Submodular Function Minimization and Maximization in Discrete Convex Analysis

By

Kazuo MUROTA*

Abstract

This paper sheds a new light on submodular function minimization and maximization from the viewpoint of discrete convex analysis. $L^\natural$-convex functions and $M^\natural$-concave functions constitute subclasses of submodular functions on an integer interval. Whereas $L^\natural$-convex functions can be minimized efficiently on the basis of submodular (set) function minimization algorithms, $M^\natural$-concave functions are identified as a computationally tractable subclass for maximization.

§ 1. Introduction

A function $f : S \to \mathbf{R}$ defined on a set S of integer vectors is called *submodular* if

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) \qquad (\forall x, y \in S),$$

where $x \vee y$ and $x \wedge y$ denote, respectively, the vectors of componentwise maxima and minima, and S is assumed to be closed under the operations of $\vee$ and $\wedge$. In the special case of $S = \{0, 1\}^n$ such f can be identified with a submodular set function.

This paper sheds a new light on submodular function minimization and maximization from the viewpoint of discrete convex analysis. Discrete convex analysis [27, 29, 30, 32] is a general theoretical framework for solvable discrete optimization problems by means of a combination of the ideas in continuous optimization and combinatorial optimization. The theory extends the direction set forth by J. Edmonds, A.

*Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo, Tokyo 113-8656, Japan.

Frank, S. Fujishige, and L. Lovász [6, 10, 11, 22]; see also [12, Chapter VII]. The reader is referred to [12, 41] for submodular function theory.

The objective of this paper is to explain the following:

1. $L^\natural$-convex functions and $M^\natural$-concave functions constitute subclasses of submodular functions on an integer interval.

2. $L^\natural$-convex functions can be minimized efficiently on the basis of submodular (set) function minimization algorithms.

3. Functions represented as the sum of a small number of $M^\natural$-concave functions form computationally tractable subclasses for maximization.

We denote the set of all real numbers by $\mathbf{R}$, and put $\overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$ and $\underline{\mathbf{R}} = \mathbf{R} \cup \{-\infty\}$. Similarly, we denote the set of all integers by $\mathbf{Z}$, and put $\overline{\mathbf{Z}} = \mathbf{Z} \cup \{+\infty\}$ and $\underline{\mathbf{Z}} = \mathbf{Z} \cup \{-\infty\}$.

§ 2.   Discrete Convex Functions

In this section we describe two convexity concepts defined for functions in discrete variables $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$. We denote its effective domain and the set of its minimizers by

$$\mathrm{dom}_{\mathbf{Z}} f = \{x \in \mathbf{Z}^n \mid f(x) \in \mathbf{R}\},$$
$$\mathrm{argmin}_{\mathbf{Z}} f = \{x \in \mathbf{Z}^n \mid f(x) \leq f(y) \ (\forall y \in \mathbf{Z}^n)\}.$$

For any function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ that can be qualified as a "discrete convex function" it would be natural to expect the following properties:

1. Function f is extensible to a convex function on $\mathbf{R}^n$.

2. Local optimality (or minimality) guarantees global optimality.

3. Duality theorems such as min-max relation and separation hold.

It should be clear that $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is said to be *convex-extensible* if there exists a convex function $\overline{f} : \mathbf{R}^n \to \overline{\mathbf{R}}$ such that $\overline{f}(x) = f(x)$ for all $x \in \mathbf{Z}^n$. Similarly, $f : \mathbf{Z}^n \to \underline{\mathbf{R}}$ is said to be *concave-extensible* if there exists a concave function $\overline{f} : \mathbf{R}^n \to \underline{\mathbf{R}}$ such that $\overline{f}(x) = f(x)$ for all $x \in \mathbf{Z}^n$.

§ 2.1.   L-convex functions

The concept of L-convex functions [13, 27] is explained here by featuring an equivalent variant thereof, called $L^\natural$-convex functions ("$L^\natural$" should be read "el natural").

First recall that a function $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is called *submodular* if

$$(2.1) \qquad g(p) + g(q) \geq g(p \vee q) + g(p \wedge q) \qquad (p, q \in \mathbf{Z}^n),$$

where $p \vee q$ and $p \wedge q$ denote the vectors of componentwise maxima and minima, respectively, i.e.,

$$(2.2) \qquad (p \vee q)_i = \max(p_i, q_i), \qquad (p \wedge q)_i = \min(p_i, q_i).$$

As a strengthening of submodularity we consider *translation submodularity*:

$$(2.3) \qquad g(p) + g(q) \geq g((p - \alpha \mathbf{1}) \vee q) + g(p \wedge (q + \alpha \mathbf{1})) \quad (\alpha \in \mathbf{Z}_+, \; p, q \in \mathbf{Z}^n),$$

where $\mathbf{1} = (1, 1, \ldots, 1)$ and $\mathbf{Z}_+$ denotes the set of nonnegative integers. Then we say that a function $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is $L^\natural$-*convex* if it satisfies (2.3) and $\mathrm{dom}_{\mathbf{Z}} g \neq \emptyset$.

With this definition we can actually have the following expected statement. Note here that submodularity (2.1) alone does not imply convex-extensibility.

Theorem 2.1.   *An $L^\natural$-convex function $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is convex-extensible.*

An *L-convex function* is defined as a function $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$ that satisfies submodularity (2.1) and

$$(2.4) \qquad g(p + \mathbf{1}) = g(p) + r \qquad (p \in \mathbf{Z}^n)$$

for some $r \in \mathbf{R}$ (which is independent of p), where $\mathrm{dom}_{\mathbf{Z}} g \neq \emptyset$ is assumed. It is known that g is L-convex if and only if it is an $L^\natural$-convex function that satisfies (2.4). Thus L-convex functions form a subclass of $L^\natural$-convex functions. However, they are essentially the same, in that $L^\natural$-convex functions in n variables can be identified, up to the constant r in (2.4), with L-convex functions in $n + 1$ variables; see [30, Section 7.1] for details.

Remark.   For a function $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$ in discrete variables, the translation submodularity (2.3) is known to be equivalent to *discrete midpoint convexity* of [7]:

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p + q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p + q}{2} \right\rfloor\right) \qquad (p, q \in \mathbf{Z}^n),$$

where, for $z \in \mathbf{R}$ in general, $\lceil z \rceil$ denotes the smallest integer not smaller than z (rounding-up to the nearest integer) and $\lfloor z \rfloor$ the largest integer not larger than z (rounding-down to the nearest integer), and this operation is extended to a vector by componentwise applications.   ■

L-convexity can also be defined for functions in continuous variables [34, 35, 36]. A convex function $g : \mathbf{R}^n \to \overline{\mathbf{R}}$ is called $L^\natural$-*convex* if

$$(2.5) \qquad g(p) + g(q) \geq g((p - \alpha \mathbf{1}) \vee q) + g(p \wedge (q + \alpha \mathbf{1})) \quad (\alpha \in \mathbf{R}_+, \; p, q \in \mathbf{R}^n),$$

where $\mathbf{R}_+$ denotes the set of nonnegative reals, and the effective domain of g,

$$\mathrm{dom}_{\mathbf{R}}g = \{x \in \mathbf{R}^n \mid g(x) \in \mathbf{R}\},$$

is assumed to be nonempty. An *L-convex function* (in continuous variables) is defined as an $\mathrm{L}^\natural$-convex function $g : \mathbf{R}^n \to \overline{\mathbf{R}}$ that satisfies

$$(2.6) \qquad\qquad g(p + \alpha\mathbf{1}) = g(p) + \alpha r \qquad (\alpha \in \mathbf{R}, \ p \in \mathbf{R}^n)$$

for some $r \in \mathbf{R}$ (which is independent of p and α). L-convex functions and $\mathrm{L}^\natural$-convex functions are essentially the same, in that $\mathrm{L}^\natural$-convex functions in n variables can be identified, up to the constant r in (2.6), with L-convex functions in $n + 1$ variables.

