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# Packing Arborescences

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## Abstract

In [7], Edmonds proved a fundamental theorem on packing arborescences that has become the base of several subsequent extensions. Recently, Japanese researchers found an unexpected further generalization which gave rise to many interesting questions about this subject [29], [20]. Another line of researches focused on covering intersecting families which generalizes Edmonds' theorems in a different way. The two approaches was united in [1] by introducing the notion of mixed intersecting bi-set families.

The purpose of this paper is to overview recent developments and to present some new results. We give a polyhedral description of arborescence packable subgraphs based on a connection with bi-set families, and by using this description we prove TDI-ness of the corresponding system of inequalities. We also consider the problem of independent trees and arborescences, and give a simple algorithm that decomposes a maximal planar graph into three independent trees.

## § 1. Introduction

In 1973, J. Edmonds [7] proved the following fundamental theorem.

**Theorem 1.1** (Edmonds' disjoint arborescences: weak form). *Let $D = (V, A)$ be a directed graph with a designated root-node $r_0$. $D$ has $k$ disjoint spanning arborescences of root $r_0$ if and only if $D$ is rooted $k$-edge-connected, that is,*

$$\varrho(X) \geq k \textit{ whenever } X \subseteq V - r_0, X \neq \emptyset. \quad \square \tag{1.1}$$

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Here an **arborescence of root** $r_0$ means a directed tree in which every node is reachable from a specified root-node $r_0$. We sometimes identify an arborescence $(U, F)$ with its edge-set $F$ and will say that the arborescence $F$ spans $U$. This result inspired great many extensions in the last three decades. Here we overview recent advances.

Sub- and supermodular set functions are known to be useful tools in graph optimization but in the last fifteen years it turned out that several results can be extended to functions defined on pairs of sets or on bi-sets. Given a ground-set $V$, we call a pair $X = (X_O, X_I)$ of subsets a **bi-set** if $X_I \subseteq X_O \subseteq V$ where $X_O$ is the outer member and $X_I$ is the inner member of $X$. By a bi-set function we mean a function defined on the set of bi-sets of $V$. We will tacitly identify a bi-set $X = (X_O, X_I)$ for which $X_O = X_I$ with the set $X_I$ and hence bi-set functions may be considered as straight generalizations of set functions. The set of all bi-sets on ground-set $V$ is denoted by $\mathcal{P}_2(V) = \mathcal{P}_2$. The **intersection** $\cap$ and the **union** $\cup$ of bi-sets is defined in a straightforward manner: for $X, Y \in \mathcal{P}_2$ let $X \cap Y := (X_O \cap Y_O, X_I \cap Y_I)$, $X \cup Y := (X_O \cup Y_O, X_I \cup Y_I)$. We write $X \subseteq Y$ if $X_O \subseteq Y_O, X_I \subseteq Y_I$ and this relation is a partial order on $\mathcal{P}_2$. Accordingly, when $X \subseteq Y$ or $Y \subseteq X$, we call $X$ and $Y$ **comparable**. A family of pairwise comparable bi-sets is called a **chain**. Two bi-sets $X$ and $Y$ are **independent** if $X_I \cap Y_I = \emptyset$ or $V = X_O \cup Y_O$. A set of bi-sets is independent if its members are pairwise independent. We call a set of bi-sets a **ring-family** if it is closed under taking union and intersection. Two bi-sets are **intersecting** if $X_I \cap Y_I \neq \emptyset$ and **properly intersecting** if, in addition, they are not comparable. Note that $X_O \cup Y_O = V$ is allowed for two intersecting bi-sets. In particular, two sets $X$ and $Y$ are properly intersecting if none of $X \cap Y, X - Y, Y - X$ is empty. A family of bi-sets is called **laminar** if it has no two properly intersecting members. A family $\mathcal{F}$ of bi-sets is **intersecting** if both the union and the intersection of any two intersecting members of $\mathcal{F}$ belong to $\mathcal{F}$. In particular, a family $\mathcal{L}$ of subsets is intersecting if $X \cap Y, X \cup Y \in \mathcal{L}$ whenever $X, Y \in \mathcal{L}$ and $X \cap Y \neq \emptyset$. A laminar family of bi-sets is obviously intersecting. Two bi-sets are **crossing** if $X_I \cap Y_I \neq \emptyset$ and $X_O \cup Y_O \neq V$ and **properly crossing** if they are not comparable. A bi-set $(X_O, X_I)$ is **trivial** if $X_I = \emptyset$ or $X_O = V$. We will assume throughout the paper that the bi-set functions in question are integer-valued and that their value on trivial bi-sets is always zero. In particular, set functions are also integer-valued and zero on the empty set and on the ground-set.

A directed edge **enters** or **covers** $X$ if its head is in $X_I$ and its tail is outside $X_O$. The set of edges entering a bi-set $X$ is denoted by $\Delta_D^-(X) = \Delta^-(X)$. An edge **covers** a family of bi-sets if it covers each member of the family. For a bi-set function $p$, a digraph $D = (V, A)$ is said to **cover** $p$ if $\varrho_D(X) \geq p(X)$ for every $X \in \mathcal{P}_2(V)$ where $\varrho_D(X)$ denotes the number of edges of $D$ covering $X$. For a vector $z : A \to \mathbb{R}$, let $\varrho_z(X) := \sum[z(a) : \ a \in A, a \text{ covers } X]$. A vector $z : A \to \mathbb{R}$ **covers** $p$ if $\varrho_z(X) \geq p(X)$

for every $X \in \mathcal{P}_2(V)$.

A bi-set function $p$ is said to satisfy the **supermodular inequality** on $X, Y \in \mathcal{P}_2$ if

$$p(X) + p(Y) \le p(X \cap Y) + p(X \cup Y). \tag{1.2}$$

If the reverse inequality holds, we speak of the **submodular** inequality. $p$ is said to be **fully supermodular** or supermodular if it satisfies the supermodular inequality for every pair of bi-sets $X, Y$. If (1.2) holds for intersecting (crossing) pairs, we speak of **intersecting** (**crossing**) **supermodular** functions. Analogous notions can be introduced for submodular functions. Sometimes (1.2) is required only for pairs with $p(X) > 0$ and $p(Y) > 0$ in which case we speak of **positively supermodular** functions. Positively intersecting or crossing supermodular functions are defined analogously. A typical way to construct a positively supermodular function is replacing each negative value of a fully supermodular function by zero.

**Proposition 1.2.** *The in-degree function $\varrho_D$ on $\mathcal{P}_2$ is submodular.* $\square$

Throughout we use the following notation. For an undirected graph $G = (V, E)$ and a subset $X \subseteq V$ we denote the number of edges having exactly one end in $X$ by $d(X)$. Given a directed graph $D = (V, A)$, $\varrho_D(X) = \varrho_A(X) = \varrho(X)$ and $\delta_D(X) = \delta_A(X) = \delta(X)$ denote the number of edges entering and leaving $X$, respectively. For an edge set $E' \subseteq E$ (resp. arc set $A' \subseteq A$), we use $i_{E'}(X)$ and $e_{E'}(X)$ (resp. $i_{A'}(X)$ and $e_{A'}(X)$) to denote the number of edges in $E'$ (resp. arcs in $A'$) induced by and adjacent to $X$, respectively. Often we do not distinguish between a one-element set and its only element. For example, the in-degree $\varrho(\{v\})$ of a singleton $\{v\}$ is abbreviated by $\varrho(v)$. The new results are emphasized by using capital letters.

The rest of the paper is organized as follows. Section 2 gives an overview of covering results derived from Edmonds' theorem. Concrete and abstract extensions are presented in Section 3, while Section 4 contains the polyhedral and algorithmic aspects of packing branchings. Finally, Section 5 deals with independent arborescences, and an algorithm for decomposing a maximal planar graph into trees is given.

## § 2. Extensions and consequences

In this section we exhibit results obtained by applying Edmonds' disjoint arborescences theorem (Theorem 1.1).

## § 2.1. Covering branchings and trees

**Theorem 2.1** ([12]). *The edge set of a digraph $D = (V, A)$ can be covered by $k$ branchings if and only if*

(2.1) *the in-degree of each node is at most $k$*

*and*

(2.2) $i(X) \leq k(|X| - 1)$ *for every* $\emptyset \subset X \subseteq V$.

*Proof.* Since the in-degree of each node in one branching is at most one, condition (2.1) is necessary. Since a forest can have at most $|X|-1$ edges induced by $X$, $k$ forests, and hence $k$ branchings, can have at most $k(|X|-1)$, that is, (2.2) is also necessary.

To prove the sufficiency we use the following elementary construction. Extend the digraph by adding a new node $r_0$ and by adding $k - \varrho(v)$ parallel edges from $r_0$ to $v$ for each node $v \in V$. In the resulting digraph $D'$, we have

$$\varrho'(X) = \varrho(X) + \sum[k - \varrho(v) : v \in X] = \varrho(X) - \varrho(X) - i(X) + k|X| \geq k$$

for every non-empty set $X \subseteq V$ and also $\varrho'(v) = k$ for every node $v \in V$. By Theorem 1.1, the edge set of $D'$ partitions into $k$ edge-disjoint spanning arborescences of root $r_0$. By restricting these arborescences on the edge set of $D$ we obtain the requested partition of $A$ into $k$ branchings. □

It is not difficult to see that if a rooted $k$-edge-connected digraph is minimal in point of leaving edges, then the in-degree of each non-root node is exactly $k$ and so Edmonds' Theorem 1.1 can be easily derived from Theorem 2.1. The following interesting consequence was proved in [31].

**Corollary 2.2** (Z.A. Kareyan). *The edge set of a digraph $D$ not containing loops or parallel edges can be covered by $K + 1$ branchings where $K$ denotes the maximum in-degree of a node of $D$.* □

The following result follows easily from network flows and was formulated in [16].

**Theorem 2.3.** *Let $f : V \to \mathbb{Z}_+ \cup \{-\infty\}$ and $g : V \to \mathbb{Z}_+ \cup \{\infty\}$ be two functions such that $f \leq g$. A graph $G = (V, E)$ has an orientation for which $f(v) \leq \varrho(v) \leq g(v)$ for every node $v$ if and only if*

(2.3) $e_G(X) \geq f(X)$ *for every subset* $X \subseteq V$,

*and*

(2.4) $i_G(X) \leq g(X)$ *for every subset* $X \subseteq V$. □

This immediately implies the following earlier result.

**Theorem 2.4** (Orientation lemma, S.L. Hakimi [21]). *For an undirected graph $G = (V, E)$ and a function $m : V \to \mathbb{Z}$ the following statements are equivalent.*

(2.5) *$G$ has an orientation so that $\varrho(v) = m(v)$ for every node $v$;*

(2.6) *$e_G(X) \geq m(X)$ for every subset $X \subseteq V$ and $m(V) = |E|$;*

(2.7) *$i_G(Y) \leq m(Y)$ for every subset $Y \subseteq V$ and $m(V) = |E|$.* □

Another interesting consequence is the following.

