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Polarizations on limiting mixed Hodge structures (announcement)

By

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§ 1. Introduction

§ 1.1. The aim

This is an announcement of my forthcoming paper [7]. In this article, I will explain the idea to construct a polarization on the limiting mixed Hodge structure. The following is the situation which I will consider in this article:

Setting 1.1. Let X be a complex manifold, Δ the unit disc with the coordinate function t and $f : X \longrightarrow \Delta$ a *projective* surjective morphism with connected fibers. Moreover we assume that f is smooth over $\Delta \setminus \{0\}$, and that the fiber $Y = f^{-1}(0)$ is a *reduced* simple normal crossing divisor on X. I simply call such morphism f a semistable reduction over the unit disc.

Under the situation above, Steenbrink [17] construct a mixed Hodge structure on the cohomology group $\mathrm{H}^q(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$ for every integer q, which is so called the limiting mixed Hodge structure. My aim is to construct a polarization on the limiting mixed Hodge structure above in the following sense:

Definition 1.2 ([1], Definition 2.26)**.** Let $V = (V_{\mathbb{R}}, W, F)$ be a mixed Hodge structure over $\mathbb{R}$, N an endomorphism of $V_{\mathbb{R}}$, and S a bilinear form on $V_{\mathbb{R}}$. The data (V, N, S) is said to be a polarized mixed Hodge structure if there exists a non-negative integer q such that the following conditions are satisfied:

1. $N^{q+1} = 0$

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2. $N(F^p) \subset F^{p-1}$ for every integer p
3. $N(W_m) \subset W_{m-2}$ for every integer m
4. $N^l : V_{\mathbb{R}} \longrightarrow V_{\mathbb{R}}$ induces an isomorphism from $\mathrm{Gr}^W_{q+l} V_{\mathbb{R}}$ to $\mathrm{Gr}^W_{q-l} V_{\mathbb{R}}$ for every positive integer l
5. S is $(-1)^q$-symmetric
6. $S(F^p, F^{q-p+1}) = 0$ for every integer p
7. $S(Nx, y) + S(x, Ny) = 0$ for every $x, y \in V_{\mathbb{R}}$
8. the bilinear form $S(-, N^l -)$ induces a polarization on the Hodge structure $P_{q+l} = \mathrm{Ker}(N^{l+1} : \mathrm{Gr}^W_{q+l} V_{\mathbb{R}} \longrightarrow \mathrm{Gr}^W_{q-l-2} V_{\mathbb{R}})$ for every non-negative integer l.

We remark that P_{q+l} is a Hodge structure of weight $q + l$ by the conditions 2 and 3.

§ 1.2. The motivation

My motivation comes from the log geometry.

For a semistable reduction over the unit disc, the limiting mixed Hodge structure on the cohomology group $\mathrm{H}^q(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$ can be regarded as the limit of the variation of Hodge structures on the Betti cohomology groups of general fibers. In other words, the limiting mixed Hodge structure is the limit of a nilpotent orbit of one variable as in Schmid [16].

From the viewpoint of log geometry, the singular fiber of a semistable reduction can be replaced by a log deformation. Under some suitable conditions, Steenbrink [18] shows that we can directly construct mixed Hodge structures on the relative log de Rham cohomology groups which are natural analogues of the limiting mixed Hodge structures. (see also Kawamata-Namikawa [11], Fujisawa-Nakayama [5], Nakkajima [12].) Namely, the log geometry enables us to construct the "limiting" mixed Hodge structure directly with no nilpotent orbit nor variation of Hodge structure. One of my motivation is to reconstruct nilpotent orbits from the limiting mixed Hodge structure from the viewpoint of log geometry. For this purpose, it is sufficient to construct a polarization on the limiting mixed Hodge structure by the following result:

Theorem 1.3 ([1], Corollary 3.13). *For a polarized mixed Hodge structure (V, N, S), the map $z \mapsto \exp(zN)F$ over $\mathbb{C}$ is a nilpotent orbit, where F denotes the Hodge filtration of the Hodge structure V.*

The other motivation concerns the theory of polarized log Hodge structures. (see [10] for definition.) As far as I understand, the notion of polarized log Hodge structure

is a generalization of the notion of nilpotent orbit from the viewpoint of log geometry. It is expected that the geometric objects, such as a semistable reduction and as a log deformation, give rise to polarized log Hodge structures on the base space by considering the higher direct images of the relative log de Rham complexes, because they satisfy the smoothness condition in the sense of log geometry. In fact, Kato-Matsubara-Nakayama [9] proved that the expectation above is true for a log smooth projective morphism over a log smooth base space. As a special case of Kato-Matsubara-Nakayama's result, a projective semistable reduction over the unit disc gives us a polarized log Hodge structures on the unit disc.

On the other hand, the base space of a log deformation is the standard log point, which is not log smooth at all. So, the other motivation is to prove that the expectation above is true for the case of a log deformation. Once we obtain a polarization on the limiting mixed Hodge structure, Theorem 1.3 will enable us to construct a polarized log Hodge structure on the relative log de Rham cohomology group.

Taking these two motivation into account, it is necessary to consider the case of log deformation instead of the case of semistable reduction. Nevertheless, I restrict myself to the case of semistable reduction in this article, because I hesitate to explain the generalities on the log geometry. Modifying the contents of this article to the case of a log deformation is not a difficult task (see [7]).

Throughout this article, I omit the underlying $\mathbb{Q}$-structure (or $\mathbb{R}$-structure) of the (mixed) Hodge structures in question for simplicity. Moreover, we omit the Tate twist of the Hodge structure because we consider the $\mathbb{C}$-structure only.