§ 2.2.   M-convex functions

Another kind of discrete convex functions, called M-convex functions, [26, 27, 33], is explained here by featuring an equivalent variant thereof, called $\mathrm{M}^\natural$-convex functions ("$\mathrm{M}^\natural$" should be read "em natural").

The characteristic vector of a subset X of $V = \{1, \ldots, n\}$ is denoted by $\chi_X \in \{0,1\}^n$. For $i \in V$, we write χ_i for $\chi_{\{i\}}$, which is the ith unit vector, and $\chi_0 = \mathbf{0}$ (zero vector). For a vector $z \in \mathbf{R}^n$, we define the *positive* and *negative supports* of z as

$$\mathrm{supp}^+(z) = \{i \mid z_i > 0\}, \qquad \mathrm{supp}^-(z) = \{j \mid z_j < 0\}.$$

For a function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ in discrete variables we consider the following condition: For any $x, y \in \mathrm{dom}_{\mathbf{Z}}f$ and any $i \in \mathrm{supp}^+(x - y)$, there exists $j \in \mathrm{supp}^-(x - y) \cup \{0\}$ such that

$$(2.7) \qquad\qquad f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j),$$

which is referred to as the *exchange property*. A function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ having this exchange property is called an $\mathrm{M}^\natural$-*convex function*, where $\mathrm{dom}_{\mathbf{Z}}f \neq \emptyset$ is assumed. The effective domain $\mathrm{dom}_{\mathbf{Z}}f$ of an $\mathrm{M}^\natural$-convex function f is a g-polymatroid. Naturally, a function $f : \mathbf{Z}^n \to \underline{\mathbf{R}}$ is called an $\mathrm{M}^\natural$-*concave function* if $-f$ is an $\mathrm{M}^\natural$-convex function.

With this definition we can obtain the following statement, comparable to Theorem 2.1.

Theorem 2.2.   *An $\mathrm{M}^\natural$-convex function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is convex-extensible.*

An *M-convex function* is defined as an $\mathrm{M}^\natural$-convex function f that satisfies (2.7) with $j \in \mathrm{supp}^-(x - y)$ (i.e., $j \neq 0$). This is equivalent to saying that f is an M-convex function if and only if it is $\mathrm{M}^\natural$-convex and $\mathrm{dom}_{\mathbf{Z}}f \subseteq \{x \in \mathbf{Z}^n \mid \sum_{i=1}^n x_i = r\}$ for some $r \in \mathbf{Z}$. Thus M-convex functions form a subclass of $\mathrm{M}^\natural$-convex functions. However, they are essentially the same, in that $\mathrm{M}^\natural$-convex functions in n variables can be obtained as projections of M-convex functions in $n + 1$ variables; see [30, Section 6.1] for details.

Remark.   *Valuated matroids*, introduced in [4, 5], can be identified with M-concave set functions. To be more specific, a set function $\omega : 2^V \to \underline{\mathbf{R}}$ is a valuated matroid if and only if the function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ defined by $f(\chi_X) = -\omega(X)$ for $X \subseteq V$ with $\mathrm{dom}_{\mathbf{Z}} f \subseteq \{0,1\}^n$, is an M-convex function. See [24, 25] and [28, Chapter 5] for more on valuated matroids.   ■

M-convexity can also be defined for functions in continuous variables [34, 35, 36]. We say that a convex function $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ is $M^\natural$-*convex* if, for any $x, y \in \mathrm{dom}_{\mathbf{R}} f$ and any $i \in \mathrm{supp}^+(x - y)$, there exist $j \in \mathrm{supp}^-(x - y) \cup \{0\}$ and a positive real number α_0 such that

$$(2.8) \qquad f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j))$$

for all $\alpha \in \mathbf{R}$ with $0 \leq \alpha \leq \alpha_0$.

An *M-convex function* (in continuous variables) is defined as an $M^\natural$-convex function $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ that satisfies (2.8) with $j \in \mathrm{supp}^-(x - y)$ (i.e., $j \neq 0$). This is equivalent to saying that f is M-convex if and only if it is $M^\natural$-convex and $\mathrm{dom}_{\mathbf{R}} f \subseteq \{x \in \mathbf{R}^n \mid \sum_{i=1}^n x_i = r\}$ for some $r \in \mathbf{R}$. M-convex functions and $M^\natural$-convex functions are essentially the same, in that $M^\natural$-convex functions in n variables can be obtained as projections of M-convex functions in $n + 1$ variables.

§2.3.   Conjugacy

Conjugacy under the Legendre transformation is one of the most appealing facts in convex analysis. In discrete convex analysis, the discrete Legendre transformation gives a one-to-one correspondence between L-convex functions and M-convex functions.

For a function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ with $\mathrm{dom}_{\mathbf{Z}} f \neq \emptyset$, the discrete version of the Legendre transformation is defined as

$$(2.9) \qquad f^\bullet(p) = \sup\{\langle p, x \rangle - f(x) \mid x \in \mathbf{Z}^n\} \qquad (p \in \mathbf{R}^n),$$

where $\langle p, x \rangle = \sum_{i=1}^n p_i x_i$ is the inner product of two vectors $p = (p_i)$ and $x = (x_i)$. We call (2.9) the *discrete Legendre(–Fenchel) transformation*, and the function $f^\bullet : \mathbf{R}^n \to \overline{\mathbf{R}}$ the *conjugate* of f.

Theorem 2.3.   *For an $M^\natural$-convex function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$, the conjugate function $f^\bullet : \mathbf{R}^n \to \overline{\mathbf{R}}$ is a (locally polyhedral) $L^\natural$-convex function. For an $L^\natural$-convex function $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$, the conjugate function $g^\bullet : \mathbf{R}^n \to \overline{\mathbf{R}}$ is a (locally polyhedral) $M^\natural$-convex function. Similarly for M-convex and L-convex functions.*

For an integer-valued function $f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$, the conjugate function $f^\bullet(p)$ is integer for an integer vector p. Hence (2.9) with $p \in \mathbf{Z}^n$ defines a transformation of $f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$ to $f^\bullet : \mathbf{Z}^n \to \overline{\mathbf{Z}}$. We refer to (2.9) with $p \in \mathbf{Z}^n$ as $(2.9)_{\mathbf{Z}}$.

The conjugacy theorem for discrete M-convex and L-convex functions reads as follows.