**Theorem 2.5** (Nash-Williams). *The edge set of an undirected graph $G = (V, E)$ can be covered by $k$ forests if and only if*

(2.8) $$i_G(X) \leq k(|X| - 1) \text{ for every } \emptyset \neq X \subseteq V.$$

*Proof.* The necessity is clear since any forest can have at most $|X| - 1$ edges induced by $X$.

For the sufficiency, we claim that $G = (V, E)$ has an orientation $D$ in which the in-degree of each node is at most $k$. Indeed, (2.8) implies that $i_G(X) \leq k|X|$ holds for $X \subseteq V$ and then Theorem 2.3, when applied to $g \equiv k$, $f \equiv 0$, states the existence of such an orientation. By applying Theorem 2.1 to $D$ we are done. □

## § 2.2. Covering arborescences

When can a digraph $D = (V, A)$ be covered by $k$ spanning arborescences of root $r_0$? For any subset $X$ of nodes, let $\Gamma^-(X) := \{v \in X\colon$ there is an edge $uv \in A$ for which $u \in V - X\}$ and call this set the **entrance** of $X$. That is, the entrance consists of the head nodes of edges entering $X$. The following result may be considered as a counterpart of the disjoint arborescences theorem.

**Theorem 2.6** (K. Vidyasankar [39]). *Let $r_0$ be a root node of a digraph $D = (V, A)$ so that no edge enters $r_0$. It is possible to cover the edge set of $D$ by $k$ spanning arborescences of root $r_0$ if and only if*

(2.9) $$\varrho(v) \leq k \text{ for every } v \in V - r_0$$

*and*

(2.10) $$k - \varrho(X) \leq \sum[k - \varrho(v) : v \in \Gamma^-(X)] \text{ for every } X \subseteq V - r_0$$

*where $\Gamma^-(X)$ is the entrance of $X$.* □

In Section 3.1 we give a generalization of this theorem.

### § 2.3. Packing trees

The following was shown in [11].

**Theorem 2.7** ([11]). *Let $G = (V, E)$ be an undirected graph and $r_0 \in V$ a designated root node. There is a rooted $k$-edge-connected orientation of $G$ if and only if $G$ is $k$-partition-connected, that is,*

$$e(\mathcal{F}) \geq k(t-1) \tag{2.11}$$

*holds for every partition $\mathcal{F} := \{V_1, ..., V_t\}$ of $V$ where $e(\mathcal{F})$ denotes the edges connecting distinct parts (that is, $e(\mathcal{F}) = \sum_i d(V_i)/2$).* □

The theorem combined with Edmonds' disjoint arborescences theorem immediately implies the following result.

**Theorem 2.8** (W.T. Tutte [38]). *An undirected graph $G = (V, E)$ is $k$-tree-connected if and only if it is $k$-partition-connected. In other words, $G$ contains $k$ edge-disjoint spanning trees if and only if*

$$e(\mathcal{F}) \geq k(t-1) \text{ for every partition } \mathcal{F} := \{V_1, ..., V_t\} \text{ of } V. \tag{2.12}$$

*Proof.* Necessity. From a connected graph we obtain a connected graph by contracting each part of a given partition into single nodes. Therefore each spanning tree must contain at least $t-1$ cross edges and hence $k$ edge-disjoint spanning trees contains $k(t-1)$ cross edges from which the necessity of (2.12) follows.

To see the sufficiency, observe first that, for an arbitrarily chosen root node $r_0 \in V$, Theorem 2.7 implies the existence of a rooted $k$-edge-connected orientation $D$ of $G$. Second, by applying Edmonds' Theorem 1.1, one obtains $k$ edge-disjoint spanning arborescences of $D$ which correspond to $k$ edge-disjoint spanning trees of $G$. □

The following orientation theorem will be used.

**Theorem 2.9** ( [11]). *Let $G = (V, E)$ be an undirected graph and $h$ an integer-valued intersecting supermodular function (with possible negative values). There is an orientation of $G$ covering $h$ if and only if*

$$e'(\mathcal{P}) \geq \sum [h(V_i) : i = 1, ..., t] \tag{2.13}$$

*holds for every subpartition $\mathcal{P} = \{V_1, ..., V_t\}$ of $V$ where $e'(\mathcal{P})$ denotes the number of edges of $G$ entering at least one member of $\mathcal{P}$.* □

This 'abstract' theorem can be specialized to obtain the following connectivity orientation result [11].

**Theorem 2.10.** *Let $M = (V, A+E)$ be a mixed graph consisting of an undirected graph $G = (V, E)$ and a directed graph $D = (V, A)$, and let $r_0 \in V$ be a designated root node. $M$ has a rooted $k$-edge-connected orientation with respect to $r_0$ if and only if*

$$e(\mathcal{P}) \geq \sum_{i=1}^{p} [k - \varrho_D(V_i)] \tag{2.14}$$

*holds for every partition $\mathcal{P} := \{V_0, V_1, ..., V_p\}$ of $V$ where $r_0 \in V_0$ and $e(\mathcal{P})$ denotes the number of edges of $G$ connecting distinct parts of $\mathcal{P}$.* □

By this result, Tutte's theorem can be extended to mixed graphs. We call a mixed tree $T$ a **mixed arborescence** of root $r_0$ if it is possible to orient its undirected edges so that the resulting directed tree is an arborescence. This is clearly equivalent to requiring for each directed edge of $T$ to be oriented away from $r_0$.

**Theorem 2.11** ([11]). *In a mixed graph $M = (V, A + E)$ with a root node $r_0$ there are $k$ edge-disjoint spanning mixed arborescences of root $r_0$ if and only if*

$$e(\mathcal{P}) \geq \sum_{i=1}^{p} [k - \varrho_D(V_i)] \tag{2.15}$$

*holds for every partition $\mathcal{P} := \{V_0, V_1, ..., V_p\}$ of $V$ where $r_0 \in V_0$ and $e(\mathcal{P})$ denotes the number of edges of $G$ connecting distinct parts of $\mathcal{P}$.*

*Outline of the proof.* The necessity is left to the reader. The sufficiency follows immediately by combining Theorems 1.1 and 2.10. □

## § 2.4. Root-vectors

Call a vector $z : V \to \{0, 1, ..., k\}$ a **root-vector** if there are $k$ edge-disjoint spanning arborescences in $D$ so that each node $v$ is the root of $z(v)$ arborescences. From Edmonds' theorem one easily gets the following characterization of root-vectors.

**Theorem 2.12.** *Given a digraph $D' = (V', A')$, a vector $z$ is a root-vector if and only if $z(V') = k$ and $z(X) \geq k - \varrho'(X)$ for every non-empty subset $X \subseteq V'$.*

*Proof.* The necessity of both conditions is evident. For the sufficiency, extend $D'$ with a node $r_0$ and $z(v)$ parallel edges from $r_0$ to $v$ for each $v \in V$. In the resulting digraph $D$ the out-degree of $r_0$ is exactly $k$ and $\varrho_D(X) = z(X) + \varrho'(X) \geq k$ holds for every non-empty $X \subseteq V'$. By Edmonds' theorem, $D$ contains $k$ edge-disjoint spanning arborescences of root $r_0$. Since $\delta_D(r_0) = k$, each of these arborescences must have exactly one edge leaving $r_0$ and therefore their restrictions to $A'$ form $k$ arborescences of $D'$ of root-vector $z$. $\square$

For an intersecting supermodular function $p$ with finite $p(S)$, let

$$B'(p) := \{x \in \mathbb{R}^S : \ x(S) = p(S), \ x(A) \geq p(A) \text{ for every } A \subseteq S\}.$$

This is called a base-polyhedron. The following result appeared in an equivalent form in [18].

**Theorem 2.13** (A. Frank, É. Tardos). *Let $p$ be an intersecting supermodular function for which $p(S)$ finite and let $f : S \to \mathbb{R} \cup \{-\infty\}$, $g : S \to \mathbb{R} \cup \{\infty\}$ be two functions for which $f \leq g$.*

*(i) The polyhedron $\{x \in B'(p) : f \leq x\}$ is non-empty if and only if*

$$f(S) \leq p(S) \tag{2.16}$$

*and*

$$f(X_0) + \sum_{i=1}^{t} p(X_i) \leq p(S) \tag{2.17}$$

*for every partition $\{X_0, X_1, ..., X_t\}$ $(t \geq 1)$ of $S$ in which only $X_0$ may be empty.*

*(ii) The polyhedron $\{x \in B'(p) : x \leq g\}$ is non-empty if and only if*

$$g(X) \geq p(X) \text{ for every } X \subseteq S. \tag{2.18}$$

*(iii) The base-polyhedron $\{x \in B'(p) : f \leq x \leq g\}$ is non-empty if and only if neither $\{x \in B'(p) : f \leq x\}$ nor $\{x \in B'(p) : x \leq g\}$ is empty.*

*If, in addition, each of $p$, $f$ and $g$ is integer-valued, then the corresponding polyhedra are integral.* $\square$

Let $D = (V, A)$ be a digraph. Define the set function $p$ by $p(X) = k - \varrho(X)$ for non-empty subsets $X$. Then $p$ is intersecting supermodular and Theorem 2.12 implies that the root vectors of $D$ are exactly the integral elements of the bases polyhedron $B'(p)$. By combining this with Theorem 2.13, one arrives at the following result.

**Theorem 2.14** (M.C. Cai [2], A. Frank [13]). *In a digraph $D = (V, A)$ there exist $k$ edge-disjoint spanning arborescences so that* *(i) each node $v$ is the root of at most $g(v)$ of them if and only if*

$$\sum_{i=1}^{t} \varrho_D(X_i) \geq k(t-1) \tag{2.19}$$

*holds for every subpartition $\{X_1, ..., X_t\}$ of $V$, and*

$$g(X) \geq k - \varrho_D(X) \tag{2.20}$$

*for every $\emptyset \subset X \subseteq V$;*

*(ii) each node $v$ is the root of at least $f(v)$ of them if and only if $f(V) \leq k$ and*

$$\sum_{i=1}^{t} \varrho_D(X_i) \geq k(t-1) + f(X_0) \tag{2.21}$$

*holds for every partition $\{X_0, X_1, ..., X_t\}$ of $V$ for which $t \geq 1$ and only $X_0$ may be empty;*

*(iii) each node $v$ is the root of at least $f(v)$ and at most $g(v)$ of them if and only if the lower bound problem and the upper bound problem have separately solutions.* □

Two interesting special cases are as follows.

