{\varphi}_j\|_{L^{p_0}} = 2^{jn/p_0'} \|\widehat{\varphi}_0\|_{L^{p_0}} \sim 2^{j(s_2-s_1)}. \tag{7.14}$$

Thus we obtain (I).

For (II) we have

$$|\varphi_j * \nabla f| = |\nabla \widehat{\varphi}_j * (\varphi_j * f)| \le |\nabla \widehat{\varphi}_j| * |\varphi_j * f| \lesssim |\partial_r \widehat{\varphi}_j| * |\varphi_j * f|, \tag{7.15}$$

where in the last step we used the radial symmetry of $\widehat{\varphi}_j$. Then by the above lemma we get

$$\|\varphi_j * \nabla f\|_{L^p_r L^q_\sigma} \lesssim \|\partial_r \widehat{\varphi}_j\|_{L^1} \|\varphi_j * f\|_{L^p_r L^q_\sigma} \sim 2^j \|\varphi_j * f\|_{L^p_r L^q_\sigma}. \tag{7.16}$$

For the reverse inequality, fix a cut-off function $\Gamma \in C^\infty(\mathbb{R})$ satisfying $\Gamma(r) = 0$ for $|r| \le 1/(5n)$ and $\Gamma(r) = 1$ for $|r| \ge 1/(4n)$. For $\theta \in S^{n-1}$ and $j \in \mathbb{Z}$, we define $G^\theta_j(x)$ by

$$\mathcal{F} G^\theta_j(2^j \xi) = (i\theta \cdot \xi)^{-1} \frac{\Gamma(\theta\cdot\xi)}{\int_{S^{n-1}} \Gamma(\omega\cdot\xi) d\omega} [\chi(\xi/2) - \chi(4\xi)], \tag{7.17}$$

Since $\int_{S^{n-1}} \Gamma(\omega\cdot\xi)d\omega \gtrsim 1$ on $\operatorname{supp}[\chi(\xi/2) - \chi(4\xi)]$ and $|\theta\cdot\xi| \ge 1/(5n)$ on $\operatorname{supp}\Gamma(\theta\cdot\xi)$, the right hand side is in $C^\infty_0(\mathbb{R}^n)$. Moreover, we have

$$\varphi_j = \int_{S^{n-1}} \theta \cdot \nabla G^\theta_j * \varphi_j d\theta. \tag{7.18}$$

Hence

$$|\varphi_j * f| = \left| \int_{S^{n-1}} \theta \cdot G^\theta_j * \varphi_j * \nabla f d\theta \right| \le \int_{S^{n-1}} |G^\theta_j| d\theta * |\varphi_j * \nabla f|. \tag{7.19}$$

Since $G_j := \int_{S^{n-1}} |G_j^\theta| d\theta$ is radially symmetric, by the above lemma and by the scaling property we get

$$\|\varphi_j * f\|_{L^p_r L^q_\sigma} \leq \|G_j\|_{L^1} \|\varphi_j * \nabla f\|_{L^p_r L^q_\sigma} \lesssim 2^{-j} \|\varphi_j * \nabla f\|_{L^p_r L^q_\sigma}. \tag{7.20}$$

Thus we obtain (II).

For the interpolation (III) and (IV), we only need a universal retraction from $2^{-sj}\ell^\rho_j L^p_r L^q_\sigma$, which is the same as in the isotropic case. For any $F \in (\mathcal{S}'_0)^{\mathbb{Z}}$ we define $RF(x)$, and for any $f \in \mathcal{S}'_0$ we define $S_j f(x)$, respectively by

$$RF = \sum_{j\in\mathbb{Z}} \widehat{\varphi}_j * F_j, \quad S_j f = \varphi_j * f. \tag{7.21}$$

Then we have

$$RSf = \sum_{j\in\mathbb{Z}} \widehat{\varphi}_j * \varphi_j * f = \sum_{j\in\mathbb{Z}} \varphi_j * f = f, \tag{7.22}$$

which is convergent in $\mathcal{S}'_0$ provided that $2^{-N|j|}\varphi_j * f$ is bounded in $\mathcal{S}'$ for some $N \in \mathbb{N}$. Moreover we have by the above lemma

$$\begin{aligned} &\|Sf\|_{2^{-sj}\ell^\rho_j L^p_r L^q_\sigma} = \|f\|_{\dot{B}^s_{p,q,\rho}}, \\ &\|RF\|_{\dot{B}^s_{p,q,\rho}} \leq \sum_{k=-3}^{3} \|\widehat{\varphi}_{j+k} * \varphi_j * F_{j+k}\|_{2^{-sj}\ell^\rho_j L^p_r L^q_\sigma} \lesssim \|F\|_{2^{-sj}\ell^\rho_j L^p_r L^q_\sigma}. \end{aligned} \tag{7.23}$$

(III) and (IV) follow immediately from the above bounds together with the corresponding identities for $2^{-js}\ell^\rho_j L^p_r L^q_\sigma$ spaces (cf. [20]). $\square$

Remark 8. The Sobolev inequality (I) implies in particular that the Littlewood-Paley decomposition $f = \sum_{j\in\mathbb{Z}} \varphi_j * f$ is convergent in $\mathcal{S}'$ if $s < n/p$ or $s = n/p$ and $\rho = 1$. In addition, we have

$$\||\nabla|^\alpha f\|_{\dot{B}^s_{p,q,\rho}} \sim \|f\|_{\dot{B}^{s+\alpha}_{p,q,\rho}}, \tag{7.24}$$

and so in particular

$$\dot{B}^s_{p,q,1} \subset \dot{H}^s_{p,q} \subset \dot{B}^s_{p,q,\infty}. \tag{7.25}$$

The equivalence (7.24) is proved by using Lemma 7.1,

$$\begin{aligned} &\||\nabla|^\alpha \varphi_j * f\|_{L^p_r L^q_\sigma} \leq \||\nabla|^\alpha \widehat{\varphi}_j\|_{L^1} \|\varphi_j * f\|_{L^p_r L^q_\sigma} \lesssim 2^{j\alpha} \|\varphi_j * f\|_{L^p_r L^q_\sigma}, \\ &\|\varphi_j * f\|_{L^p_r L^q_\sigma} \leq \||\nabla|^{-\alpha} \widehat{\varphi}_j\|_{L^1} \||\nabla|^\alpha \varphi_j * f\|_{L^p_r L^q_\sigma} \lesssim 2^{-j\alpha} \|\varphi_j * |\nabla|^\alpha f\|_{L^p_r L^q_\sigma}. \end{aligned}$$

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