Theorem 2.4 ([27]).   *The discrete Legendre transformation* $(2.9)_{\mathbf{Z}}$ *gives a one-to-one correspondence between the classes of all integer-valued* $M^{\natural}$-*convex functions and* $L^{\natural}$-*convex functions in discrete variables. Similarly for M-convex and L-convex functions.*

It should be clear that the first statement above means that, for an integer-valued $M^{\natural}$-convex function $f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$, the function $f^{\bullet}$ in $(2.9)_{\mathbf{Z}}$ is an integer-valued $L^{\natural}$-convex function and $f^{\bullet\bullet} = f$ holds, where $f^{\bullet\bullet}$ is a short-hand notation for $(f^{\bullet})^{\bullet}$ using the discrete Legendre transformation $(2.9)_{\mathbf{Z}}$, and similarly when f is $L^{\natural}$-convex.

The conjugacy between M-convex and L-convex functions is also valid for functions in continuous variables. For a function $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ with $\mathrm{dom}_{\mathbf{R}} f \neq \emptyset$, the *conjugate* $f^{\bullet} : \mathbf{R}^n \to \overline{\mathbf{R}}$ is defined by

$$(2.10) \qquad\qquad f^{\bullet}(p) = \sup\{\langle p, x \rangle - f(x) \mid x \in \mathbf{R}^n\} \qquad (p \in \mathbf{R}^n).$$

Theorem 2.5 ([34]).   *The Legendre transformation* (2.10) *gives a one-to-one correspondence between the classes of all polyhedral* $M^{\natural}$-*convex functions and* $L^{\natural}$-*convex functions. Similarly for M-convex and L-convex functions.*

Theorem 2.6 ([35]).   *The Legendre transformation* (2.10) *gives a one-to-one correspondence between the classes of all closed proper* $M^{\natural}$-*convex functions and* $L^{\natural}$-*convex functions. Similarly for M-convex and L-convex functions.*

§3.   Convexity, Concavity, and Submodularity

In this section we consider functions $f : [a, b]_{\mathbf{Z}} \to \mathbf{R}$ defined on an integer interval $[a, b]_{\mathbf{Z}} = \{x \in \mathbf{Z}^n \mid a \leq x \leq b\}$, where $a \in \underline{\mathbf{Z}}^n$ and $b \in \overline{\mathbf{Z}}^n$. We are particularly concerned with submodular functions, which satisfy, by definition, the following inequality:

$$(3.1) \qquad\qquad f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) \qquad (x, y \in [a, b]_{\mathbf{Z}}).$$

Both $L^{\natural}$-convex functions and $M^{\natural}$-concave functions constitute subclasses of submodular functions on $[a, b]_{\mathbf{Z}}$.

Theorem 3.1.
(1) *An* $L^{\natural}$-*convex function* $f : [a, b]_{\mathbf{Z}} \to \mathbf{R}$ *is submodular.*
(2) *An* $M^{\natural}$-*concave function* $f : [a, b]_{\mathbf{Z}} \to \mathbf{R}$ *is submodular.*

Proof. (1) This follows from the fact that translation submodularity (2.3) with $\alpha = 0$ coincides with submodularity (2.1).

(2) For $x \in [a,b]_{\mathbf{Z}}$, the exchange property (2.7) for $(x + \chi_i + \chi_j, x)$ yields

$$f(x + \chi_i) + f(x + \chi_j) \geq f(x + \chi_i + \chi_j) + f(x) \qquad (i \neq j).$$

This is the so-called local submodularity, from which (3.1) follows by induction on $||x - y||_1$. See the proof of Theorem 6.19 in [30] for detail. $\qquad \square$

$L^{\natural}$-convex functions are convex-extensible submodular functions by Theorem 2.1 and Theorem 3.1 (1). By Theorem 2.2 and Theorem 3.1 (2), on the other hand, $M^{\natural}$-concave functions are concave-extensible submodular functions. Recall that submodularity (3.1) alone does not imply convex-extensibility nor concave-extensibility. It is also mentioned that there exist convex-extensible submodular functions that are not $L^{\natural}$-convex, and that there exist concave-extensible submodular functions that are not $M^{\natural}$-concave.

In the case of $a = \mathbf{0}$ and $b = \mathbf{1}$ we have $f : [\mathbf{0}, \mathbf{1}]_{\mathbf{Z}} \to \mathbf{R}$, with which a set function $\rho : 2^V \to \mathbf{R}$ is associated naturally by

$$(3.2) \qquad\qquad \rho(X) = f(\chi_X) \qquad (X \subseteq V).$$

Then the submodularity (3.1) is translated to

$$(3.3) \qquad \rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y) \qquad (X, Y \subseteq V),$$

which is the submodular inequality for set functions. We say a set function $\rho : 2^V \to \mathbf{R}$ is $L^{\natural}$-convex if the function f associated with ρ by (3.2) is $L^{\natural}$-convex on $[\mathbf{0}, \mathbf{1}]_{\mathbf{Z}}$. Similarly for an $M^{\natural}$-concave set function.

Theorem 3.2.
(1) *An $L^{\natural}$-convex set function is submodular, and the converse is also true.*
(2) *An $M^{\natural}$-concave set function is submodular.*

Not every submodular set function is $M^{\natural}$-concave, as the following example shows. Thus $M^{\natural}$-concave set functions form a proper subclass of submodular set functions.

Example 3.3.   This is an example of a submodular set function that is not $M^{\natural}$-concave. Let ρ be defined on $V = \{1, 2, 3\}$ as $\rho(\emptyset) = 0$, $\rho(\{2,3\}) = 2$, $\rho(\{1\}) = \rho(\{2\}) = \rho(\{3\}) = \rho(\{1,2\}) = \rho(\{1,3\}) = \rho(\{1,2,3\}) = 1$. The exchange property (2.7) fails for the associated f with $x = \chi_{\{2,3\}}$, $y = \chi_{\{1\}}$ and $i = 2$. $\qquad \blacksquare$

Theorem 3.1 carries over to functions in continuous variables, as follows, where $[a,b]_{\mathbf{R}} = \{x \in \mathbf{R}^n \mid a \leq x \leq b\}$.

Theorem 3.4.

(1) *An $L^\natural$-convex function $f : [a, b]_\mathbf{R} \to \mathbf{R}$ is submodular.*

(2) *An $M^\natural$-concave function $f : [a, b]_\mathbf{R} \to \mathbf{R}$ is submodular.*

§4. Submodular Function Minimization

Minimization of submodular functions is one of the most fundamental problems in discrete optimization. General submodular functions on integer lattices are computationally tractable for minimization, and $L^\natural$-convex functions form a subclass of submodular functions that admits natural approaches such as the descent method and the scaling technique.

§4.1. Submodular functions on integer lattices

Importance of the submodular set function minimization problem seems to have been recognized around 1970 by Edmonds [6] and others, and combinatorial strongly polynomial algorithms were found in 1999 by Iwata–Fleischer–Fujishige [16] and Schrijver [39]. The (presently) fastest algorithm is due to Orlin [38]. It is worth noting that these algorithms can cope with submodular set functions $\rho : 2^V \to \overline{\mathbf{R}}$ defined effectively on a ring family. See surveys [15, 23] for details.