§ 1.3. The strategy

First, I recall the previous results concerning the mixed Hodge structure on the relative log de Rham cohomology group $\mathrm{H}^q(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$ under the situation in Setting 1.1. Steenbrink [17] constructed a cohomological mixed Hodge complex which is quasi-isomorphic to the relative log de Rham complex $\Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y$. Thus the limiting mixed Hodge structure on the relative log de Rham cohomology group is constructed. The monodromy logarithm gives the endomorphisms N on the relative log de Rham cohomology groups. The conditions 1, 2 and 3 in Definition 1.2 are easily seen from the construction of the cohomological mixed Hodge complex in [17]. Steenbrink also claimed that he proved the condition 4 in [17], but there was a gap in his proof. Morihiko Saito [15], Guillén-Navarro Aznar [8] and Usui [19] filled the gap independently. Thus the condition 4 is established for the relative log de Rham cohomology group. Moreover, [15] and [8] proved the weaker version of the condition 8, that is, P_{q+l} is polarizable for every integer l by using the polarization on the Betti cohomology groups of strata $Y_{\underline{\lambda}}$ for subsets $\underline{\lambda}$ of Λ (see (2.1) for the definition of $Y_{\underline{\lambda}}$).

Hence the remaining part is to construct the bilinear form S on $\mathrm{H}^q(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes$

$\mathcal{O}_Y$) which satisfies the condition 6 and which induces the polarization on P_{q+l} given in [15] and [8]. In order to construct a natural bilinear form S above, I follow the way in the classical Hodge theory. Namely, I expect that the following steps will be carried out successfully:

1. Construct a cup product

$$\cup : \mathrm{H}^p(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \otimes \mathrm{H}^q(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \longrightarrow \mathrm{H}^{p+q}(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$$

   for every integer a, b.

2. Find a trace morphism

$$\mathrm{Tr} : \mathrm{H}^{2n}(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \longrightarrow \mathbb{C}$$

   where n is the relative dimension of f.

3. Define a pairing

$$Q : \mathrm{H}^q(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \otimes \mathrm{H}^{2n-q}(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \longrightarrow \mathbb{C}$$

   by $Q(x, y) = \pm \mathrm{Tr}(x \cup y)$, where the sign should be chosen appropriately.

4. Define a morphism

$$l : \mathrm{H}^q(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \longrightarrow \mathrm{H}^{q+2}(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$$

   by taking a cup product with the ample class. and prove the hard Lefschetz theorem for l.

5. Construct a polarization S from the pairing Q and the morphism l by using the Lefschetz decomposition.

In Step 1, we can easily find a candidate of the cup product: the exterior product on the complex $\Omega^\bullet_{X/\Delta}(\log Y)$ induces a cup product on the relative log de Rham cohomology groups. However, we have to relate it to the weight filtration W in order to prove the desired properties for Q in Step 3 and for S in Step 5. Therefore, we have to lift the exterior product to the cohomological mixed Hodge complex which gives the weight and Hodge filtrations on $\mathrm{H}^q(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$. Unfortunately, it seems impossible to lift the exterior product to Steenbrink's cohomological mixed Hodge complex $A_{\mathbb{C}}$ in [17] (see Definition 2.9 below). Navvaro Aznar [13] constructed another cohomological mixed Hodge complex which is quasi-isomorphic to the complex $\Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y$ and which admits a lifting of the exterior product of the complex $\Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y$.

However, I prefer to another way of using the complex $K_{\mathbb{C}}$ defined in [6], which is similar to El Zein's cohomological mixed Hodge complex in [4] (see Definition 2.15 below). This approach seems much simpler than Navarro Azanr's method. Since the complex $K_{\mathbb{C}}$ is constructed from the simplicial data, we can construct a product

$$\Phi : K_{\mathbb{C}} \otimes K_{\mathbb{C}} \longrightarrow K_{\mathbb{C}}$$

by using Alexander-Whitney formula. It is easy to see that this product is compatible with the exterior product on the relative log de Rham complex.

In order to define a trace morphism, I define a morphism at the level of E_1-terms of the weight spectral sequence in terms of integration on strata $Y_{\underline{\lambda}}$. Then I prove that this morphism can "descend" to $\mathrm{H}^{2n}(Y, \Omega_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$. To check the condition for "descent" above, I have to compute the morphism of E_1-terms in the very precise form, which is almost the same formula as in Nakkajima [12].

As for the pairing Q in Step 3, it is the key point for the whole procedure above that the induced pairing from Q on

$$\mathrm{Gr}^W_m \mathrm{H}^q(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \otimes \mathrm{Gr}^W_{-m} \mathrm{H}^{2n-q}(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$$

coincides with the one considered in [15] and in [8]. This fact enables us to apply the results in [15] and [8]. In order to check the coincidence above, I construct a comparison morphism from the Steenbrink's cohomological mixed Hodge complex $A_{\mathbb{C}}$ to the complex $K_{\mathbb{C}}$, which enables us to relate the product in Step 1 to the graded pieces of the weight filtration of the cohomological mixed Hodge complex $A_{\mathbb{C}}$. Via this comparison morphism I can prove the coincidence mentioned above. The construction of S from the data Q and l is similar to the standard procedure as in the case of classical Hodge theory.

In this article, I only give sketches of the proofs. See [7] for the detail.

§ 2. Preliminaries

§ 2.1. Notation

First, we fix the notation which we will use in this article.