**Corollary 2.15.** *A digraph $D = (V, A)$ includes $k$ edge-disjoint spanning arborescences* (with no restriction on their roots) *if and only if $\sum_{i=1}^{t} \varrho_D(X_i) \geq k(t-1)$ for every subpartition $\{X_1, ..., X_t\}$ of $V$.* □

**Corollary 2.16.** *A digraph $D = (V, A)$ includes $k$ edge-disjoint spanning arborescences whose roots are distinct if and only if $|X| \geq k - \varrho_D(X)$ holds for every non-empty subset $X \subseteq V$ set and $\sum_{i=1}^{t} \varrho_D(X_i) \geq k(t-1)$ for every subpartition $\{X_1, ..., X_t\}$ of $V$.* □

## § 3. Evolution of disjoint arborescences

### § 3.1. Concrete extensions

Edmonds actually proved his theorem in a stronger form where the goal was packing $k$ edge-disjoint branchings of given root-sets. A **branching** is a directed forest in which the in-degree of each node is at most one. The set of nodes of in-degree 0 is called the **root-set** of the branching. Note that a branching with root-set $R$ is the union of $|R|$ node-disjoint arborescences (where an arborescence may consist of a single node and no edge but we always assume that an arborescence has at least one node). For a digraph $D=(V,A)$ and root-set $\emptyset \subset R \subseteq V$ a branching $(V,B)$ is called a **spanning** $R$**-branching** of $D$ if its root-set is $R$. In particular, if $R$ is a singleton consisting of an element $s$, then a spanning branching is a spanning arborescence of root $s$.

**Theorem 3.1** (Edmonds' disjoint branchings). *In a digraph $D=(V,A)$, let $\mathcal{R}=\{R_1,...,R_k\}$ be a family of $k$ non-empty* (not necessarily disjoint or distinct) *subsets of $V$. There are $k$ edge-disjoint spanning branchings of $D$ with root-sets $R_1,...,R_k$, respectively, if and only if*

$$\varrho(X) \geq p(X) \text{ whenever } \emptyset \subset X \subseteq V \tag{3.1}$$

*where $p(X)$ denotes the number of root-sets $R_i$ disjoint from $X$.* □

*Remark.* In the special case of Theorem 3.1 when each root-set $R_i$ is a singleton consisting of the same node $r_0$, we are back at Theorem 1.1. Conversely, when the $R_i$'s are singletons (which may or may not be distinct), then Theorem 3.1 easily follows from Theorem 1.1. However, for general $R_i$'s no reduction is known.

**Theorem 3.2** (Edmonds' disjoint arborescences: strong form). *Let $D=(V,A)$ be a digraph whose node set is partitioned into a root-set $R=\{r_1,...,r_k\}$* (of distinct roots) *and a terminal set $T$. Suppose that no edge of $D$ enters any node of $R$. There are $k$ disjoint arborescences $F_1,...,F_k$ in $D$ so that $F_i$ is rooted at $r_i$ and spans $T+r_i$ for each $i=1,...,k$ if and only if $\varrho_D(X) \geq |R-X|$ for every subset $X \subseteq V$ for which $X \cap T \neq \emptyset$.* □

This follows easily by applying Theorem 3.1 to the subgraph $D'$ of $D$ induced by $T$ with the choice $R_i=\{v : \text{there is an edge } r_iv \in A\}$ $(i=1,...,k)$. The same construction shows the reverse implication, too.

The weak form of Edmonds' theorem could be used to derive Nash-Williams theorem on covering graphs by forests. The strong form gives rise to the following sharpening.

**Theorem 3.3.** *Suppose that the edge set of a connected undirected graph $G=(V,E\cup F)$ is partitioned into subsets $E$ and $F$ where $F$ is the union of $k$ trees $T_i=(R_i,F_i)$ $(i=1,...,k)$ for which $\emptyset \neq R_i \subseteq V$* (allowing $F_i=\emptyset$, $|R_i|=1$)*. It is possible to extend the $k$ trees from the elements of $E$ into $k$ spanning trees covering $E\cup F$ if and only if*

$$i_E(X) \leq \sum_{v\in X} p(v) - p(X) \text{ for every } \emptyset \subset X \subseteq V \tag{3.2}$$

*where $i_E(X)$ denotes the number of edges in $E$ induced by $X$, $p(X)$ denotes the number of sets $R_i$ disjoint from $X$.*

*Proof.* Necessity. Let $X$ be a non-empty subset of $V$. If $R_i \cap X \neq \emptyset$, then a forest including $T_i$ may contain at most $|X - R_i|$ elements of $E$ induced by $X$. Therefore the total number of these type of edges is at most

$$\sum_{R_i \cap X \neq \emptyset} |X - R_i| = \sum_{1}^{k} |X - R_i| - p(X)|X| = \sum_{v \in X} p(v) - p(X)|X|.$$

If $R_i \cap X = \emptyset$, then a forest including $T_i$ may contain at most $|X| - 1$ elements of $E$ induced by $X$. Therefore the total number of these type of edges is at most $p(X)(|X|-1)$. By combining the two upper bounds, (3.2) follows.

Sufficiency. We may assume that $G$ is maximal in the sense that the addition of any possible new edge to $E$ would destroy (3.2). Define a set function $b$ by

$$b(X) := \sum_{v \in X} p(v) - p(X).$$

Then $b$ is intersecting submodular. Call a set $X$ tight if $i_E(X) = b(X)$. The submodularity of $b$ and (3.2) imply that the union (and intersection) of two intersecting tight sets $X$ and $Y$ is tight since

$$i_E(X) + i_E(Y) = b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \geq$$

$$\geq i_E(X \cap Y) + i_E(X \cup Y) \geq i_E(X) + i_E(Y)$$

from which one must have $i_E(X \cup Y) = b(X \cup Y)$. Therefore the maximal tight sets are disjoint and hence the maximality of $G$ implies that $V$ itself is tight, that is, $|E| = \sum[p(v) : v \in V]$.

Let $m(v) := p(v)$ $(v \in V)$. We have $|E| = m(V)$ and $i_E(X) \leq m(X)$ for every $X \subseteq V$ by (3.2). By the Orientation lemma (Theorem 2.4), there is an orientation $\vec{E}$ of $E$ for which $\varrho_{\vec{E}}(v) = m(v)$ for every $v \in V$. Condition (3.2) implies for this orientation that $\varrho(X) = \sum[\varrho(v) : v \in X] - i_E(X) = \sum[p(v) : v \in X] - i_E(X) \geq p(X)$. By Theorem 3.1, there are $k$ edge-disjoint spanning branchings $\vec{B}_1, ..., \vec{B}_k$ with root sets $R_1, ..., R_k$, respectively. By this construction, each underlying forest $B_i$ along with the initial tree $T_i$ form a spanning tree of $G$. Since $\varrho(v) = p(v)$ for each node $v$, every edge of $G$ must belong to one of these trees. □

Notice that Nash-Williams' theorem is indeed a special case: let $T_i = (\{r\}, \emptyset)$ where $r$ is an arbitrary node. Then $\sum[p(v) : v \in X] - p(X) = k(|X| - 1) - 0 = k(|X| - 1)$

whenever $r \in X$ while $\sum[p(v) : v \in X] - p(X) = k|X| - k = k(|X|-1)$ whenever $v \notin X$. In other words, (3.2) and (2.8) are equivalent in this case.

It should be noted that the matroid partition theorem provides a good characterization for the problem formulated in Theorem 3.3 even in the more general case when the initial subgraphs are not necessarily trees but only forests. The point is that in our case the necessary and sufficient condition is simpler than the one arising from the matroid approach, similarly to the situation occurred already in the special case of Theorem 2.5 of Nash-Williams.

The following proper extension of Theorem 3.2 was derived in [1] with the help of a theorem of Frank and Tardos [19] on covering supermodular functions by digraphs.

**Theorem 3.4** ([1]). *Let $D = (V, A)$ be a digraph whose node set is partitioned into a root-set $R = \{r_1, ..., r_q\}$ and a terminal set $T$. Suppose that no edge of $D$ enters any node of $R$. Let $m : R \to \mathbb{Z}_+$ be a function and let $k = m(R)$. There are $k$ disjoint arborescences in $D$ so that $m(r)$ of them are rooted at $r$ and spanning $T + r$ for each $r \in R$ if and only if*

(3.3) $\quad \varrho_D(X) \geq m(R - X)$ *for every subset $X \subseteq V$ for which $X \cap T \neq \emptyset$.* □

On the other hand the following closely related problem proves to be NP-complete. Theorem 2.14 characterized root-vectors satisfying upper and lower bounds. One may be interested in a possible generalization for the framework described in Theorem 3.4. We show that this problem is NP-complete. Indeed, let $D = (V, A)$ be a digraph whose node set is partitioned into a root-set $R = \{r_1, ..., r_q\}$ and a terminal set $T$. Suppose that no edge of $D$ enters any node of $R$.

**Theorem 3.5.** *The problem of deciding whether there are $k$ disjoint arborescences so that they are rooted at distinct nodes in $R$ and each of them spans $T$ is NP-complete.*

*Proof.* Let $T$ be a set with even cardinality and let $\mathcal{R} = \{R_1, ..., R_q\}$ be subsets of $T$ so that $|R_i| \geq 2$ for $i = 1, ..., q$. It is well-known that the problem of deciding whether $T$ can be covered with $k$ members of $\mathcal{R}$ is NP-complete. Let $D_T$ be a directed graph on $T$ with $\varrho_{D_T}(Z) = k - 1$ for each $Z \subseteq T$, $|Z| = 1$ or $|Z| = |T| - 1$ and $\varrho_{D_T}(Z) \geq k$ otherwise. Such a graph can be constructed easily as follows. Take the same directed Hamilton cycle on the nodes $k-2$ times, then add the arcs $v_i v_{i+\frac{|T|}{2}}$ to the graph for each $i = 0, ..., |T| - 1$ where $v_0, ..., v_{|T|-1}$ denote the nodes according to their order around the cycle (the indices are meant modulo $|T|$). The arising digraph satisfies the in-degree conditions.

Extend the graph with $R = \{r_1, ..., r_q\}$ and with a new arc $r_i v$ for each $v \in R_i$. Let $r_{i_1}, ..., r_{i_k} \in R$ be a set of distinct root-nodes. Edmonds' disjoint branchings theorem implies that there are edge-disjoint $r_i$-arborescences $F_i$ spanning $r_i + T$ for $i = i_1, ..., i_k$ if and only if $\varrho_{D_T}(Z) \geq p(Z)$ for each $\emptyset \subset Z \subseteq T$ where $p(Z)$ denotes the number of $R_i$'s (with $i \in \{i_1, ..., i_k\}$) disjoint from $Z$. For a subset $Z$ with $|Z| \geq 2$ the inequality holds automatically because of the structure of $D_T$ and $|R_i| \geq 2$. Hence one only has to care about sets containing a single node and so the existence of the arborescences is equivalent to cover $T$ with $R_{i_1}, ..., R_{i_k}$.