Let $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$ be a submodular function, which implies that $\mathrm{dom}_\mathbf{Z} g$ is a distributive sublattice of $\mathbf{Z}^n$ with respect to $(\vee, \wedge)$ of (2.2). It is assumed that $\mathrm{dom}_\mathbf{Z} g$ is a finite set with ℓ_1-size $K_1 = \max\{\|p - q\|_1 \mid p, q \in \mathrm{dom}_\mathbf{Z} g\}$. By Birkhoff's representation theorem the distributive lattice $(\mathrm{dom}_\mathbf{Z} g, \vee, \wedge)$ can be represented in the form of a ring family on some underlying set $\tilde{V}$, where the size of $\tilde{V}$ is equal to the length of a maximal chain of $\mathrm{dom}_\mathbf{Z} g$. Thus, with an appropriate representation of $(\mathrm{dom}_\mathbf{Z} g, \vee, \wedge)$, the function g can be minimized using a submodular function minimization algorithm, where the time complexity is a polynomial in n and $|\tilde{V}|$. Note that $|\tilde{V}| = K_1$.

§4.2. $L^\natural$-convex functions

$L^\natural$-convex functions form a subclass of submodular functions that admits a local characterization of global minimality and a natural steepest descent algorithm.

Theorem 4.1 ([30, Theorem 7.14]). *Let $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$ be an $L^\natural$-convex function. A point $p \in \mathrm{dom}_\mathbf{Z} g$ is a global minimum of g if and only if it is a local minimum in the sense that*

$$(4.1) \qquad g(p) \le \min\{g(p - q), \ g(p + q)\} \qquad (\forall q \in \{0, 1\}^n).$$

Steepest descent algorithm for L$^\natural$-convex function g

S0: Find a vector $p \in \text{dom}_{\mathbf{Z}}g$.

S1: Find $\varepsilon \in \{1, -1\}$ and $X \subseteq V$ that minimize $g(p + \varepsilon\chi_X)$.

S2: If $g(p) \le g(p + \varepsilon\chi_X)$, then stop ($p$ is a minimizer of g).

S3: Set $p := p + \varepsilon\chi_X$ and go to S1.

Step S1 amounts to minimizing a pair of submodular set functions

$$\rho_p^+(X) = g(p + \chi_X) - g(p), \qquad \rho_p^-(X) = g(p - \chi_X) - g(p).$$

This can be done by using the existing algorithms for submodular set function minimization. Assuming that a minimizer of a submodular set function can be computed with $\text{O}(\sigma(n))$ function evaluations and $\text{O}(\tau(n))$ arithmetic operations[1], and denoting by F an upper bound on the time to evaluate g, we can perform Step S1 in $\text{O}(\sigma(n)F + \tau(n))$ time.

As to the number of iterations of the above algorithm a sharp upper bound has been shown by Kolmogorov–Shioura [19] as an improvement upon [31]. We denote by K_∞ the ℓ_∞-size of the effective domain of g, i.e.,

$$K_\infty = \max\{\|p - q\|_\infty \mid p, q \in \text{dom}_{\mathbf{Z}}g\}.$$

Theorem 4.2 ([19]).   *The number of iterations of the steepest descent algorithm for an L$^\natural$-convex function is bounded by $2K_\infty + 1$.*

The steepest descent algorithm can be made more efficient with the aid of a scaling technique. Efficiency of the resulting algorithm is guaranteed by the complexity bound in Theorem 4.2 and a proximity theorem (Theorem 4.3 below).

Steepest descent-scaling algorithm for L$^\natural$-convex function g

S0: Find a vector $b \in \text{dom}_{\mathbf{Z}}g$, and set $p^* := \mathbf{0}$, $\alpha := 2^{\lceil \log_2 K_\infty \rceil}$.

S1: If $\alpha < 1$, then stop ($b + p^*$ is a minimizer of g).

S2: Find an integer vector p that minimizes $g(\alpha p + b)$ in the range of
$2p^* - n\mathbf{1} \le p \le 2p^* + n\mathbf{1}$.

S3: Set $p^* := p$, $\alpha := \alpha/2$, and go to S1.

Note that the function $\tilde{g}(p) = g(\alpha p + b)$ is an L$^\natural$-convex function. By Theorem 4.3 below, there exists a minimizer p of $\tilde{g}$ satisfying $2p^* - n\mathbf{1} \le p \le 2p^* + n\mathbf{1}$. Then, by Theorem 4.2, the minimizer in Step S2 can be found in $\text{O}(n(\sigma(n)F + \tau(n)))$ time by the steepest descent algorithm. The number of executions of Step S2 is bounded by $\lceil \log_2 K_\infty \rceil$. Thus the above algorithm finds a minimizer of g in $\text{O}(n(\sigma(n)F + \tau(n))\lceil \log_2 K_\infty \rceil)$ time.

---

[1] $\sigma(n) = n^5$ and $\tau(n) = n^6$ by Orlin's algorithm [38].

Theorem 4.3 ([17], [30, Theorem 7.18]).   *Let $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$ be an $L^\natural$-convex function, α a positive integer, and $p \in \mathrm{dom}_\mathbf{Z} g$. If*

$$g(p) \le \min\{g(p - \alpha q), \ g(p + \alpha q)\} \qquad (\forall q \in \{0, 1\}^n),$$

then $\mathrm{argmin}_\mathbf{Z} g \ne \emptyset$ *and there exists* $p^* \in \mathrm{argmin}_\mathbf{Z} g$ *such that*

$$p - n(\alpha - 1)\mathbf{1} \le p^* \le p + n(\alpha - 1)\mathbf{1}.$$

§5.   Submodular Function Maximization

Maximization of a submodular set function is a difficult task in general. Many NP-hard problems can be reduced to this problem. Also known is that no polynomial algorithm exists in the ordinary oracle model (and this statement is independent of the P$\ne$NP conjecture) [18, 21, 22]. For approximate maximization under matroid constraints the performance bounds of greedy or ascent type algorithms were analyzed in [3, 8, 37] and, recently, a pipage rounding algorithm [1] has been designed for a subclass of submodular functions in [2], which is extended in [42] to general nondecreasing submodular functions with the aid of randomization.

$M^\natural$-concave functions on $\{0, 1\}$-vectors form a subclass of submodular set functions that are algorithmically tractable for maximization. This fact, being compatible with our general understanding that concave functions are easy to maximize, explains why certain submodular functions treated in the literature are easier to maximize. To be specific, we have the following.

1. The greedy algorithm can be generalized for maximization of a single $M^\natural$-concave set function.

2. The matroid intersection algorithm can be generalized for maximization of a sum of two $M^\natural$-concave set functions.

3. The pipage rounding algorithm can be generalized for approximate maximization of a sum of nondecreasing $M^\natural$-concave set functions under a matroid constraint.

Note that a sum of $M^\natural$-concave set functions is not necessarily $M^\natural$-concave, though it is submodular. It is also mentioned that maximization of a sum of three $M^\natural$-concave set functions is NP-hard, since it includes the three-matroid intersection problem as a special case.

In the following we dwell on the above three points by focusing on set functions, although much of the argument carries over to functions on integers.