Notation 2.1. Under the situation in Setting 1.1, the irreducible components of Y are denoted by $\{Y_\lambda\}_{\lambda \in \Lambda}$, where Λ is a finite set. We fix an total order $<$ on Λ. Moreover, we often use symbols $\underline{\lambda}, \underline{\mu}, \ldots$ for subsets of Λ. We set

$$Y_{\underline{\lambda}} = \bigcap_{\lambda \in \underline{\lambda}} Y_\lambda \tag{2.1}$$

for a subset $\underline{\lambda}$ of Λ. For the case of $\underline{\lambda} = \{\lambda_0, \lambda_1, \cdots, \lambda_k\}$ we sometimes use the symbol $Y_{\lambda_0\lambda_1\cdots\lambda_k}$ for $Y_{\underline{\lambda}}$. We denote by Y_k the disjoint union of $Y_{\underline{\lambda}}$ for all subsets $\underline{\lambda}$ with $(k+1)$-elements, that is,

$$Y_k = \coprod_{|\underline{\lambda}|=k+1} Y_{\underline{\lambda}}$$

for a non-negative integer k. By the usual way, the data $\{Y_k\}_{n\in\mathbb{Z}_{\geq 0}}$ form a semi-simplicial complex manifold, which is denoted by $Y_\bullet$. (For the notion of semi-simplicial object, see e.g. Peters-Steenbrink [14, Definition 5.1].) The canonical morphism from Y_k to Y is denoted by a_k.

Remark. Note that the notation above is slightly different from the one in Steenbrink [17]. In [17], Y_k above is denoted by $Y^{(k+1)}$.

Remark. We often omit the symbol $(a_k)_*$ for direct images of sheaves.

Notation 2.2. For a non-empty subset

$$\underline{\mu} = \{\mu_1, \mu_2, \cdots, \mu_k\} \quad (\mu_1 < \mu_2 < \cdots < \mu_k)$$

of Λ, we set

$$\underline{\mu}_i = \{\mu_1 < \mu_2 < \cdots \not{\mu}_i \cdots < \mu_k\} \subset \underline{\mu}$$

for $i = 1, 2, \cdots, k$, where $\not{\mu}_i$ means to omit the element $\underline{\mu}_i$. Then $Y_{\underline{\mu}}$ is a smooth divisor on $Y_{\underline{\mu}_i}$ for $i = 1, 2, \cdots, k$.

Notation 2.3. For an abelian sheaf $\mathcal{F}^\bullet$ on the semi-simplicial complex manifold $Y_\bullet$, the data $\{(a_k)_*\mathcal{F}^k\}_{k\in\mathbb{Z}_{\geq 0}}$ form a co-semi-simplicial abelian sheaf on Y, from which we obtain a complex of abelian sheaves on Y by using the Čech type morphism as usual. We denote it by $\mathrm{C}(\mathcal{F}^\bullet)$. By the similar procedure, we obtain the associated single complex on Y for a complex of abelian sheaves $\mathcal{F}^{\bullet,\bullet}$ on $Y_\bullet$, which is denoted by $\mathrm{C}(\mathcal{F}^{\bullet,\bullet})$.

Example 2.4. The data $\{\mathcal{O}_{Y_k}\}_{k\in\mathbb{Z}_{\geq 0}}$ are an abelian sheaf on $Y_\bullet$, which we denote by $\mathcal{O}_{Y_\bullet}$. The complex $\mathrm{C}(\mathcal{O}_{Y_\bullet})$ is nothing but the complex

$$0 \longrightarrow \mathcal{O}_{Y_0} \xrightarrow{\delta} \mathcal{O}_{Y_1} \xrightarrow{\delta} \cdots\cdots \xrightarrow{\delta} \mathcal{O}_{Y_k} \xrightarrow{\delta} \cdots$$

where δ denotes the usual Čech type morphism. It is well-known (or easy to check) that this complex is quasi-isomorphic to the sheaf $\mathcal{O}_Y$, that is, the sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{Y_0} \xrightarrow{\delta} \mathcal{O}_{Y_1} \xrightarrow{\delta} \cdots\cdots \xrightarrow{\delta} \mathcal{O}_{Y_k} \xrightarrow{\delta} \cdots \tag{2.2}$$

is exact.

Example 2.5. The data $\{\Omega^\bullet_X(\log Y)\otimes\mathcal{O}_{Y_k}\}_{k\in\mathbb{Z}_{\geq 0}}$ form a complex of abelian sheaves $\Omega^\bullet_X(\log Y)\otimes\mathcal{O}_{Y_\bullet}$ on $Y_\bullet$. The complex $\mathrm{C}(\Omega^\bullet_X(\log Y)\otimes\mathcal{O}_{Y_\bullet})$ is given by

$$\mathrm{C}(\Omega^\bullet_X(\log Y)\otimes\mathcal{O}_{Y_\bullet})^n=\bigoplus_{k\geq 0}\Omega^{n-k}_X(\log Y)\otimes\mathcal{O}_{Y_k}$$

for every integer n. The differential d of the complex $\mathrm{C}(\Omega^\bullet_X(\log Y)\otimes\mathcal{O}_{Y_\bullet})$ is given by

$$\delta+(-1)^k d$$

on the direct summand $\Omega^{n-k}_X(\log Y)\otimes\mathcal{O}_{Y_k}$, where δ denotes the Čech type morphism and d the differential of log differential forms. By the exact sequence (2.2), the canonical morphism $\Omega^\bullet_X(\log Y)\otimes\mathcal{O}_Y\longrightarrow\mathrm{C}(\Omega^\bullet_X(\log Y)\otimes\mathcal{O}_{Y_\bullet})$ is a quasi-isomorphism. Similarly, we have a quasi-isomorphism $\Omega^\bullet_{X/\Delta}(\log Y)\otimes\mathcal{O}_Y\longrightarrow\mathrm{C}(\Omega^\bullet_{X/\Delta}(\log Y)\otimes\mathcal{O}_{Y_\bullet})$.