The observation above means that $T$ can be covered with $k$ members of $\mathcal{R}$ if and only if the digraph includes $k$ arborescences rooted at different nodes in $R$. $\square$

Recently, Kamiyama, Katoh and Takizawa [29] were able to find a surprising new proper extension of the strong Edmonds theorem which implies Theorem 3.4 as well.

**Theorem 3.6** (N. Kamiyama, N. Katoh, A. Takizawa). *Let $D = (V, A)$ be a digraph and $R = \{r_1, ..., r_k\} \subseteq V$ a list of $k$ (possibly not distinct) root-nodes. Let $S_i$ denote the set of nodes reachable from $r_i$. There are edge-disjoint $r_i$-arborescences $F_i$ spanning $S_i$ for $i = 1, ..., k$ if and only if*

$$\varrho_D(Z) \geq p_1(Z) \text{ for every subset } Z \subseteq V \tag{3.4}$$

*where $p_1(Z)$ denotes the number of sets $S_i$ for which $S_i \cap Z \neq \emptyset$ and $r_i \notin Z$.* $\square$

The original proof is more complicated than that of Theorem 3.1 due to the fact that the corresponding set function $p_1$ in the theorem is no more supermodular. Based on Theorem 3.6, S. Fujishige [20] recently found a further extension. For two disjoint subsets $X$ and $Y$ of $V$ of a digraph $D = (V, A)$, we say that $Y$ is **reachable** from $X$ if there is a directed path in $D$ whose first node is in $X$ and last node is in $Y$. We call a subset $U$ of nodes **convex** if there is no node $v$ in $V - U$ so that $U$ is reachable from $v$ and $v$ is reachable from $U$.

**Theorem 3.7** (S. Fujishige). *Let $D = (V, A)$ be a directed graph and let $R = \{r_1, ..., r_k\} \subseteq V$ be a list of $k$ (possibly not distinct) root-nodes. Let $U_i \subseteq V$ be convex sets with $r_i \in U_i$. There are edge-disjoint $r_i$-arborescences $F_i$ spanning $U_i$ for $i = 1, ..., k$ if and only if*

$$\varrho_D(Z) \geq p_1(Z) \text{ for every subset } Z \subseteq V \tag{3.5}$$

*where $p_1(Z)$ denotes the number of sets $U_i$'s for which $U_i \cap Z \neq \emptyset$ and $r_i \notin Z$.* $\square$

*Remark.* Convexity plays an essential role in the proof of the theorem. It can be showed that even an apparently slight weakening of the reachability conditions results in NP-complete problems. Namely, let $D = (V, A)$ be a digraph with $u_1, u_2, v_1, v_2 \in V$ and let $U_1 = V,\ U_2 = V - v_1$. The problem of finding two edge-disjoint arborescences rooted at $u_1, u_2$ and spanning $U_1, U_2$, respectively, is NP-complete.

To show this, let $D'$ be a digraph with $u_1, u_2, v_1, v_2 \in V$. It is well-known that the problem of finding disjoint $u_1v_1$ and $u_2v_2$ paths is NP-complete. We may suppose that the in-degree of $v_1$ and $v_2$ is one. Let $D$ denote the graph arising from $D'$ by adding arcs $v_1v$ and $v_2v$ to $A$ for each $v \in V$. Clearly, there are edge-disjoint directed $u_1v_1$ and $u_2v_2$ paths in $D'$ if and only if there are two arborescences $F_1, F_2$ in $D$ such that $F_i$ is rooted at $u_i$ and spans $U_i$.

Vidyasankar's theorem can be considered as a covering counterpart of Edmonds' packing theorem. One may be interested in a covering counterpart of Fujishige's theorem. We show that Theorem 3.7 implies an extension of Vidyasankar's result.

**Theorem 3.8.** *Let $D = (V, A)$ be a digraph and $\{r_1, ..., r_k\} = R \subseteq V$ be a set of (not necessary distinct) root-nodes. Let $U_i \subseteq V$ be convex sets with $r_i \in U_i$. The edge set $A$ can be covered by $r_i$-arborescences $F_i$ spanning $U_i$ if and only if*

$$\varrho(v) \leq p_1(v) \text{ for each } v \in V \tag{3.6}$$

*and*

$$p_1(X) - \varrho(X) \leq \sum[p_1(v) - \varrho(v) :\ v \in \Gamma^-(X)] \tag{3.7}$$

*for every $\emptyset \subset X \subseteq V$, where $\Gamma^-(X)$ denotes the entrance of $X$ and $p_1(X)$ denotes the number of sets $U_i$'s for which $U_i \cap X \neq \emptyset$ and $r_i \notin X$.*

*Proof.* Necessity. Suppose that there are $k$ proper arborescences covering $A$. We may suppose that $F_i$ spans $U_i$ for each $i \in \{1, ..., k\}$. Since an arborescence $F_i$ contains no edge entering $v$ if $v = r_i$ or $v \notin U_i$, and one edge entering $v$ if $v \neq r_i$ and $v \in U_i$, the necessity of (3.6) follows immediately.

Necessity of (3.7) can be seen as follows. For each $e \in A$ let $z(e)$ denote the number of arborescences covering $e$ minus 1. Then $z \geq 0$, moreover $\varrho_z(X) + \varrho(X) \geq p_1(X)$ for each $\emptyset \subset U \subseteq V$ and $\varrho_z(v) + \varrho(v) = p_1(v)$ for each $v \in V$. Since each edge entering $X$ has its head in $\Gamma^-(X)$, we have $\varrho_z(X) \leq \sum[\varrho_z(v) :\ v \in \Gamma^-(X)]$ and these imply

$$p_1(X) - \varrho(X) \leq \varrho_z(X) \leq \sum[\varrho_z(v) :\ v \in \Gamma^-(X)] = \sum[p_1(v) - \varrho(v) :\ v \in \Gamma^-(X)].$$

Now we turn to sufficiency. For every node $v \in V$, give a copy of $v$ to $D$ denoted by $v'$. For a subset $X$ of $V$ let $X'$ be the copy of $X$. Add $p_1(v)$ parallel edges from $v$ to $v'$, $p_1(v) - \varrho(v)$ parallel edges from $v'$ to $v$, and finally $p_1(v)$ parallel edges from $u$ to $v'$ for every edge $uv \in A$. Let $D'$ denote the directed graph thus arising.

If there exist $F'_1, ..., F'_k$ disjoint arborescences in $D'$ such that $F'_i$ is rooted at $r_i$ and $F'_i$ is spanning $U_i \cup U'_i$ (where $U'_i$ denotes the copy of $U_i$), then these determine $k$ proper arborescences of $D$ covering $A$. It is easy to see that for every convex set $X \subseteq V$ in $D$ the union $X \cup X' \subseteq V \cup V'$ is also convex in $D'$.

In other case, by Fujishige's theorem, there is a subset $W$ of $V \cup V'$ such that $p'(W) > \varrho'(W)$ where $p'(W) = |\{i \in \{1, ..., k\} : (U_i \cup U_i') \cap W \neq \emptyset, \ r_i \notin W\}|$ and $\varrho' = \varrho_{D'}$. We define the following subsets of $W$: $X = \{v \in V : v \in W\}$, $Y = \{v \in V : v' \notin W\}$, and $Z = \{v' \in W : v \notin W\}$. We have

$$p'(W) \leq p_1(X) + \sum [p_1(v) : v' \in Z].$$

On the other hand

$$\varrho_{D'}(W) \geq \varrho(X) + \sum [p_1(v) - \varrho(v) : v \in Y] + \sum [p_1(v) : v \in \Gamma^-(X) - Y] + \sum [p_1(v) : v' \in Z].$$

The explanation of the second sum is that if $v \in \Gamma^-(X) - Y$, then $v' \in W$ also holds. Moreover, there exists, since $v$ is in the entrance, $u \notin W$ such that $uv \in A$, hence there are $p_1(v)$ arcs from $u$ to $v'$.

From these inequalities we get

$$p_1(X) > \varrho(X) + \sum [p_1(v) - \varrho(v) : v \in Y] + \sum [p_1(v) : v \in \Gamma^-(X) - Y] \geq$$

$$\geq \varrho(X) + \sum [p_1(v) - \varrho(v) : v \in \Gamma^-(X)],$$

contradicting condition (3.7). □

## § 3.2. Abstract extensions

There is another line of extending Theorem 1.1 in which, rather than working directly with arborescences, one considers disjoint edge-coverings of certain families of sets or bi-sets. We say that a set $F$ of directed edges covers a set or bi-set $X$ if at least one element of $F$ enters $X$. A family of sets or bi-sets is covered by $F$ if each member of it is covered.

**Theorem 3.9** ([12]). *Let $D = (V, A)$ be a digraph and $\mathcal{F}$ an intersecting family of subsets of $V$. It is possible to partition $A$ into $k$ coverings of $\mathcal{F}$ if and only if the in-degree of every member of $\mathcal{F}$ is at least $k$.* □

Obviously, when $\mathcal{F}$ consists of every non-empty subset of $V - r_0$, we obtain the weak form of Edmonds' theorem. A disadvantage of Theorem 3.9 is that it does not imply the strong version of Edmonds' theorem. The following result of L. Szegő [36], however, overcame this difficulty.

**Theorem 3.10** (L. Szegő). *Let $\mathcal{F}_1, ..., \mathcal{F}_k$ be intersecting families of subsets of nodes of a digraph $D = (V, A)$ with the following mixed intersection property:*

$$X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X \cap Y \neq \emptyset\ \Rightarrow\ X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j.$$

*Then $A$ can be partitioned into $k$ subsets $A_1, ..., A_k$ such that $A_i$ covers $\mathcal{F}_i$ for each $i = 1, ..., k$ if and only if $\varrho_D(X) \geq p_1(X)$ for all non-empty $X \subseteq V$ where $p_1(X)$ denotes the number of $\mathcal{F}_i$'s containing $X$.* $\square$

However, Theorem 3.10 does not imply Theorem 3.6. In [1], we derived an extension of Szegő's theorem to bi-set families.

The bi-set families $\mathcal{F}_1, ..., \mathcal{F}_k$ said to satisfy the **mixed intersection** property if

$$X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X_I \cap Y_I \neq \emptyset\ \Rightarrow\ X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j.$$

For a bi-set $X$, let $p_2(X)$ denote the number of indices $i$ for which $\mathcal{F}_i$ contains $X$. For $X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j$, the inclusion $X \subseteq Y$ implies $X = X \cap Y \in \mathcal{F}_j$ and hence $p_2$ is monotone non-increasing in the sense that $X \subseteq Y$, $p_2(X) > 0$ and $p_2(Y) > 0$ imply $p_2(X) \geq p_2(Y)$.