§5.1.   M$^\natural$-concave set functions

A set function $\mu : 2^V \to \underline{\mathbf{R}}$ is said to be M$^\natural$-concave[2] if, for any $X, Y \subseteq V$ and $i \in X \setminus Y$, we have

$$(5.1) \qquad \mu(X) + \mu(Y) \le \mu(X - i) + \mu(Y + i)$$

or else

$$(5.2) \qquad \mu(X) + \mu(Y) \le \mu(X - i + j) + \mu(Y + i - j)$$

for some $j \in Y \setminus X$. Here we use short-hand notations $X - i = X \setminus \{i\}$, $Y + i = Y \cup \{i\}$, $X - i + j = (X \setminus \{i\}) \cup \{j\}$, $Y + i - j = (Y \cup \{i\}) \setminus \{j\}$. We refer to this property, consisting of (5.1) and (5.2) above, as the *exchange property*. The effective domain of μ, denoted dom μ, is assumed to be nonempty. Note that dom μ is a g-matroid.

.

An M$^\natural$-concave set function is submodular (cf. Theorem 3.2), but, not every submodular set function is M$^\natural$-concave (cf. Example 3.3). Thus M$^\natural$-concave set functions form a proper subclass of submodular set functions. M$^\natural$-concavity is known [14] to be equivalent to gross substitutes property, which is used in economics; see, e.g., [20].

A linear function μ on a g-matroid on V, defined by a weight vector $w \in \mathbf{R}^V$ as

$$(5.3) \qquad \mu(X) = \begin{cases} \sum \{w_i \mid i \in X\} & \text{if } X \text{ is a feasible (or independent) set,} \\ -\infty & \text{otherwise,} \end{cases}$$

is an M$^\natural$-concave set function.

Another simple example of an M$^\natural$-concave set function is given by $\mu(X) = \varphi(|X|)$, where φ is a univariate concave function. This is a classical example of a submodular function that connects submodularity and concavity [6, 22].

For a family of univariate concave functions $\{\varphi_A \mid A \in \mathcal{T}\}$ indexed by a family $\mathcal{T}$ of subsets of V, the function

$$\mu(X) = \sum_{A \in \mathcal{T}} \varphi_A(|A \cap X|) \qquad (X \subseteq V)$$

is submodular. This function is M$^\natural$-concave if, in addition, $\mathcal{T}$ is a laminar family (i.e., $A, B \in \mathcal{T} \Rightarrow A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B$).

Given a set of real numbers a_i indexed by $i \in V$, the *maximum-value function*

$$\mu(X) = \max_{i \in X} a_i \qquad (X \subseteq V)$$

---

[2]In Section 3 we have defined a set function $\rho : 2^V \to \mathbf{R}$ to be M$^\natural$-concave if the function $f : [\mathbf{0}, \mathbf{1}]_{\mathbf{Z}} \to \mathbf{R}$ associated with ρ by (3.2) is M$^\natural$-concave. The definition here is consistent with this.

is an M$^\natural$-concave function, where $\mu(\emptyset)$ is defined to be sufficiently small.

Given a matroid on V in terms of the family $\mathcal{I}$ of independent sets, the *rank function* μ is defined by

$$(5.4) \qquad \mu(X) = \max\{|I| \mid I \in \mathcal{I},\ I \subseteq X\} \qquad (X \subseteq V),$$

which denotes the maximum size of an independent set contained in X.

Theorem 5.1.   *Matroid rank function* (5.4) *is $M^\natural$-concave.*

Proof.   Here we give two different proofs, although this is a special case of the M$^\natural$-concavity of the vector rank function of a matroid stated in [12, p. 51].

First proof: Take $X, Y \subseteq V$ and $i \in X \setminus Y$, and suppose that

$$\mu(X) + \mu(Y) > \mu(X - i) + \mu(Y + i).$$

This implies that $\mu(X) - 1 = \mu(X - i)$ and $\mu(Y) = \mu(Y + i)$. Hence i is not a loop and it is contained in the closure (span) of Y. Therefore, there exists a circuit C such that $i \in C \subseteq Y \cup \{i\}$. Similarly, i is not a co-loop and there exists a co-circuit D such that $i \in D \subseteq (V \setminus X) \cup \{i\}$. Since $|C \cap D| \geq 2$, there exists some $j \in (C \cap D) \setminus \{i\} \subseteq Y \setminus X$. For this j we have $\mu(X) = \mu(X - i + j)$ and $\mu(Y) = \mu(Y + i - j)$. This implies (5.2).

Second proof: Let $f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$ be such that $f(\chi_X) = \mu(X)$ for $X \subseteq V$ and $\text{dom}_{\mathbf{Z}} f = \{0, 1\}^n$, and denote by $f^\bullet$ the discrete Legendre transform of f defined by $(2.9)_{\mathbf{Z}}$ (i.e., (2.9) with $p \in \mathbf{Z}^n$). Since μ is submodular, f is L$^\natural$-convex, and hence $f^\bullet$ is M$^\natural$-convex by conjugacy (Theorem 2.4). On the other hand, for $X \subseteq V$ we have

$$\begin{aligned}
f^\bullet(\chi_X) &= \max_Y\{|X \cap Y| - \mu(Y) \mid Y \subseteq V\} \\
&= \max_Y\{|X \cap Y| - \mu(Y) \mid X \subseteq Y \subseteq V\} \\
&= \max_Y\{|X| - \mu(Y) \mid X \subseteq Y \subseteq V\} \\
&= |X| - \mu(X).
\end{aligned}$$

Since $f^\bullet$ is M$^\natural$-convex, this expression shows that μ is M$^\natural$-concave. $\square$

A *weighted matroid rank function* is a function represented as

$$(5.5) \qquad \mu(X) = \max\{\sum_{i \in I} w_i \mid I \in \mathcal{I},\ I \subseteq X\} \qquad (X \subseteq V)$$

with $w \in \mathbf{R}^V$, where w is not assumed to be nonnegative.

Theorem 5.2 ([40]).   *Weighted matroid rank function* (5.5) *is $M^\natural$-concave.*

Proof. The original proof of Shioura [40] relies on the convolution theorem [30, Theorem 6.13 (8)] for M$^\natural$-concave functions. Here we give an elementary proof on the basis of the simultaneous exchange property of independent sets:

> For any $I, J \in \mathcal{I}$ and $i \in I \setminus J$, either $I - i, J + i \in \mathcal{I}$ or $I - i + j, J + i - j \in \mathcal{I}$ for some $j \in J \setminus I$.

Take $X, Y \subseteq V$ and $i \in X \setminus Y$. Let I and J be independent subsets of X and Y, respectively, such that $\mu(X) = w(I)$ and $\mu(Y) = w(J)$. If $i \notin I$, then

$$\mu(X - i) \geq w(I) = \mu(X), \quad \mu(Y + i) \geq \mu(Y),$$

which implies (5.1). So assume $i \in I$. If $J + i \in \mathcal{I}$, then

$$\mu(X - i) \geq w(I - i) = \mu(X) - w_i, \quad \mu(Y + i) \geq w(J + i) = \mu(Y) + w_i,$$

which implies (5.1). So assume $J + i \notin \mathcal{I}$. Then we have the second case in the simultaneous exchange axiom of $\mathcal{I}$ for I, J, i. That is, there exists $j \in J \setminus I$ such that $I - i + j, J + i - j \in \mathcal{I}$. If $j \in X$, then $I - i + j \subseteq X - i$, $J + i - j \subseteq Y + i$, and hence

$$\mu(X - i) \geq w(I - i + j) = \mu(X) - w_i + w_j, \quad \mu(Y + i) \geq w(J + i - j) = \mu(Y) + w_i - w_j,$$