§ 2.2. Residue morphism

Definition 2.6. For a subset

$$\underline{\mu}=\{\mu_1,\mu_2,\cdots,\mu_k\}\quad(\mu_1<\mu_2<\cdots<\mu_k)$$

of Λ, the morphism

$$\mathrm{Res}^{\underline{\mu}}_X:\Omega^p_X(\log Y)\longrightarrow\Omega^{p-k}_X(\log Y)\otimes\mathcal{O}_{Y_{\underline{\mu}}}$$

is defined locally by the formula

$$\mathrm{Res}^{\underline{\mu}}_X(\mathrm{dlog}\,x_{\mu_1}\wedge\mathrm{dlog}\,x_{\mu_2}\wedge\cdots\wedge\mathrm{dlog}\,x_{\mu_k}\wedge\omega)=\omega|_{Y_{\underline{\mu}}},$$

where $x_{\mu_1},x_{\mu_2},\cdots,x_{\mu_k}$ are local defining functions of the divisors $Y_{\mu_1},Y_{\mu_2},\cdots,Y_{\mu_k}$, respectively. We can easily check the morphism $\mathrm{Res}^{\underline{\mu}}_X$ is well-defined, i.e., independent from the choice of the functions $x_{\mu_1},x_{\mu_2},\cdots,x_{\mu_k}$. A morphism of complexes

$$\mathrm{Res}^{\underline{\mu}}_X:\Omega^\bullet_X(\log Y)\longrightarrow\Omega^\bullet_X(\log Y)\otimes\mathcal{O}_{Y_{\underline{\mu}}}[-k]$$

is obtained by the morphism above. This morphism induces morphisms

$$\begin{aligned}\mathrm{Res}^{\underline{\mu}}_Y&:\Omega^\bullet_X(\log Y)\otimes\mathcal{O}_Y\longrightarrow\Omega^\bullet_X(\log Y)\otimes\mathcal{O}_{Y_{\underline{\mu}}}[-k]\\ \mathrm{Res}^{\underline{\mu}}_{Y_{\underline{\lambda}}}&:\Omega^\bullet_X(\log Y)\otimes\mathcal{O}_{Y_{\underline{\lambda}}}\longrightarrow\Omega^\bullet_X(\log Y)\otimes\mathcal{O}_{Y_{\underline{\lambda}\cup\underline{\mu}}}[-k]\end{aligned}$$

for a subset $\underline{\lambda}$ of Λ. These morphisms are called the residue morphisms.

The following proposition plays an important role in the next section.

Proposition 2.7. *For a subset $\underline{\mu}$ of Λ with $|\underline{\mu}| = k$, we have the equality*

$$\mathrm{Res}_X^{\underline{\mu}}(\mathrm{dlog}\, t \wedge \omega) = (-1)^k\, \mathrm{dlog}\, t \wedge \mathrm{Res}_X^{\underline{\mu}}(\omega) + \sum_{i=1}^{k} (-1)^{i-1}\, \mathrm{Res}_X^{\underline{\mu}_i}(\omega)|_{Y_{\underline{\mu}}}$$

for any local section ω of $\Omega_X^p(\log Y)$, where $\underline{\mu}_i$ is a subset of $\underline{\mu}$ defined in Notation 2.2.

Proof. See [6]. □

Definition 2.8. We have

$$\mathrm{Res}_X^k = \bigoplus_{|\underline{\mu}|=k+1} \mathrm{Res}_X^{\underline{\mu}} : \Omega_X^\bullet(\log Y) \longrightarrow \Omega_X^\bullet(\log Y) \otimes \mathcal{O}_{Y_k}[-k-1]$$

$$\mathrm{Res}_Y^k = \bigoplus_{|\underline{\mu}|=k+1} \mathrm{Res}_Y^{\underline{\mu}} : \Omega_X^\bullet(\log Y) \otimes \mathcal{O}_Y \longrightarrow \Omega_X^\bullet(\log Y) \otimes \mathcal{O}_{Y_k}[-k-1]$$

for a non-negative integer k.

§ 2.3. Steenbrink's cohomological mixed Hodge complex

In this subsection, we recall the construction of a cohomological mixed Hodge complex given by Steenbrink in [17], which is quasi-isomorphic to the relative log de Rham complex $\Omega_{X/\Delta}^\bullet(\log Y) \otimes \mathcal{O}_Y$.

Definition 2.9. Under the situation in Setting 1.1, we set

$$A_{\mathbb{C}}^n = \bigoplus_{r \geq 0} \Omega_X^{n+1}(\log Y)/W_r\Omega_X^{n+1}(\log Y) \tag{2.3}$$

for every non-negative integer n, where W denotes the increasing filtration by the order of poles along Y (see e.g. Deligne [2]). The differential d on $\Omega_X^\bullet(\log Y)$ induces a morphism

$$d : \Omega_X^{n+1}(\log Y)/W_r\Omega_X^{n+1}(\log Y) \longrightarrow \Omega_X^{n+2}(\log Y)/W_r\Omega_X^{n+2}(\log Y) \subset A_{\mathbb{C}}^{n+1}$$

for every n, r. On the other hand, we obtain a morphism

$$\mathrm{dlog}\, t\wedge : \Omega_X^{n+1}(\log Y)/W_r\Omega_X^{n+1}(\log Y) \longrightarrow \Omega_X^{n+2}(\log Y)/W_{r+1}\Omega_X^{n+2}(\log Y) \subset A_{\mathbb{C}}^{n+1}$$

by taking wedge product with $\mathrm{dlog}\, t$. Then the morphisms

$$-d + \mathrm{dlog}\, t\wedge : \Omega_X^{n+1}(\log Y)/W_r\Omega_X^{n+1}(\log Y) \longrightarrow A_{\mathbb{C}}^{n+1}$$

define a morphism $A_{\mathbb{C}}^n \longrightarrow A_{\mathbb{C}}^{n+1}$ for every integer n. Thus we obtain a complex $A_{\mathbb{C}}$. An increasing filtration W on $A_{\mathbb{C}}$ is defined by

$$W_m A_{\mathbb{C}}^n = \bigoplus_{r \geq 0} W_{m+2r+1}\Omega_X^{n+1}(\log Y)/W_r\Omega_X^{n+1}(\log Y)$$

for every integer m. A decreasing filtration F on $A_{\mathbb{C}}$ is defined by

$$F^p A^n_{\mathbb{C}} = \bigoplus_{0 \leq r \leq n-p} \Omega^{n+1}_X(\log Y)/W_r\Omega^{n+1}_X(\log Y)$$

for every p.