**Theorem 3.11** ([1]). *Let $D = (V, A)$ be a digraph and $\mathcal{F}_1, ..., \mathcal{F}_k$ intersecting families of bi-sets on ground set $V$ satisfying the mixed intersection property. The edges of $D$ can be partitioned into $k$ subsets $A_1, ..., A_k$ in such a way that $A_i$ covers $\mathcal{F}_i$ for each $i = 1, ..., k$ if and only if*

(3.8) $$\varrho_D(X) \geq p_2(X) \text{ for every bi-set } X. \ \square$$

The proof of this went along the same line as Lovász' original proof for Edmonds' theorem and was based on the following property.

**Lemma 3.12.** *If $p_2(X) > 0,\ p_2(Y) > 0$ and $X_I \cap Y_I \neq \emptyset$, then $p_2(X) + p_2(Y) \leq p_2(X \cap Y) + p_2(X \cup Y)$. Moreover, if there is an $\mathcal{F}_i$ for which $X \cap Y \in \mathcal{F}_i$ and $X, Y \notin \mathcal{F}_i$, then strict inequality holds.* $\square$

Here we are going to show that Theorem 3.11 implies Fujihige's theorem, as well.

*Proof of Theorem 3.7.* If the node set of an arborescence $F$ of root $r_i$ intersects a subset $Z \subseteq V - r_i$, then $F$ contains an element entering $Z$. Therefore if the $k$ edge-disjoint arborescences exist, then $Z$ admits as many entering edges as the number of sets $U_i$ for which $Z \cap U_i \neq \emptyset$ and $r_i \notin Z$, that is, (4.2) is indeed necessary.

Sufficiency. For brevity, we call a strongly connected component of $D$ an **atom**. It is known that the atoms form a partition of the node set of $D$ and that there is a so-called topological ordering of the atoms so that there is no edge from a later atom to an earlier one. By a **subatom** we mean a subset of an atom. Clearly, a subset $X \subseteq V$ is a subatom if and only if any two elements of $X$ are reachable in $D$ from each other. The following observation is obvious from the definitions.

**Proposition 3.13.** *If a subatom $X$ intersects a convex set $U$, then $X \subseteq U$.*

Define $k$ bi-set families $\mathcal{F}_i$ for $i = 1, ..., k$ as follows. Let $\mathcal{F}_i := \{(X_O, X_I) : X_O \subseteq V - r_i,\ X_I = X_O \cap U_i \neq \emptyset,\ X_I \text{ is a subatom}\}$. For each bi-set $X$, let $p_2(X)$ denote the number of $\mathcal{F}_i$'s containing $X$. It follows immediately that $\mathcal{F}_i$ is an intersecting bi-set family.

**Proposition 3.14.** *The bi-set families $\mathcal{F}_i$ satisfy the mixed intersecting property.*

*Proof.* Let $X = (X_O, X_I)$ and $Y = (Y_O, Y_I)$ be members of $\mathcal{F}_i$ and $\mathcal{F}_j$, respectively, and suppose that $X$ and $Y$ are intersecting, that is, $X_I \cap Y_I \neq \emptyset$. By Proposition 3.13, we have that $X_I = X_O \cap U_i \subseteq U_i \cap U_j$ and $Y_I = Y_O \cap U_j \subseteq U_i \cap U_j$. This implies for the sets $Z_O := X_O \cap Y_O$ and $Z_I := X_I \cap Y_I$ that $Z_O \cap U_i = X_O \cap U_i \cap Y_O = X_O \cap U_i \cap Y_O \cap U_j = Z_I$ and also $Z_O \cap U_j = X_O \cap Y_O \cap U_j = X_O \cap U_i \cap Y_O \cap U_j = Z_I$ from which $Z_I \subseteq U_i \cap U_j$ and $(Z_O - Z_I) \cap (U_i \cup U_j) = \emptyset$. Hence $X \cap Y = (Z_O, Z_I) \in \mathcal{F}_i \cap \mathcal{F}_j$, as required. $\square$

**Proposition 3.15.** *$\varrho(X) \geq p_2(X)$ for each bi-set $X$.*

*Proof.* Let $q := p_2(X)$ and suppose that $X$ belongs to $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_q$. Let $V' := V - (U_1 \cup ... \cup U_q)$ and $Z := X_I \cup \{v \in V' : X_I \text{ is reachable from } v\}$.

Let $e = uv$ be an edge of $D$ entering the set $Z$. Then $u$ cannot be in $V' - Z$ for otherwise $X_I$ would be reachable from $u$ and then $u$ should belong to $Z$. Therefore $u$ is in $(U_1 \cup ... \cup U_q) - Z$. Let $U_i$ be one of the sets $U_1, ..., U_q$ containing $u$. We claim that the head $v$ of $e$ must be in $X_I$. For otherwise we are in a contradiction with the hypothesis that $U_i$ is convex since $v$ is reachable from $U_i$ (along the edge $uv$) and $U_i$ is also reachable from $v$ since $X_I \subseteq U_i$ is reachable from $v$.

It follows that the edge $e$ entering the set $Z$ also enters the bi-set $X = (X_O, X_I)$. Therefore $\varrho(X) \geq \varrho(Z)$. By (4.2), we have $\varrho(Z) \geq p_1(Z)$. It follows from the definition of $Z$ that $p_1(Z) \geq q = p_2(X)$, and hence $\varrho(X) \geq p_2(X)$ $\square$

Therefore Theorem 3.11 applies and hence the edges of $D$ can be partitioned into subsets $A_1, ..., A_k$ so that $A_i$ covers $\mathcal{F}_i$ for $i = 1, ..., k$.

**Proposition 3.16.** *Each $A_i$ includes an $r_i$-arborescence $F_i$ which spans $U_i$.*

*Proof.* If the requested arborescence does not exist for some $i$, then there is a non-empty subset $Z$ of $U_i - r_i$ so that $A_i$ contains no edge from $U_i - Z$ to $Z$. Consider a topological ordering of the atoms and let $Q$ be the earliest one intersecting $Z$. Since no edge leaving a later atom can enter $Q$, no edge with tail in $Z$ enters $Q$.

Let $X_O := (V - U_i) \cup (Z \cap Q)$ and $X_I := X_O \cap U_i$. Then $X_I = Z \cap Q$ is a subatom and $X = (X_O, X_I)$ belongs to $\mathcal{F}_i$. Therefore there is an edge $e = uv$ in $A_i$ which enters $X$. It follows that $v \in X_I \subseteq Z$ and that $u \in U_i - X_I$. Since $u$ is not in $Z$ and not in $V - U_i$, it must be in $U_i - Z$, that is, $e$ is an edge from $U_i - Z$ to $X_I \subseteq Z$, contradicting the assumption that no such edge exists. □

□

It is worth mentioning that Theorem 3.11 has an equivalent form that uses $T$-intersecting families instead of bi-sets [1]. For a subset $T$ of $V$, we call the set families $\mathcal{F}_1, ..., \mathcal{F}_k$ *$T$-intersecting* if

$$X, Y \in \mathcal{F}_i,\ X \cap Y \cap T \neq \emptyset \ \Rightarrow\ X \cap Y, X \cup Y \in \mathcal{F}_i.$$

We say that $\mathcal{F}_1, ..., \mathcal{F}_k$ satisfy the *mixed $T$-intersection property* if

$$X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X \cap Y \cap T \neq \emptyset \ \Rightarrow\ X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j.$$

Then the equivalent form is as follows.

**Theorem 3.17** ([1]). *Let $D = (V, A)$ be a digraph and $T$ a subset of $V$ that contains the head of every edge of $D$. Let $\mathcal{F}_1, ..., \mathcal{F}_k$ be $T$-intersecting families also satisfying the mixed $T$-intersection property. Then $A$ can be partitioned into subsets $A_1, ..., A_k$ so that $A_i$ covers $\mathcal{F}_i$ if and only if $\varrho(X) \geq p(X)$ for each non-empty subset $X$ of $V$ where $p(X)$ denotes the number of $\mathcal{F}_i$'s containing $X$.* □

## § 4. Polyhedral and algorithmic aspects

### § 4.1. Cheapest packing of arborescences

In [6], Edmonds pointed out that a digraph $D = (V, B)$ is the union of $k$ edge-disjoint spanning arborescences of root $r_0$ if and only if

$$\varrho(r_0) = 0 \text{ and } \varrho(v) = k \text{ for every } v \in V - r_0 \tag{4.1}$$

and the underlying undirected graph of $D$ is the union of $k$ edge-disjoint spanning trees.

J. Edmonds and D.R. Fulkerson [9] proved that the sum (or union) of some matroids forms a matroid, in particular, the subsets of edges of a graph which are the union of $k$ edge-disjoint spanning trees form the set of bases of a matroid denoted by $M_1$. Let $M_2$ denote the partition matroid whose set of bases is defined by (4.1). Therefore finding a cheapest subgraph of a digraph which is the union of $k$ edge-disjoint arborescences is equivalent to computing a cheapest common basis of matroids $M_1$ and $M_2$. This can be done with the help of any weighted matroid intersection algorithm. A matroid intersection algorithm can only be applied if the independence oracles (or equivalent) for the two matroids are indeed available. This is obviously the case for the partition matroid $M_2$. As far as $M_1$ is concerned, Edmonds' [5] polynomial-time algorithm for computing the rank of the sum of matroids provides the requested oracle.

Fujishige observed that a similar approach works for his extension above. Let $D = (V, A)$ be a directed graph and let $R = \{r_1, ..., r_k\} \subseteq V$ be a list of $k$ (possibly not distinct) root-nodes. Let $U_i \subseteq V$ be convex sets with $r_i \in U_i$. Recall that $p_1(X)$ denotes the number of sets $U_i$'s for which $U_i \cap Z \neq \emptyset$ and $r_i \notin Z$.

Let $G = (V, E)$ denote the underlying undirected graph of $D$ and suppose that each $U_i$ induces a connected subgraph. Let $N_i$ be a matroid on $A$ in which a subset is a bases if the corresponding undirected set of edges forms a tree of $G$ spanning $U_i$. Let $M_1$ be the sum of matroids $N_1, ..., N_k$. Let $M_2$ be a partition matroid on $A$ in which a subset $A'$ is a bases $\varrho_{A'}(v) = p_1(v)$ for every node $v \in V$.

**Theorem 4.1** (S. Fujishige). *A subgraph $D' = (V, A')$ of $D$ is a minimal subgraph (with respect to edge-deletion) including disjoint $r_i$-arborescences $F_i$ spanning $U_i$ for $i = 1, ..., k$ if and only of $A'$ is a common bases of matroids $M_1$ and $M_2$.*

*Proof.* For completeness, the proof of the theorem is also presented. If $D'$ is a minimal subgraph including the requested $k$ arborescences, then $A'$ partitions into those arborescences implying that $A'$ is indeed a common basis of $M_1$ and $M_2$.