which implies (5.1). If $j \notin X$, then $j \in Y \setminus X$, and

$$\mu(X - i + j) \geq w(I - i + j) = \mu(X) - w_i + w_j, \quad \mu(Y + i - j) \geq w(J + i - j) = \mu(Y) + w_i - w_j,$$

which implies (5.2). $\qquad\square$

Example 5.3. A polymatroid rank function is not necessarily M$^\natural$-concave. Let $V = \{1, 2, 3, 4\}$ and define $\mu : 2^V \to \mathbf{R}$ by $\mu(\emptyset) = 0$, $\mu(\{i\}) = 2$ $(i \in V)$, $\mu(\{1, 2\}) = \mu(\{3, 4\}) = 4$, $\mu(\{1, 3\}) = \mu(\{1, 4\}) = \mu(\{2, 3\}) = \mu(\{2, 4\}) = 3$, $\mu(X) = 4$ if $|X| \geq 3$. The exchange property fails, since for $X = \{1, 2\}$, $Y = \{3, 4\}$ there exists no $i \in X \setminus Y$ such that $\mu(X) + \mu(Y) \leq \mu(X - i) + \mu(Y + i)$, nor (i, j) such that $i \in X \setminus Y$, $j \in Y \setminus X$, and $\mu(X) + \mu(Y) \leq \mu(X - i + j) + \mu(Y + i - j)$. This example is due to A. Shioura. $\blacksquare$

§5.2. Greedy algorithm

M$^\natural$-concave set functions admit the following local characterization of global maximum.

Theorem 5.4. *Let $\mu : 2^V \to \underline{\mathbf{R}}$ be an $M^\natural$-concave set function. For a subset $X \in \mathrm{dom}\,\mu$, we have $\mu(X) \geq \mu(Y)$ $(\forall\, Y \subseteq V)$ if and only if*

$$\mu(X) \geq \max_{i \in X, j \in V \setminus X} \{\mu(X - i + j),\ \mu(X - i),\ \mu(X + j)\}.$$

For maximization of an $M^\natural$-concave set function μ the following natural greedy algorithm works. It is assumed that $\emptyset \in \operatorname{dom} \mu$.

S0: Put $X := \emptyset$.
S1: Find $j \in V \setminus X$ that maximizes $\mu(X + j)$.
S2: If $\mu(X) \geq \mu(X + j)$, then stop (X is a maximizer of μ).
S3: Set $X := X + j$ and go to S1.

This algorithm may be regarded as a variant of the algorithm of Dress–Wenzel [4] for valuated matroids, and the validity can be shown similarly.

Theorem 5.5.   *For an $M^\natural$-concave set function $\mu : 2^V \to \underline{\mathbf{R}}$, the family of maximizers of μ, $\operatorname{argmax} \mu$, is a g-matroid.*

Proof.   Let X and Y be maximizers of μ, and let $i \in X \setminus Y$. By (5.1) and (5.2) we see that both $X - i$ and $Y + i$ are maximizers, or else there exists some $j \in Y \setminus X$ such that both $X - i + j$ and $Y + i - j$ are maximizers of μ. This shows that $\operatorname{argmax} \mu$ is a g-matroid. $\square$

It is mentioned that Theorems 5.4 and 5.5 as well as the greedy algorithm carry over to an $M^\natural$-concave function $f : \mathbf{Z}^n \to \underline{\mathbf{R}}$ on integers. In particular we have the following local characterization of global maximum.

Theorem 5.6 ([26], [30, Theorem 6.26]).   *Let $f : \mathbf{Z}^n \to \underline{\mathbf{R}}$ be an $M^\natural$-concave function. A point $x \in \operatorname{dom}_{\mathbf{Z}} f$ is a global maximum of f if and only if it is a local maximum in the sense that*

$$f(x) \geq \max_{1 \leq i,j \leq n} \{f(x - \chi_i + \chi_j), \ f(x - \chi_i), \ f(x + \chi_j)\}.$$

§5.3.   Intersection algorithm

In this section we shed a new light on the matroid intersection/union from the viewpoint of discrete convex analysis.

Let ρ_1 and ρ_2 be the rank functions of two matroids on ground set V. For the maximum size of a common independent set of matroids (V, ρ_1) and (V, ρ_2) we have a well-known formula:

$$(5.6) \qquad \max_X \{\rho_1(X) + \rho_2(X) - |X|\} = \min_Y \{\rho_1(Y) + \rho_2(V \setminus Y)\}.$$

For the rank of the union of two matroids (V, ρ_1) and (V, ρ_2) we have another well-known formula:

$$(5.7) \qquad \max_X \{\rho_1(X) + \rho_2(V \setminus X)\} = \min_Y \{\rho_1(Y) + \rho_2(Y) - |Y|\} + |V|.$$

Edmonds's matroid intersection/union algorithms show that we can efficiently maximize the submodular functions appearing on the left-hand sides of (5.6) and (5.7). Why can we maximize these submodular functions efficiently?

In Theorem 5.1 we have seen that matroid rank functions are $M^\natural$-concave. Hence both $\rho_1(X) + (\rho_2(X) - |X|)$ and $\rho_1(X) + \rho_2(V \setminus X)$, to be maximized in (5.6) and (5.7), are submodular functions that are represented as a sum of two $M^\natural$-concave set functions. This is a crucial observation here. In fact, a sum of arbitrary two $M^\natural$-concave set functions can be maximized in strongly polynomial time by an adaptation of the valuated matroid intersection algorithm [24, 25]; see also [28, Chapter 5].

The following is an optimality criterion, of duality nature, for the problem of maximizing a sum of two $M^\natural$-concave set functions. Note that a sum of $M^\natural$-concave set functions is no longer $M^\natural$-concave in general and the optimality criterion in Theorem 5.4 does not apply. For a vector $p \in \mathbf{R}^n$ we use the notations $\mu_1 - p$ and $\mu_2 + p$ to mean the set functions defined, respectively, by $(\mu_1 - p)(X) = \mu_1(X) - p(X)$ and $(\mu_2 + p)(X) = \mu_2(X) + p(X)$ for $X \subseteq V$, where $p(X) = \sum_{i \in X} p_i$.

Theorem 5.7. *For $M^\natural$-concave set functions $\mu_1, \mu_2 : 2^V \to \underline{\mathbf{R}}$ and a subset $X \in \operatorname{dom} \mu_1 \cap \operatorname{dom} \mu_2$ we have $X \in \operatorname{argmax}(\mu_1 + \mu_2)$ if and only if there exists $p \in \mathbf{R}^n$ such that $X \in \operatorname{argmax}(\mu_1 - p) \cap \operatorname{argmax}(\mu_2 + p)$. In addition, for such p we have*

$$\operatorname{argmax}(\mu_1 + \mu_2) = \operatorname{argmax}(\mu_1 - p) \cap \operatorname{argmax}(\mu_2 + p).$$

Moreover, if μ_1 and μ_2 are integer-valued, we can choose integer-valued $p \in \mathbf{Z}^n$.