Remark. The sign on the differential d above is slightly different from the original one in [17] (see [19]). From this change, the sign of the morphisms in what follows are slightly changed in the original ones in [17].

Definition 2.10. We can easily see that the morphism

$$\mathrm{dlog}\, t \wedge : \Omega^n_X(\log Y) \longrightarrow \Omega^{n+1}_X(\log Y)/W_0\Omega^{n+1}_X(\log Y) \subset A^n_{\mathbb{C}}$$

factors through the surjection $\Omega^n_X(\log Y) \longrightarrow \Omega^n_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y$. Thus a morphism of complexes

$$\Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y \longrightarrow A_{\mathbb{C}} \tag{2.4}$$

is obtained.

Theorem 2.11 ([17])**.** *Under the situation above, we have the following*:

1. *The morphism* (2.4) *induces an isomorphism*

$$\mathrm{H}^q(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \simeq \mathrm{H}^q(Y, A_{\mathbb{C}}) \tag{2.5}$$

   for every q.

2. *The triple* $(A_{\mathbb{C}}, W, F)$ *is* (*the* $\mathbb{C}$*-structure of*) *a cohomological mixed Hodge complex on* Y.

Remark. The E_1-terms of the weight spectral sequence for $(A_{\mathbb{C}}, W)$ are given by

$$E_1^{p,q}(A_{\mathbb{C}}, W) = \bigoplus_{r \geq \max(0,p)} \mathrm{H}^{2p+q-2r}(Y_{-p+2r}, \Omega^\bullet_{Y_{-p+2r}}) \tag{2.6}$$

for every p, q.

Definition 2.12. The canonical morphism

$$\Omega^{n+1}_X(\log Y)/W_r\Omega^{n+1}_X(\log Y) \longrightarrow \Omega^{n+1}_X(\log Y)/W_{r+1}\Omega^{n+1}_X(\log Y) \text{ in } A^n_{\mathbb{C}}$$

for every r defines a morphism of complexes $\nu : A_{\mathbb{C}} \longrightarrow A_{\mathbb{C}}$.

Lemma 2.13. *Under the identification* (2.5)*, the morphism* ν *induces the monodromy logarithm* N *on* $\mathrm{H}^q(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$.

Proof. See [17]. □

§ 2.4. Simplicial method

In this subsection, we recall the mixed Hodge structures on the relative log de Rham cohomology groups constructed in [6]. This approach is very similar to the construction by El Zein in [4].

Definition 2.14. We denote by $\mathbb{C}[u]$ the polynomial ring over $\mathbb{C}$ of the one indeterminate u. Fix a non-negative integer k. A morphism

$$\mathbb{C}[u] \otimes \Omega^p_X(\log Y) \otimes \mathcal{O}_{Y_k} \longrightarrow \mathbb{C}[u] \otimes \Omega^{p+1}_X(\log Y) \otimes \mathcal{O}_{Y_k}$$

is defined by sending a local section $P(u) \otimes \omega$ of $\mathbb{C}[u] \otimes \Omega^p_X(\log Y) \otimes \mathcal{O}_{Y_k}$ to the local section

$$P(u) \otimes d\omega + \frac{dP}{du} \otimes \mathrm{dlog}\, t \wedge \omega$$

of $\mathbb{C}[u] \otimes \Omega^{p+1}_X(\log Y) \otimes \mathcal{O}_{Y_k}$, where $P(u)$ is an element of $\mathbb{C}[u]$ and ω a local section of $\Omega^p_X(\log Y) \otimes \mathcal{O}_{Y_k}$. We denote it by d again by abuse of the language. We can easily check the equality $d^2 = 0$, that is, we obtain a complex $\mathbb{C}[u] \otimes \Omega^{\bullet}_X(\log Y) \otimes \mathcal{O}_{Y_k}$ on Y_k for every non-negative integer k.

We set $u^{[r]} = u^r/r!$ and identify

$$\mathbb{C}[u] \otimes \Omega^{\bullet}_X(\log Y) \otimes \mathcal{O}_{Y_k} = \bigoplus_{r \geq 0} u^{[r]} \otimes \Omega^{\bullet}_X(\log Y) \otimes \mathcal{O}_{Y_k}$$

trivially. Then we define an increasing filtration W and a decreasing filtration F on $\mathbb{C}[u] \otimes \Omega^{\bullet}_X(\log Y) \otimes \mathcal{O}_{Y_k}$ by

$$W_m(\mathbb{C}[u] \otimes \Omega^{\bullet}_X(\log Y) \otimes \mathcal{O}_{Y_k}) = \bigoplus_{r \geq 0} u^{[r]} \otimes W_{m-2r}\Omega^{\bullet}_X(\log Y) \otimes \mathcal{O}_{Y_k}$$

$$F^p(\mathbb{C}[u] \otimes \Omega^{\bullet}_X(\log Y) \otimes \mathcal{O}_{Y_k}) = \bigoplus_{r \geq 0} u^{[r]} \otimes F^{p-r}\Omega^{\bullet}_X(\log Y) \otimes \mathcal{O}_{Y_k}$$

for every m, p. Moreover we can check that the data

$$\{(\mathbb{C}[u] \otimes \Omega^{\bullet}_X(\log Y) \otimes \mathcal{O}_{Y_k}, W, F)\}_{k \in \mathbb{Z}_{\geq 0}}$$

form a (filtered) complex on a semi-simplicial complex manifold $Y_{\bullet}$, which we denote by $(\mathbb{C}[u] \otimes \Omega^{\bullet}_X(\log Y) \otimes \mathcal{O}_{Y_{\bullet}}, W, F)$.