Conversely, suppose that $A'$ is a common basis. Without loss of generality, we may assume that the roots $r_i$ are distinct. Let $R = \{r_1, ..., r_k\}$. Observe that

$$i_{A'}(Z) \leq \sum_{r_i \in Z} (|Z \cap U_i| - 1) + \sum_{r_i \notin Z, Z \cap U_i \neq \emptyset} (|Z \cap U_i| - 1) = \sum_{Z \cap U_i \neq \emptyset} (|Z \cap U_i|) - |Z \cap R| - p_1(Z)$$

and also

$$\sum_{v \in Z} p_1(v) = \sum_{Z \cap U_i \neq \emptyset} (|Z \cap U_i|) - |Z \cap R|$$

from which

$$\varrho_{A'}(Z) = \sum [\varrho_{A'}(v) : v \in Z] - i_{A'}(Z) = \sum [p_1(v) : v \in Z] - i_{A'}(Z) \geq$$

$$\geq \sum [p_1(v) : v \in Z] - \sum_{Z \cap U_i \neq \emptyset} (|Z \cap U_i|) + |Z \cap R| + p_1(Z) = p_1(Z).$$

By Theorem 3.7, $A'$ includes $k$ disjoint $r_i$-arborescences spanning respectively $U_i$ ($i = 1, ..., k$). Due to the assumption that $\varrho_{A'}(v) = p_1(v)$ for every node, these arborescences partition $A'$, that is, $A'$ is a minimal subset of $A$ including the $k$ disjoint arborescences. □

Due to this result, a minimum total cost of the disjoint $r_i$-arborescences spanning $U_i$ can be computed with the help of a weighted matroid intersection algorithm.

## § 4.2. The capacitated case

Fujishige's theorem can also be reformulated in terms of root-sets and branchings.

**Theorem 4.2.** *Let $D = (V, A)$ be a directed graph and let $\mathcal{R} = \{R_1, ..., R_k\}$ be a list of $k$ (possibly not distinct) root-sets. Let $U_i \subseteq V$ be convex sets with $R_i \subseteq U_i$. There are edge-disjoint $R_i$-branchings $B_i$ spanning $U_i$ for $i = 1, ..., k$ if and only if*

(4.2) $$\varrho_D(Z) \geq p_1(Z) \text{ for every subset } Z \subseteq V$$

*where $p_1(Z)$ denotes the number of sets $U_i$'s for which $U_i \cap Z \neq \emptyset$ and $R_i \cap Z = \emptyset$.* □

In [35] (pp. 920–921), Schrijver presented a strongly polynomial time algorithm to find maximum number of $r$-arborescences under capacity restrictions. By following his approach, one can find disjoint branchings satisfying the conditions of Theorem 4.2 in strongly polynomial time even in the more general case when a demand function is given on the set of root-sets.

**Theorem 4.3.** *Let $D = (V, A)$ be a digraph, $g : A \to \mathbb{Z}_+$ a capacity function, $\mathcal{R} = \{R_1, ..., R_k\}$ a list of root-sets, $\mathcal{U} = \{U_1, ..., U_k\}$ a set of convex sets with $R_i \subseteq U_i$, and $m : \mathcal{R} \to \mathbb{Z}_+$ a demand function. There is a strongly polynomial time algorithm that finds (if there exist) $m(\mathcal{R})$ disjoint branchings so that $m(R_i)$ of them are spanning $U_i$ with root-set $R_i$ and each edge $e \in A$ is contained in at most $g(e)$ branchings.*

*Proof.* For every $Z \subseteq V$, let $p_1(Z) = \sum [m(R_i) : R_i \in \mathcal{R}, R_i \cap Z = \emptyset, U_i \cap Z \neq \emptyset]$. By replacing every arc $a$ by $g(a)$ parallel arcs, it follows from Theorem 4.2 that the required branchings exist if and only if

(4.3) $$\varrho_g(Z) \geq p_1(Z) \text{ for every } Z \subseteq V.$$

The root-sets are gradually increased during the algorithm, and also the set of root-sets may become larger. We always assign one of the convex sets to the newly appearing root-sets. We may assume that $g$ and $m$ are strictly positive everywhere and (4.3) is satisfied.

We are done if $R_i = U_i$ for each $i$ so assume that, say, $R_1 \subset U_1$. Let $e = uv$ be an arc with $u \in R_1,\ v \in U_1 - R_1$ and define the following parameter.

$$\mu = \min\left\{g(e),\ m(R_1),\ \min\{\varrho_g(W) - p_1(W) :\ e \text{ enters } W,\ R_1 \cap W \neq \emptyset\}\right\}. \tag{4.4}$$

The value of $\mu$ can be determined in strongly polynomial time by computing a maximum flow in an auxiliary graph.

By Theorem 4.2, there is an arc $e$ for which $\mu$ is strictly positive. Add $e$ to $\mu$ copies of the $m(R_1)$ branchings to be rooted at $R_1$, $m(R_1) := m(R_1) - \mu$. Moreover, add a copy of $R := R_1 + v$ to $\mathcal{R}$ (even if it was already the member of the root-sets), define $m(R) := m(R) + \mu$ and assign the same convex set to $R$ as to $R_1$. Finally, revise $g(e)$ by $g(e) - \mu$. Due to the definition of $\mu$, the revised problem also meets (4.3) and we can apply the basic step recursively.

Now we turn to the running time. First we consider phases when the minimum in (4.4) is taken on $g(e)$ or $m(R_1)$. If the minimum is taken on $g(e)$ for some arc $e$, then the number of arcs with positive capacity decreases which may happen at most $|A|$ times. Note that the set of root-sets may increase only in these phases. Otherwise, the minimum is taken on $m(R_1)$ meaning that $R_1$ gets out from the set of root-sets. The size of each root-set increases at most $|V|$ times and the set of root-sets may increase, according to the above, at most $|A|$ times, hence the total number of phases is bounded by $|A||V|$.

It only remains to take into account those phases when the minimum is taken on $min\{\varrho_g(W) - p_1(W) :\ e \text{ enters } W,\ R_1 \cap W \neq \emptyset\}$. The approach of [35] does not work directly here as it strongly relies on the supermodularity of the set function $p(Z) = \sum[m(R_i) :\ R_i \in \mathcal{R},\ R_i \cap Z = \emptyset]$. As we already mentioned in Section 3, $p_1$ is no more supermodular (for that very reason the original proof of Theorem 4.2 was far more complicated than the one Lovász gave to Edmonds' theorem). Define the bi-set function $p_2(X) = \sum[m(R_i) :\ R_i \in \mathcal{R},\ X_O \cap R_i = \emptyset,\ X_I = X_O \cap U_i \neq \emptyset]$ if $X_I$ is a subatom, and 0 otherwise.

Recall that, by the proof of Theorem 3.7, (4.3) is equivalent to requiring that $\varrho_g(X) \geq p_2(X)$ for each bi-set $X \in \mathcal{P}_2$, hence the latter inequality also holds throughout the algorithm. The advantage of using bi-sets is that $p_2$ is positively intersecting supermodular on $\mathcal{P}_2$ (this can be seen similarly to Lemma 3.12). The collection $\mathcal{C} = \{X \in \mathcal{P}_2 :\ \varrho_g(X) = p_2(X) > 0\}$ of tight bi-sets increases in the considered phases ($\varrho_g(X) > 0$ may be assumed, otherwise the minimum in (4.4) is also taken on $g(e)$ and such phases are already counted).

Let $\mathcal{C}_O(e) = \{X_O :\ X \in \mathcal{C},\ e \text{ enters } X\}$ for each $e \in A$. However, $|\mathcal{C}_O(e)| = |\{X \in \mathcal{C} :\ e \text{ enters } X\}|$ holds for each $e$. To see this, let $X$ be a bi-set that becomes tight during the revision step and assume that $e = uv$ enters $X$. For an arbitrary set $Z_O$ containing $v$, there is at most one set $Z_I$ such that $v \in Z_I$ and $p_2((Z_O, Z_I)) > 0$. Namely, $Z_I$ must be a subatom and it must arise as the intersection of $Z_O$ and the atom containing $v$. Hence for each $Z_O \in \mathcal{C}_O(e)$ the corresponding inner set $Z_I$ is uniquely determined. This implies that $X_O \notin \mathcal{C}_O(e)$ before the revision step as otherwise $X \in \mathcal{C}$, a contradiction.

The above immediately implies that if $\mathcal{C}$ increases then also $\mathcal{C}_O(e)$ increases for some $e \in A$. If an edge $e$ enters both $X, Y \in \mathcal{C}$, then $\varrho_g(X \cap Y) > 0$ and $\varrho_g(X \cup Y) > 0$. The submodularity of $\varrho_g$ and positively intersecting supermodularity of $p_2$ implies that $\mathcal{C}_O(e)$ is a lattice family. As a lattice family $\mathcal{L}$ is uniquely determined by the preorder defined as

$$s \preceq t \;\Leftrightarrow\; \text{each set in } \mathcal{L} \text{ containing } t \text{ also contains } s,$$

if $\mathcal{L}$ increases then $\preceq$ decreases, which can happen at most $|V|^2$ times. Hence $\mathcal{C}_O$ increases at most $|V|^2$ times for each $e \in A$, and the total number of phases is $O(|A||V|^2)$. $\square$

## § 4.3. Polyhedral description

Let $D = (V, A)$ be a digraph, $R = \{r_1, ..., r_k\}$ a set of root-nodes and $\mathcal{U} = \{U_1, ..., U_k\}$ a set of convex sets with $r_i \in U_i$ for each $i$. We say that the digraph is **arborescence-packable** (with respect to $\mathcal{U}$) if there are $k$ disjoint arborescences $F_1, ..., F_k$ so that $F_i$ is an $r_i$-arborescences spanning $U_i$. Our next goal is to describe the convex hull of the incidence vectors of arborescence-packable subgraphs of $D$.

We may suppose that the root nodes $r_1, ..., r_k$ are distinct, each having exactly one leaving edge and no entering ones. Let $R = \{r_1, ..., r_k\}$ and $T = V - R$, so $U_i \cap R = \{r_i\}$ for each $r_i \in R$. For every non-empty subset $Z$ of $T$, let $p_1(Z)$ denote the number of roots $r_i$ for which $Z \cap U_i \neq \emptyset$. In particular, for every $v \in T$, $p_1(v)$ is the number of roots $r_i$ for which $v \in U_i$.

**Theorem 4.4.** *Let $D = (V, A)$ be a digraph in which $R$ is a set of $k$ root-nodes so that the out-degree and the in-degree of each root-node is one and zero, respectively. Let $T = V - R$ and for each root-node $r_i$ let $U_i$ be a convex set for which $U_i \cap R = \{r_i\}$. Then $D$ is arborescence-packable if and only if $\varrho(Z) \geq p_1(Z)$ for every subset $Z \subseteq T$.* $\square$

Define $k$ bi-set families $\mathcal{F}_i$ for $i = 1, ..., k$ as follows. Let

$$\mathcal{F}_i := \{(X_O, X_I) : \; X_O \subseteq T, \; X_I = X_O \cap U_i \neq \emptyset, \; X_I \textit{ is a subatom}\}.$$

For each bi-set $X$, let $p_2(X)$ denote the number of $\mathcal{F}_i$'s containing $X$. It follows immediately that $\mathcal{F}_i$ is an intersecting bi-set family.