The duality nature of Theorem 5.7 is revealed by rewriting the claim as

$$(5.8) \qquad \max_X \{\mu_1(X) + \mu_2(X)\} = \min_p \{\max(\mu_1 - p) + \max(\mu_2 + p)\},$$

where the minimum on the right-hand side may be taken over integer vectors p when μ_1 and μ_2 are integer-valued. The formula (5.6) of matroid intersection can be understood as a special case of (5.8) with $\mu_1(X) = \rho_1(X)$ and $\mu_2(X) = \rho_2(X) - |X|$. On assuming that p is of the form $p = \chi_Z - \chi_Y$ for some disjoint Y and Z, we can easily see that $\max(\mu_1 - p) = \rho_1(V \setminus Z)$ and $\max(\mu_2 + p) = \rho_2(Z)$, and hence the right-hand side of (5.8) reduces to that of (5.6). Similarly, the formula (5.7) of matroid union can be understood as a special case of (5.8) with $\mu_1(X) = \rho_1(X)$ and $\mu_2(X) = \rho_2(V \setminus X)$.

Theorem 5.7 is a generalization of Frank's "weight splitting" theorem [9] for the weighted matroid intersection problem. Given a weight vector w, define $\mu_1(X) = w(X)$ if X is independent in the first matroid, and $= -\infty$ otherwise (see (5.3)); and $\mu_2(X) = 0$ if X is independent in the second matroid, and $= -\infty$ otherwise. Then a splitting $w = w_1 + w_2$ corresponds to $p = w_2$.

We now turn to the family of maximizers in the formula (5.6) of matroid intersection. Let $\mathcal{F}$ be the family of maximizers on the left-hand side, i.e.,

$$\mathcal{F} = \operatorname{argmax}_X (\rho_1(X) + \rho_2(X) - |X|).$$

Note that the minimal elements of $\mathcal{F}$ are exactly the maximum common independent sets. Theorem 5.7 implies the following.

Theorem 5.8. $\mathcal{F} = \mathcal{G}_1 \cap \mathcal{G}_2$ for some g-matroids $\mathcal{G}_1$ and $\mathcal{G}_2$.

We prove this by establishing a more general statement involving

$$\mathcal{F}(a_1, a_2, w) = \operatorname{argmax}_X (a_1 \rho_1(X) + a_2 \rho_2(X) + w(X)),$$

which contains parameters, $a_1 \geq 0$ and $a_2 \geq 0$, and a modular function (or a weight vector) w.

Theorem 5.9. $\mathcal{F}(a_1, a_2, w) = \mathcal{G}_1 \cap \mathcal{G}_2$ for some g-matroids $\mathcal{G}_1$ and $\mathcal{G}_2$.

Proof.   Take $\mu_1(X) = a_1 \rho_1(X)$ and $\mu_2(X) = a_2 \rho_2(X) + w(X)$ in Theorem 5.7, and put $\mathcal{G}_1 = \operatorname{argmax}(\mu_1 - p)$ and $\mathcal{G}_2 = \operatorname{argmax}(\mu_2 + p)$, both of which are g-matroids by Theorem 5.5. $\square$

It is mentioned that Theorem 5.7 as well as the valuated matroid intersection algorithm carries over to a pair of M$^\natural$-concave functions $f_1, f_2 : \mathbf{Z}^n \to \underline{\mathbf{R}}$ on integers. In particular we have the following theorem, a version of the *M-convex intersection theorem* ([26, 27], [30, Theorem 8.17]). Note that the sum of M-concave functions is no longer M-concave in general and Theorem 5.6 does not apply. We use notations $f_1 - p$, $f_2 + p$ for functions defined by $(f_1 - p)(x) = f_1(x) - \langle p, x \rangle$, $(f_2 + p)(x) = f_2(x) + \langle p, x \rangle$ for $x \in \mathbf{Z}^n$.

Theorem 5.10.   *For M$^\natural$-concave functions $f_1, f_2 : \mathbf{Z}^n \to \underline{\mathbf{R}}$ and a point $x \in \operatorname{dom}_{\mathbf{Z}} f_1 \cap \operatorname{dom}_{\mathbf{Z}} f_2$ we have $x \in \operatorname{argmax}_{\mathbf{Z}}(f_1 + f_2)$ if and only if there exists $p \in \mathbf{R}^n$ such that $x \in \operatorname{argmax}_{\mathbf{Z}}(f_1 - p) \cap \operatorname{argmax}_{\mathbf{Z}}(f_2 + p)$. In addition, for such p we have*

$$\operatorname{argmax}_{\mathbf{Z}}(f_1 + f_2) = \operatorname{argmax}_{\mathbf{Z}}(f_1 - p) \cap \operatorname{argmax}_{\mathbf{Z}}(f_2 + p).$$

Moreover, if f_1 and f_2 are integer-valued, we can choose integer-valued $p \in \mathbf{Z}^n$.

§5.4.   Pipage rounding algorithm

Let ρ be a nondecreasing submodular set function on V and $(V, \mathcal{I})$ be a matroid on V with the family $\mathcal{I}$ of independent sets. We consider the problem of maximizing

$\rho(X)$ subject to $X \in \mathcal{I}$. It is assumed that the function evaluation oracle for ρ and the membership oracle for $\mathcal{I}$ are available.

A recent paper of Calinescu–Chekuri–Pál–Vondrák [2] proposes a pipage rounding framework for approximate solution of this problem, showing that it works if the function ρ is represented as a sum of weighted matroid rank functions (5.5). Subsequently, it is pointed out by Shioura [40] that this approach can be extended to the class of functions ρ represented as a sum of nondecreasing M$^\natural$-concave functions.

The framework of [2] consists of three major steps.

1. Define a continuous relaxation: maximize $f(x)$ subject to $x \in P$, where P is the matroid polytope (convex hull of the characteristic vectors of independent sets) of $(V, \mathcal{I})$, and $f(x)$ is a nondecreasing concave function on P such that $f(\chi_X) = \rho(X)$ for all $X \subseteq V$.

2. Find an (approximately) optimal solution $x^* \in P$ of the continuous relaxation.

3. Round the fractional vector $x^* \in P$ to a $\{0,1\}$-vector $\hat{x} \in P$ by applying the "pipage rounding scheme," and output the corresponding subset $\hat{X}$ (such that $\chi_{\hat{X}} = \hat{x}$) as an approximate solution to the original problem.

This algorithm, if computationally feasible at all, is guaranteed to output a $(1 - 1/e)$-approximate solution, where e denotes the base of natural logarithm.

In the case where $\rho = \sum_{k=1}^{m} \rho_k$ with nondecreasing M$^\natural$-concave set functions ρ_k, the above algorithm can be executed in polynomial time. As the concave extension f we may take the sum of the concave closures, say, $\overline{\rho}_k$ of ρ_k for $k = 1, \ldots, m$. The continuous relaxation can be solved by the ellipsoid method, which uses subgradients of $\overline{\rho}_k$. The subgradients of $\overline{\rho}_k$ can in turn be computed in polynomial time by exploiting the combinatorial structure of M$^\natural$-concave functions.

Acknowledgements

References

[1] Ageev, A. and Sviridenko, M., Pipage rounding: A new method of constructing algorithms with proven performance guarantee, *Journal of Combinatorial Optimization*, **8** (2004), 307–328.

[2] Calinescu, G., Chekuri, C., Pál, M. and Vondrák, J., Maximizing a submodular set function subject to a matroid constraint (extended abstract), in: M. Fischetti and D. P. Williamson, eds., *Integer Programming and Combinatorial Optimization*, Lecture Notes in Computer Science, **4513**, Springer-Verlag, Berlin, 2007, 182–196.