Definition 2.15. As explained in Notation 2.3, we obtain a complex of sheaves

$$\mathrm{C}(\mathbb{C}[u] \otimes \Omega_X^\bullet(\log Y) \otimes \mathcal{O}_{Y_\bullet})$$

on Y, which we denote by $K_{\mathbb{C}}$ in this article. As for the filtration F, the data

$$\{F^p(\mathbb{C}[u] \otimes \Omega_X^\bullet(\log Y) \otimes \mathcal{O}_{Y_k})\}_{k \in \mathbb{Z}_{\geq 0}}$$

form a complex on $Y_\bullet$. By setting

$$F^p K_{\mathbb{C}} = \mathrm{C}(\{F^p(\mathbb{C}[u] \otimes \Omega_X^\bullet(\log Y) \otimes \mathcal{O}_{Y_k})\}_{k \in \mathbb{Z}_{\geq 0}}),$$

for every p, we obtain a decreasing filtration F on $K_{\mathbb{C}}$. On the other hand, the data

$$\{W_{m+k}(\mathbb{C}[u] \otimes \Omega_X^\bullet(\log Y) \otimes \mathcal{O}_{Y_k})\}_{k \in \mathbb{Z}_{\geq 0}}$$

form a complex on $Y_\bullet$. Then we obtain an increasing filtration W on $K_{\mathbb{C}}$ by

$$W_m K_{\mathbb{C}} = \mathrm{C}(\{W_{m+k}(\mathbb{C}[u] \otimes \Omega_X^\bullet(\log Y) \otimes \mathcal{O}_{Y_k})\}_{k \in \mathbb{Z}_{\geq 0}})$$

for every integer m.

Remark. We note that the filtrations W and F are not finite filtrations. So we have to be careful to apply usual results on finite filtrations. In particular, we do not claim the triple $(K_{\mathbb{C}}, W, F)$ is (the $\mathbb{C}$-structure of) a cohomological mixed Hodge complex on Y.

Although the filtration W and F are not finite filtration as mentioned above, we obtain the following results in [6].

Theorem 2.16 ([6])**.** *Under the above situation, we obtain the followings.*

1. *The filtrations W and F induces finite filtration on $\mathrm{H}^q(Y, K_{\mathbb{C}})$ for every integer q.*
2. *The data $(\mathrm{H}^q(Y, K_{\mathbb{C}}), W[q], F)$ is (the $\mathbb{C}$-structure of) a mixed Hodge structure for every q.*
3. *There exists an isomorphism*

$$\mathrm{H}^q(Y, K_{\mathbb{C}}) \simeq \mathrm{H}^q(Y, \Omega_{X/\Delta}^\bullet(\log Y) \otimes \mathcal{O}_Y), \tag{2.7}$$

   for every q.
4. *The spectral sequence associated to the filtration F on $K_{\mathbb{C}}$ degenerates at E_1-terms.*
5. *The spectral sequence associated to the filtration W on $K_{\mathbb{C}}$ degenerates at E_2-terms.*

Remark. In order to illustrate the complexity of the E_1-terms $E_1^{p,q}(K_{\mathbb{C}}, W)$ of the weight spectral sequence for $K_{\mathbb{C}}$, I present a result of the computation in [6]. The E_1-terms are given by

$$E_1^{p,q}(K_{\mathbb{C}}, W) \simeq \bigoplus_{r\geq 0} \bigoplus_{\substack{\underline{\lambda}\subset\Lambda \\ \underline{\lambda}\neq\emptyset}} \bigoplus_{\substack{\underline{\mu}\subset\Lambda \\ |\underline{\mu}|=-p+|\underline{\lambda}|-1-2r}} u^{[r]} \otimes \mathrm{H}^{2p+q+2-2|\underline{\lambda}|+2r}(Y_{\underline{\lambda}\cup\underline{\mu}}, \Omega^{\bullet}_{Y_{\underline{\lambda}\cup\underline{\mu}}}) \tag{2.8}$$

for every integers p, q.

Definition 2.17. The morphism of complexes

$$\frac{d}{du} \otimes \mathrm{id} : \mathbb{C}[u] \otimes \Omega_X^{\bullet}(\log Y) \otimes \mathcal{O}_{Y_k} \longrightarrow \mathbb{C}[u] \otimes \Omega_X^{\bullet}(\log Y) \otimes \mathcal{O}_{Y_k}$$

induces a morphism of complexes $K_{\mathbb{C}} \longrightarrow K_{\mathbb{C}}$, which we denote by d/du for simplicity.

Lemma 2.18. *The morphism d/du induces the monodromy logarithm N on $\mathrm{H}^q(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$ under the identification* (2.7).

Proof. See [7]. □

§ 3. Comparison morphism

In this section, we construct a morphism from Steenbrink's cohomological mixed Hodge complex $A_{\mathbb{C}}$ to the complex $K_{\mathbb{C}}$.

Definition 3.1. The residue morphism

$$\mathrm{Res}_X^k : \Omega_X^{n+1}(\log Y) \longrightarrow \Omega_X^{n-k}(\log Y) \otimes \mathcal{O}_{Y_k}$$

satisfies the condition $\mathrm{Res}_X^k(W_r\Omega_X^{n+1}(\log Y)) = 0$ if $r \leq k$. Therefore the morphism Res_X^k induces a morphism

$$\begin{aligned}\varphi_{k,r} : \Omega_X^{n+1}(\log Y)/W_r\Omega_X^{n+1}(\log Y) \\ \longrightarrow \Omega_X^{n-k}(\log Y) \otimes \mathcal{O}_{Y_k} \simeq u^{[k-r]} \otimes \Omega_X^{n-k}(\log Y) \otimes \mathcal{O}_{Y_k} \subset K_{\mathbb{C}}^n\end{aligned}$$

for every integer r with $0 \leq r \leq k$. Then we have a morphism

$$\varphi = \sum_{0\leq r\leq k} \varphi_{k,r} : A_{\mathbb{C}}^n \longrightarrow K_{\mathbb{C}}^n$$

for every n. Proposition 2.7 implies that this φ is compatible with the differentials on both sides. Thus we obtain a morphism of complexes $\varphi : A_{\mathbb{C}} \longrightarrow K_{\mathbb{C}}$.