*Remark.* Suppose that the out-degree of the root-nodes in $R$ may be larger than one. Let $\mathcal{U} = \{U_1, ..., U_k\}$ be a set of convex sets so that $U_i \cap R = \{r_i\}$ for each $r_i \in R$. Furthermore, let $m : R \to \mathbb{Z}_+$ be a demand function on the root-nodes so that $m(R) = t$. By Fujishige's theorem, there are $t$ disjoint arborescences so that $r_i$ is the root of $m_i$ arborescences spanning $U_i$ if and only if

$$\varrho(Z) \geq p_1(Z)$$

for every subset $Z \subseteq V$ where

$$p_1(Z) = \sum_{r_i \notin Z,\ Z \cap U_i \neq \emptyset} m(r_i).$$

In this case the bi-set families should be defined as follows. Let

$$\mathcal{F}_i^j := \{(X_O, X_I) :\ X_O \cap T \neq \emptyset,\ X_I = X_O \cap U_i,\ \emptyset \neq X_I \subseteq T \ \textit{is a subatom}\}$$

where $i = 1, ..., k$ and $j = 1, ..., m(r_i)$. It is easy to see that $\mathcal{F}_i^j$ is an intersecting bi-set family. However, this form follows from Theorem 4.4 by an easy construction. Since the statements are simpler when root-nodes has out-degree one, we will use this special form when formulating our result.

Before formulating our result, we prove two useful lemmas exhibiting an interrelation between sets and bi-sets.

**Lemma 4.5.** *For every bi-set $X = (X_O, X_I)$ there is a subset $Z \subseteq T$ for which $p_1(Z) \geq p_2(X)$ and $\Delta^-(Z) \subseteq \Delta^-(X)$.*

*Proof.* Let $q := p_2(X)$. If $q = 0$, then $Z := \emptyset$ will do. Suppose that $q \geq 1$ and $X$ belongs to $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_q$. Let $V' := V - (U_1 \cup ... \cup U_q)$. We claim that the set $Z := X_I \cup \{v \in V' : X_I$ is reachable from $v\}$ satisfies the properties required by the lemma.

One obviously has $p_1(Z) \geq q = p_2(X)$ since $Z$ intersects each of $U_1, ..., U_q$. Consider now an edge $e = uv$ of $D$ entering $Z$. The tail $u$ of $e$ cannot be in $V' - Z$ for otherwise $X_I$ would be reachable from $u$ and then $u$ should belong to $Z$. Therefore $u$ must be in $(U_1 \cup ... \cup U_q) - Z$. Let $U_i$ be one of the sets $U_1, ..., U_q$ containing $u$. Then the head $v$ of $e$ must be in $X_I$, for otherwise $v$ is reachable from $U_i$ (along the edge $uv$) and $X_I$ is also reachable from $v$ by the definition of $Z$ but this contradicts the convexity of $U_i$ since $X_I \subseteq U_i$. Hence the edge $e$ entering the set $Z$ also enters the bi-set $X = (X_O, X_I)$. □

**Lemma 4.6.** *For every subset $Z \subseteq T$, there are bi-sets $X_1, ..., X_t$ so that $\sum[p_2(X_j) : j = 1, ..., t] = p_1(Z)$ and $\{\Delta^-(X_j) : j = 1, ..., t\}$ is a partition of $\Delta^-(Z)$.*

*Proof.* Let $\mathcal{C}_Z := \{C_1, ..., C_t\}$ denote the set of atoms of $D$ intersecting $Z$ and assume that its members are arranged in a topological ordering, that is, no edge of $D$ leaving a $C_j$ enters a $C_i$ for which $i < j$. For each $j = 1, ...., t$, let $X_j = (X_O^j, X_I^j)$ where $X_O^j := Z \cap (C_1 \cup ... \cup C_j)$ and $X_I^j := Z \cap C_j$. We claim that these bi-sets $X_j$ satisfy the properties required by the lemma.

If an edge $e = uv$ enters a bi-set $X_j$, then its head $v$ is in $Z \cap C_j$ while its tail $u$ must be outside $Z$ by the property of the topological ordering, that is, $e$ enters $Z$, too. This and the obvious fact that $\{X_I^j : \ j = 1, ..., t\}$ forms a partition of $Z$ imply $\{\Delta^-(X_j) : j = 1, ..., t\}$ forms a partition of $\Delta^-(Z)$.

Let $\mathcal{U}_Z := \{U \in \mathcal{U} : U$ intersects $Z\}$. Note that $|\mathcal{U}_Z|$ has been denoted by $p_1(Z)$ and recall that an atom is either disjoint from or included by a convex set. For $j = 1, ..., t$, let $\mathcal{U}_Z^j := \{U \in \mathcal{U}_Z : j$ is the smallest subscript for which $C_j \in \mathcal{C}_Z$ and $C_j \subseteq U\}$. Some of the $\mathcal{U}_Z^j$'s may be empty but the non-empty ones form a partition of $\mathcal{U}_Z$. For each $j = 1, ..., t$, one has $p_2(X_j) = |\mathcal{U}_Z^j|$ and hence

$$p_1(Z) = |\mathcal{U}_Z| = \sum_{j=1}^{t} |\mathcal{U}_Z^j| = \sum_{j=1}^{t} p_2(X_j),$$

as required. □

Consider the following two polyhedra.

(4.5) $$R_1 := \{x \in \mathbb{R}^A : 0 \leq x, \varrho_x(Z) \geq p_1(Z) \text{ for every non-empty } Z \subseteq T\}.$$

(4.6) $$R_2 := \{x \in \mathbb{R}^A : 0 \leq x, \varrho_x(X) \geq p_2(X) \text{ for every non-trivial bi-set } X = (X_O, X_I) \text{ with } X_O \subseteq T\}.$$

**Lemma 4.7.** $R_1 = R_2$.

*Proof.* Suppose first that $x \in R_1$. Let $X$ be an arbitrary bi-set for which $p(X) > 0$. By Lemma 4.5 there is a subset $Z \subseteq T$ for which $p_1(Z) \geq p_2(X)$ and $\Delta^-(Z) \subseteq \Delta^-(X)$. This and the non-negativity of $x$ imply that $\varrho_x(X) \geq \varrho_x(Z) \geq p_1(Z) \geq p_2(X)$ from which $x \in R_2$ follows.

Second, suppose that $x \in R_2$. Let $Z$ be an arbitrary set for which $p_1(Z) > 0$. By Lemma 4.6 there are bi-sets $X_1, ..., X_t$ so that $\sum[p_2(X_j) : j = 1, ..., t] = p_1(Z)$ and $\{\Delta^-(X_j) : j = 1, ..., t\}$ is a partition of $\Delta^-(Z)$. This and the non-negativity of $x$ imply that $\varrho_x(Z) \geq \sum[\varrho_x(X_j) : j = 1, ..., t] \geq [p_2(X_j) : j = 1, ..., t] = p_1(Z)$ from which $x \in R_1$ follows. □

The following result was proved in [15].

**Theorem 4.8** (A. Frank, T. Jordán). *Let $D = (V, A)$ be a digraph and $p$ a positively intersecting supermodular bi-set function on $V$. Let $g : A \to \mathbb{Z}_+ \cup \{\infty\}$ be a capacity function on $A$ so that $\varrho_g(X) \geq p(X)$ for every bi-set. The following linear system for $x \in \mathbb{R}_+$ is totally dual integral (TDI):*

$$\{0 \leq x \leq g, \varrho_x(X) \geq p(X) \textit{ for every bi-set } X\}. \quad \square$$

From this we derive the following.

**Theorem 4.9.** *The linear system written for $x \in \mathbb{R}^A$*

$$\{0 \leq x \leq g, \varrho_x(Z) \geq p_1(Z) \textit{ for every non-empty } Z \subseteq T\} \tag{4.7}$$

*is totally dual integral (TDI). In particular, the convex hull of arborescence-packable subgraphs of $D$ is equal to the following polyhedron*

$$\{x \in \mathbb{R}^A : 0 \leq x \leq 1, \varrho_x(Z) \geq p_1(Z) \textit{ for every non-empty } Z \subseteq T\}. \tag{4.8}$$

*Proof.* By theorem 4.8, the system

$$\{0 \leq x \leq g, \varrho_x(X) \geq p_2(X) \text{ for every bi-set } X\} \tag{4.9}$$

is TDI. By Lemma 4.7, this and 4.7 define the same polyhedron.

We say that an inequality $qx \geq \beta$ is an integer consequence of a inequality system $Qx \geq p$ if there is an integer vector $y$ so that $yQ = q$ and $yp = \beta$. By elementary properties of TDI systems, it suffices to show that each inequality from 4.9 is an integer combination of inequalities of 4.7. By Lemma 4.5, for a bi-set $X = (X_O, X_I)$, there is a subset $Z \subseteq T$ for which $p_1(Z) \geq p_2(X)$ and $\Delta^-(Z) \subseteq \Delta^-(X)$. Therefore the inequality $\varrho_x(X) \geq p_2(X)$ is indeed a integer consequence of 4.7.

A general result of Edmonds and Giles [10] implies that the polyhedron defined by 4.8 is integral and hence its vertices are $0-1$ vectors. By Theorem 4.4, these vertices correspond to the arborescence-packable subgraphs of $D$. $\square$

## § 5. Independent arborescences

The following result is due to A. Huck [22].

**Theorem 5.1** (A. Huck). *Let $D = (V, A)$ be a simple acyclic digraph in which $S = \{s_1, ..., s_k\}$ denotes the set of source nodes (that is, those of in-degree zero) while $U := V - S$ is the rest. Suppose that the in-degree of each node $u \in U$ is at least $k$. Then there are $s_i$-arborescences $F_i$ spanning $U + s_i$ for $i = 1, ..., k$ so that the $k$ unique $s_iu$-paths in the arborescences $F_i$ are openly disjoint for every node $u \in U$.*

*Proof.* The arborescences in the theorem are called **independent**.

**Lemma 5.2.** *Suppose that $D' = (U + s, A')$ is a simple acyclic digraph in which $s$ is a source node and the in-degree of each other node is at least one. Then there is an ordering of the elements of $U$ in such a way that the set of all edges going forward can be completed with some edges leaving $s$ so as to obtain a spanning $s$-arborescence.*

*Proof.* The lemma is clear when $U$ is a singleton so we may assume that $|U| \geq 2$. Then there is a sink node $z$. By induction, there is a requested ordering of the elements of $U - z$ with respect to the digraph $D' - z$. If $s$ is the only node of $D'$ from which there is an edge entering $z$, then by putting $z$ at the beginning of the existing ordering we are done.