[3] Conforti, M. and Cornuéjols, G., Submodular set functions, matroids and the greedy algo-rithm: tight worst-case bounds and some generalizations of the Rado–Edmonds theorem, *Discrete Applied Mathematics*, **7** (1984), 251–274.

[4] Dress, A. W. M. and Wenzel, W., Valuated matroid: A new look at the greedy algorithm, *Applied Mathematics Letters*, **3** (1990), 33–35.

[5] Dress, A. W. M. and Wenzel, W., Valuated matroids, *Advances in Mathematics*, **93** (1992), 214–250.

[6] Edmonds, J., Submodular functions, matroids and certain polyhedra, in: R. Guy, H. Hanani, N. Sauer, and J. Schönheim, eds., *Combinatorial Structures and Their Applica-tions*, Gordon and Breach, New York, 1970, 69–87. Also in: M. Jünger, G. Reinelt, G. Rinaldi, eds., *Combinatorial Optimization—Eureka, You Shrink!*, Lecture Notes in Com-puter Science, **2570**, Springer-Verlag, Berlin, 2003, 11–26.

[7] Favati, P. and Tardella, F., Convexity in nonlinear integer programming, *Ricerca Opera-tiva*, **53** (1990), 3–44.

[8] Fisher, M. L., Nemhauser, G. L. and Wolsey, L.A., An analysis of approximations for maximizing submodular set functions II, *Mathematical Programming Study*, **8** (1978), 73–87.

[9] Frank, A., A weighted matroid intersection algorithm, *Journal of Algorithms*, **2** (1981), 328–336.

[10] Frank, A., An algorithm for submodular functions on graphs, *Annals of Discrete Mathe-matics*, **16** (1982), 97–120.

[11] Fujishige, S., Theory of submodular programs: A Fenchel-type min-max theorem and subgradients of submodular functions, *Mathematical Programming*, **29** (1984), 142–155.

[12] Fujishige, S., *Submodular Functions and Optimization*, 2nd ed., Annals of Discrete Math-ematics, **58**, Elsevier, 2005.

[13] Fujishige, S. and Murota, K., Notes on L-/M-convex functions and the separation theo-rems, *Mathematical Programming*, **88** (2000), 129–146.

[14] Fujishige, S. and Yang, Z., A note on Kelso and Crawford's gross substitutes condition, *Mathematics of Operations Research*, **28** (2003), 463–469.

[15] Iwata, S., Submodular function minimization, *Mathematical Programming, Series B*, **112** (2007), 45–64.

[16] Iwata, S., Fleischer, L. and Fujishige, S., A combinatorial, strongly polynomial-time algo-rithm for minimizing submodular functions, *Journal of the ACM*, **48** (2001), 761–777.

[17] Iwata, S., and Shigeno, M., Conjugate scaling algorithm for Fenchel-type duality in dis-crete convex optimization, *SIAM Journal on Optimization*, **13** (2003), 204–211.

[18] Jensen, P. M. and Korte, B., Complexity of matroid property algorithms, *SIAM Journal on Computing*, **11** (1982), 184–190.

[19] Kolmogorov, V. and Shioura, A., New algorithms for convex cost tension problem with application to computer vision, *Discrete Optimization*, to appear.

[20] Lehmann, B., Lehmann, D. and Nisan, N., Combinatorial auctions with decreasing marginal utilities, *Games and Economic Behavior*, **55** (2006), 270–296.

[21] Lovász, L., Matroid matching and some applications, *Journal of Combinatorial Theory (B)*, **28** (1980), 208–236.

[22] Lovász, L., Submodular functions and convexity, in: A. Bachem, M. Grötschel and B. Ko-rte, eds., *Mathematical Programming—The State of the Art*, Springer-Verlag, Berlin, 1983, 235–257.

[23] McCormick, S. T., Submodular Function Minimization, in: K. Aardal, G. Nemhauser,

and R. Weismantel, eds., *Discrete Optimization*, Handbooks in Operations Research and Management Science, Vol.12, Elsevier Science Publishers, Berlin, 2006, Chapter 7, 321–391.

[24] Murota, K., Valuated matroid intersection, I: optimality criteria, *SIAM Journal on Discrete Mathematics*, **9** (1996), 545–561.

[25] Murota, K., Valuated matroid intersection, II: algorithms, *SIAM Journal on Discrete Mathematics*, **9** (1996), 562–576.

[26] Murota, K., Convexity and Steinitz's exchange property, *Advances in Mathematics*, **124** (1996), 272–311.

[27] Murota, K., Discrete convex analysis, *Mathematical Programming*, **83** (1998), 313–371.

[28] Murota, K., *Matrices and Matroids for Systems Analysis*, Springer-Verlag, Berlin, 2000.

[29] Murota, K., *Discrete Convex Analysis—An Introduction* (in Japanese), Kyoritsu Publishing Co., Tokyo, 2001.

[30] Murota, K., *Discrete Convex Analysis*, SIAM Monographs on Discrete Mathematics and Applications, Vol. 10, Society for Industrial and Applied Mathematics, Philadelphia, 2003.

[31] Murota, K., On steepest descent algorithms for discrete convex functions, *SIAM Journal on Optimization*, **14** (2003), 699–707.

[32] Murota, K., Recent developments in discrete convex analysis, in: W. Cook, L. Lovász and J. Vygen, eds., *Research Trends in Combinatorial Optimization*, Springer-Verlag, Berlin, 2009, Chapter 11, 219–260.

[33] Murota, K. and Shioura, A., M-convex function on generalized polymatroid, *Mathematics of Operations Research*, **24** (1999), 95–105.

[34] Murota, K. and Shioura, A., Extension of M-convexity and L-convexity to polyhedral convex functions, *Advances in Applied Mathematics*, **25** (2000), 352–427.

[35] Murota, K. and Shioura, A., Conjugacy relationship between M-convex and L-convex functions in continuous variables, *Mathematical Programming*, **101** (2004), 415–433.

[36] Murota, K. and Shioura, A., Fundamental properties of M-convex and L-convex functions in continuous variables, *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, **E87-A** (2004), 1042–1052.

[37] Nemhauser, G. L., Wolsey, L. A. and Fisher, M. L., An analysis of approximations for maximizing submodular set functions I, *Mathematical Programming*, **14** (1978), 265–294.

[38] Orlin, J. B., A faster strongly polynomial time algorithm for submodular function minimization, *Mathematical Programming, Ser.A*, **118** (2009), 237–251.

[39] Schrijver, A., A combinatorial algorithm minimizing submodular functions in strongly polynomial time, *Journal of Combinatorial Theory (B)*, **80** (2000), 346–355.

[40] Shioura, A., On the pipage rounding algorithm for submodular function maximization: A view from discrete convex analysis, *Discrete Mathematics, Algorithms and Applications*, **1** (2009), 1–23.

[41] Topkis, D. M., *Supermodularity and Complementarity*, Princeton University Press, Princeton, NJ, 1998.

[42] Vondrák, J., Optimal approximation for the submodular welfare problem in the value oracle model, *Proceedings of the 40th Annual ACM Symposium on Theory of Computing* (Victoria, May 2008), ACM, 2008, 67–74.