Lemma 3.2. *The morphism φ above preserves the filtrations W and F.*

Proof. Since we have

$$\mathrm{Res}_X^k(W_m\Omega_X^{n+1}(\log Y)) \subset W_{m-k-1}\Omega_X^{n-k}(\log Y)\otimes\mathcal{O}_{Y_k},$$

the morphism $\varphi_{k,r}$ satisfies

$$\begin{aligned}\varphi_{k,r}(W_{m+2r+1}\Omega_X^{n+1}(\log Y)/W_r\Omega_X^{n+1}(\log Y))&\\ \subset u^{[k-r]}\otimes W_{m+2r-k}\Omega_X^{n-k}(\log Y)\otimes\mathcal{O}_{Y_k}&\\ = u^{[k-r]}\otimes W_{m-2(k-r)+k}\Omega_X^{n-k}(\log Y)\otimes\mathcal{O}_{Y_k},&\end{aligned}$$

by which we can easily see the conclusion for the filtration W. For the filtration F, we can easily see the conclusion. □

The following is proved in [7].

Proposition 3.3. *The morphism $\varphi : A_{\mathbb{C}} \longrightarrow K_{\mathbb{C}}$ induces an isomorphism of mixed Hodge structures*

$$\mathrm{H}^q(Y,\varphi) : \mathrm{H}^q(Y, A_{\mathbb{C}}) \longrightarrow \mathrm{H}^q(Y, K_{\mathbb{C}})$$

for every integer q.

Proof. The above lemma implies that the morphism $\mathrm{H}^q(Y,\varphi)$ is a morphism of mixed Hodge structures. We can prove that the morphism $\mathrm{H}^q(Y,\varphi)$ is compatible with the isomorphisms (2.4) in Theorem 2.11 and (2.7) in Theorem 2.16. Thus we obtain the conclusion. See [7] for the detail. □

§ 4. Product

In this section, we give the definition of a product on the complex $K_{\mathbb{C}}$. This procedure is known as Alexander-Whitney formula.

Definition 4.1. For non-negative integers k, K with $k \leq K$, morphisms

$$h_{k,K}, t_{k,K} : Y_K \longrightarrow Y_k$$

are defined as follows: the morphism $h_{k,K}$ on the connected component $Y_{\lambda_0\lambda_1\cdots\lambda_K}$ is the closed immersion $Y_{\lambda_0\lambda_1\cdots\lambda_K} \longrightarrow Y_{\lambda_0\lambda_1\cdots\lambda_k}$ and the morphism $t_{k,K}$ on $Y_{\lambda_0\lambda_1\cdots\lambda_K}$ is the closed immersion $Y_{\lambda_0\lambda_1\cdots\lambda_K} \longrightarrow Y_{\lambda_{K-k}\lambda_{K-k+1}\cdots\lambda_K}$. (The symbols h and t are abbreviation of "head" and "tail".)

Definition 4.2. For non-negative integers k, l, p, q, a morphism

$$\Phi_{k,l}^{p,q} : (\mathbb{C}[u] \otimes \Omega_X^p(\log Y) \otimes \mathcal{O}_{Y_k}) \otimes (\mathbb{C}[u] \otimes \Omega_X^q(\log Y) \otimes \mathcal{O}_{Y_l}) \\ \longrightarrow \mathbb{C}[u] \otimes \Omega_X^{p+q}(\log Y) \otimes \mathcal{O}_{Y_{k+l}} \subset K_{\mathbb{C}}^{k+l+p+q}$$

is defined by

$$\Phi_{k,l}^{p,q}((P(u) \otimes \omega) \otimes (Q(u) \otimes \eta)) = P(u)Q(u) \otimes h_{k,k+l}^* \omega \wedge t_{l,k+l}^* \eta$$

for $P(u), Q(u) \in \mathbb{C}[u]$, and for local sections ω of $\Omega_X^p(\log Y) \otimes \mathcal{O}_{Y_k}$, η of $\Omega_X^q(\log Y) \otimes \mathcal{O}_{Y_l}$. Then we define a morphism

$$\Phi = \bigoplus_{k+l+p+q=n} (-1)^{pl} \Phi_{k,l}^{p,q} : (K_{\mathbb{C}} \otimes K_{\mathbb{C}})^n \longrightarrow K_{\mathbb{C}}^n$$

for every integer n. Direct but tedious computation shows that Φ defines a morphism of complexes.

Definition 4.3. The morphisms Φ induces a morphism of cohomology groups

$$\mathrm{H}^{p,q}(Y, \Phi) : \mathrm{H}^p(Y, K_{\mathbb{C}}) \otimes \mathrm{H}^q(Y, K_{\mathbb{C}}) \longrightarrow \mathrm{H}^{p+q}(Y, K_{\mathbb{C}})$$

for every p, q.