Suppose now that there is a node in $U$ from which there is an edge of $D'$ entering $z$ and let $u_i$ denote the earliest one of these nodes in the given ordering of $U - z$. Insert $z$ between $u_i$ and $u_{i+1}$. The resulting ordering of $U$ satisfies the requirements of the lemma since the only new edge going forward created by the insertion of $z$ is $u_i z$. □

As the theorem is obvious for $k = 1$, we may assume that $k \geq 2$. Apply the lemma to the subgraph $D' = (U + s_k, A')$ of $D$ induced by $U + s_k$. Let $u_1, ..., u_p$ be the ordering of the elements of $U$ ensured by the lemma and let $F_k$ denote the arborescence corresponding to it. Let $D''$ be a subgraph of $D$ obtained by deleting node $s_k$ and the edges of $F_k$. By induction, $D''$ admits the requested independent arborescences $F_1, ..., F_{k-1}$ for $i = 1, ..., k-1$. Since all the edges of these arborescences go backward in the ordering $u_1, ..., u_p$ while all the edges of $F_k$ go forward, it follows that the unique $s_k u$-path in $F_k$ and the unique $s_i u$-path of $F_i$ $(i = 1, ..., k-1)$ have the only node $u$ in common for every $u \in U$. □

**Theorem 5.3** (A. Huck). *Let $D = (V, A)$ be a simple acyclic digraph with a designated root node $r_0$. There are $k$ independent spanning arborescences of root $r_0$ if and only $D$ is rooted $k$-node-connected.*

*Proof.* The necessity is evident from the definition of independence. For the sufficiency, put a new node $v_e$ on each arc $e = r_0 v$, split $r_0$ into $k$ nodes $r_1, ..., r_k$ and replace each arc $r_0 v_e$ leaving $r_0$ by arcs $r_1 v_e, ..., r_k v_e$. Clearly, the digraph thus obtained contains independent $r_1, ..., r_k$-arborescences if and only if there are $k$ independent arborescences of root $r_0$ in $D$. Moreover, $D$ is rooted $k$-node-connected if and only if there exist openly disjoint $r_1 v, ..., r_k v$ paths for each node $v \neq r_1, ..., r_k$ in the resulting graph. Apply Theorem 5.1 to the new digraph. □

In [25] Huck showed that the assumption that $D$ is acyclic cannot be left out for any $k \geq 3$. For $k = 2$, however, R.W. Whitty [40] proved that it can.

**Theorem 5.4** (R.W. Whitty). *Let $D = (V, A)$ be a digraph with a root-node $r_0 \in V$. There are two independent spanning arborescences of root $r_0$ if and only if $D$ is rooted 2-node-connected.*

We may assume that $\varrho(r_0) = 0$. We prove another statement which can be seen easily to be equivalent with Theorem 5.4 by the above construction.

**Theorem 5.5.** *Let $D = (V + r_1 + r_2, A)$ be a digraph with $\varrho(r_i) = 0$ for $i = 1, 2$. Suppose that there exist openly disjoint $r_1v$ and $r_2v$ paths for each $v \in V$. The there exist two independent arborescences $F_1, F_2$ spanning $V$ and rooted at $r_1$ and $r_2$, respectively.*

*Proof.* Similarly to Huck's proof, we define a special ordering of the nodes.

**Lemma 5.6.** *There is an ordering $r_1 = v_0, v_1, ..., v_{n+1} = r_2$ of the nodes so that, for each node $v_i \in V - r_1 - r_2$, there is an edge $v_hv_i$ with $h < i$ and an edge $v_iv_j$ with $i < j$.*

*Proof.* We prove the lemma by using induction on $|V| = n$. The case when $n = 1$ is obvious. Assume that the statement is true for $n - 1$ and take a graph with $|V| = n$ satisfying the conditions. Consider an $r_1$-arborescence $F$ spanning $V$ and let $u$ be a neighbor of $s_2$ from which there is no directed path in $F$ to another neighbor of $s_2$. We will show that if we shrink $s_2$ and $u$, then there still exist openly disjoint $s_1v$ and $s_2'v$ paths for each $v \in V - u$, where $s_2'$ denotes the shrunk node.

Otherwise, there is a subset $X$ of $V - u$ and a node $y \in (V - u) + s_1 + s_2'$ such that all the $s_1 - X$ and $s_2' - X$ paths go through $y$. Then $y$ must be $s_2'$. However, this implies that $s_2$ has a neighbor in $X$ in the original graph, contradicting to the choice of $u$ since there is a directed path in $F$ from $u$ to each member of $X$. $\square$

The lemma implies that there is an ordering of the nodes in which both the set of edges going forward and the set of edges going backward determine two proper arborescences $F_1$ and $F_2$ of $D$ spanning $V$, and these two arborescences are independent since in $D$ any $r_1v$-path and $r_2v$-path share only the terminal node $v$. $\square$

In an undirected graph we call two trees $r$-**independent** for some $r \in V$ if the unique $rv$ paths in the trees are openly disjoint for every node $v \in V$. It has been verified in [3], [27] and [28] that the following theorem holds for $k = 2, 3$.

**Theorem 5.7.** *Let $G$ be a $k$-connected undirected graph for some $k \geq 1$ and let $r_0 \in V(G)$. Then there exist $k$ $r_0$-independent spanning trees in $G$.* $\square$

The case when $k = 4$ was verified in [4] but for $k \geq 5$ the problem is still open. However, in [26] and [24] Huck verified the theorem for planar graphs for each $k \geq 1$, i.e. we have the following.

**Theorem 5.8.** *Let $G$ be a rooted $k$-connected undirected planar multigraph and let $r_0 \in V(G)$. Then there exist $k$ $r_0$-independent spanning trees in $G$.* □

Planar multigraphs proved to be tractable even in the directed case. Moreover, in [23] Huck proved a strengthening of these theorems where the connectivity-condition are weakened to root-connectivity. By summarizing the results of Whitty and Huck we have the following theorem.

**Theorem 5.9.**

*(i) Let $D$ be a rooted $k$-connected directed multigraph for some root $r_0 \in V(G)$ and $k \in \{1, 2\} \cup \{6, 7, 8, ...\}$ such that $D$ is planar if $k \geq 6$. Then $D$ contains $k$ independent spanning arborescences of root $r_0$.*

*(ii) Let $G$ be a rooted $k$-connected undirected multigraph for some root $r_0 \in V(G)$ and $k \geq 1$ such that $G$ is planar if $k \geq 4$. Then $G$ contains $k$ $r_0$-independent spanning trees.* □

Although the directed case remains open for $k = 3, 4, 5$, maximal planar graphs mean an interesting special case. We call a planar graph $G = (V, E)$ (with a fixed embedding in the plane) maximal if each of its faces is bounded by a triangle. Let $r_1, r_2, r_3$ denote the three nodes of the infinite face. These are called **roots** while the other nodes are the **inner nodes**. The sets of roots and of inner nodes are denoted by $R$ and $U$, respectively. Since any subset of $j \geq 3$ nodes induces at most $3j - 6$ edges, the Orientation lemma (Theorem 2.4) easily implies the following.

**Theorem 5.10.** *Let $G = (V, E)$ be a maximal planar graph. Let $G' = (V, E')$ be the graph arising from $G$ by deleting the three edges of its infinite face. Then $G'$ has an orientation so that $\varrho(r_i) = 0$ and $\varrho(v) = 3$ for every other node.*

Consider now the three edges $e_1 = u_1v,\ e_2 = u_2v,\ e_3 = u_3v$ entering an inner node $v$. We say that an edge $e = vz$ leaving $v$ is the **cyclic successor** of $e_i$ (and that $e_i$ is the **cyclic predecessor** of $e$) if $e$ and $e_i$ are separated by $e_{i-1}$ and $e_{i+1}$ in the cyclic order of the edges at $v$ defined by the plane embedding where the indices are meant modulo 3.

**Proposition 5.11.** *There is no directed circuits in which each edge is the cyclic successor of its preceding edge.*

*Proof.* Suppose indirectly that $C$ is such a di-circuit. Let $H$ be the subgraph of $D$ induced by $C$ and its interior. We will get a contradiction by double-counting the arcs in $H$. Let $c = |C|$ and let $t$ and $l$ denote the number of nodes and arcs induced in $H$. We know that $H$ is triangular except the outer face which is bounded by $C$. With an extra node and $c$ additional edges we can triangulate the outer face, hence $l = 3(t+1) - 6 - c = 3t - c - 3$. Clearly, $\varrho_H(v) = 3$ for every node in $H - C$, $\varrho_H(v) = 2$ for each $v \in C$ except at most one $v \in C$, for which $\varrho_H(v) \geq 1$. But that would mean $l \geq 3t - c - 1$, a contradiction. □

By starting at any edge $e$ and going back along cyclic predecessors, one arrives at a node $r_i$, called the root-node of $e$.

**Proposition 5.12.** *Any two arcs with common head have different root-nodes.*

*Proof.* Suppose indirectly that there are two arcs $e$ and $f$ with common head $w$ and root-node $r_i$. Let $u$ be the first node which is reached both from $e$ and $f$ when going back along cyclic predecessors. Then we have two openly disjoint paths from $u$ to $w$ whose union is denoted by $C$. $H$ denotes the subgraph of $D$ induced by $C$ and its interior. Let $c = |C|$ and let $t$ and $l$ denote the number of nodes and arcs induced in $H$. We know that $H$ is triangular except the outer face which is bounded by $C$. With an extra node and $c$ additional edges we can triangulate the outer face, hence $l = 3(t+1) - 6 - c = 3t - c - 3$. Clearly, $\varrho_H(v) = 3$ for each node in $H - C$, $\varrho_H(v) = 2$ for each $v \in C$ except $u$ and $w$ for which $\varrho_H(w) \geq 2$ and $\varrho_H(u) \geq 0$. This implies $l \geq 3t - c - 2$, a contradiction again. □

It follows that $F_i = \{e \in E :\ r_i \text{ is the root node of } e\}$ is an arborescence of root $r_i$ spanning each inner node. Moreover, $F_1, F_2, F_3$ form a partition of the edge-set.

**Theorem 5.13.** *The arborescences $F_1, F_2, F_3$ defined above are independent.*

*Proof.* Suppose that there is a node $v \neq r_1, r_2, r_3$ for which two of the unique paths $r_i v$ are not openly disjoint. Let $u$ denote the first common node appearing on these paths while going back from $v$ along them. Then we have two openly disjoint paths from $u$ to $v$ again whose union is denoted by $C$. By double-counting the arcs in the subgraph $H$ induced by $C$ and its interior, we get the same contradiction as in Proposition 5.12. □

The trees corresponding to the arborescences $F_i$ in the underlying undirected graph have some specific properties. It was G.R. Kampen [30] who proved the existence of these undirected trees by a different method. Later W. Schnyder [33] used a similar approach to prove the independence of these trees and to construct a straight-line embedding of the graph into a small grid.

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