Proposition 4.4. *Via the isomorphism* (2.7) *in Theorem* 2.16*, the morphism* $\mathrm{H}^{p,q}(Y, \Phi)$ *coincides with the cup product*

$$\cup : \mathrm{H}^p(Y, \Omega_{X/\Delta}^\bullet(\log Y) \otimes \mathcal{O}_Y) \otimes \mathrm{H}^q(Y, \Omega_{X/\Delta}^\bullet(\log Y) \otimes \mathcal{O}_Y) \\ \longrightarrow \mathrm{H}^{p+q}(Y, \Omega_{X/\Delta}^\bullet(\log Y) \otimes \mathcal{O}_Y) \tag{4.1}$$

for every p, q*, where the cup product* $\cup$ *is the morphism induced from the exterior product on the complex* $\Omega_{X/\Delta}^\bullet(\log Y) \otimes \mathcal{O}_Y$*.*

Lemma 4.5. *The morphism* Φ *satisfies the conditions*

$$\Phi(W_a K_{\mathbb{C}} \otimes W_b K_{\mathbb{C}}) \subset W_{a+b} K_{\mathbb{C}}$$
$$\Phi(F^a K_{\mathbb{C}} \otimes F^b K_{\mathbb{C}}) \subset F^{a+b} K_{\mathbb{C}}$$

for every a, b*. Therefore the morphism* $\mathrm{H}^{p,q}(Y, \Phi)$ *satisfies the conditions*

$$\mathrm{H}^{p,q}(Y, \Phi)(W_a \mathrm{H}^p(Y, K_{\mathbb{C}}) \otimes W_b \mathrm{H}^q(Y, K_{\mathbb{C}})) \subset W_{a+b} \mathrm{H}^{p+q}(Y, K_{\mathbb{C}})$$
$$\mathrm{H}^{p,q}(Y, \Phi)(F^a \mathrm{H}^p(Y, K_{\mathbb{C}}) \otimes F^b \mathrm{H}^q(Y, K_{\mathbb{C}})) \subset F^{a+b} \mathrm{H}^{p+q}(Y, K_{\mathbb{C}})$$

for every a, b, p, q*.*

Proof. Easy by definition. □

§ 5. Trace morphism

We denote by n the relative dimension of the morphism f. Then every connected component of Y_k has dimension $n-k$.

Lemma 5.1. *We have*

$$\begin{aligned} &W_{-1}\mathrm{H}^{2n}(Y, A_{\mathbb{C}}) = 0, \\ &W_0\mathrm{H}^{2n}(Y, A_{\mathbb{C}}) = \mathrm{H}^{2n}(Y, A_{\mathbb{C}}) \simeq \mathbb{C} \end{aligned}$$

for the weight filtration W on $\mathrm{H}^{2n}(Y, A_{\mathbb{C}})$.

Proof. The formula (2.6) implies the equalities

$$\begin{aligned} &E_1^{p,2n-p}(A_{\mathbb{C}}, W) = 0 \quad \text{for } p \neq 0 \\ &E_1^{-1,2n}(A_{\mathbb{C}}, W) \simeq \mathrm{H}^{2n-2}(Y_1, \mathbb{C}) \\ &E_1^{0,2n}(A_{\mathbb{C}}, W) \simeq \mathrm{H}^{2n}(Y_0, \mathbb{C}) \\ &E_1^{1,2n}(A_{\mathbb{C}}, W) = 0 \end{aligned} \tag{5.1}$$

by using the equality $\dim Y_k = n-k$ above. Thus we obtain $E_2^{p,2n-p}(A_{\mathbb{C}}, W) = 0$ for $p \neq 0$. Moreover, we can compute the cokernel of the morphism

$$d_1 : E_1^{-1,2n}(A_{\mathbb{C}}, W) \longrightarrow E_1^{0,2n}(A_{\mathbb{C}}, W)$$

in terms of Gysin morphisms via the identification in (5.1) as in [12], and obtain $E_2^{0,2n}(A_{\mathbb{C}}, W) \simeq \mathbb{C}$. Thus we have the conclusion by E_2-degeneration of the weight spectral sequence. □

Corollary 5.2. *We have*

$$\begin{aligned} &W_{-1}\mathrm{H}^{2n}(Y, K_{\mathbb{C}}) = 0 \\ &W_0\mathrm{H}^{2n}(Y, K_{\mathbb{C}}) = \mathrm{H}^{2n}(Y, K_{\mathbb{C}}) \simeq \mathbb{C} \end{aligned}$$

for the weight filtration W on $\mathrm{H}^{2n}(Y, K_{\mathbb{C}})$. Moreover, we have an exact sequence

$$E_1^{-1,2n}(K_{\mathbb{C}}, W) \xrightarrow{d_1} E_1^{0,2n}(K_{\mathbb{C}}, W) \longrightarrow \mathrm{H}^{2n}(Y, K_{\mathbb{C}}) \longrightarrow 0 \tag{5.2}$$

for the cohomology group $\mathrm{H}^{2n}(Y, K_{\mathbb{C}})$.

Proof. Since two mixed Hodge structures $\mathrm{H}^{2n}(Y, A_{\mathbb{C}})$ and $\mathrm{H}^{2n}(Y, K_{\mathbb{C}})$ are isomorphic by Proposition 3.3, we obtain the first half of the conclusions. The formula (2.8) tells us $E_1^{1,2n}(K_{\mathbb{C}}, W) = 0$ by considering the dimension of $Y_{\underline{\lambda}}$. Thus we obtain the latter half. □

The morphisms

$$\mathrm{id} \otimes (\mathrm{dlog}\, t\wedge) : \mathbb{C}[u] \otimes \Omega_X^{\bullet}(\log Y) \otimes \mathcal{O}_{Y_k} \longrightarrow \mathbb{C}[u] \otimes \Omega_X^{\bullet}(\log Y) \otimes \mathcal{O}_{Y_k}[1]$$

for non-negative integers k induce a morphism of complexes $K_{\mathbb{C}} \longrightarrow K_{\mathbb{C}}[1]$, which we denote by $\mathrm{dlog}\, t\wedge$. We can easily check the condition $(\mathrm{dlog}\, t\wedge)(W_m K_{\mathbb{C}}) \subset W_{m+1} K_{\mathbb{C}}[1]$ for every integer m. Therefore we have a morphism of complexes

$$\mathrm{Gr}_m^W K_{\mathbb{C}} \longrightarrow \mathrm{Gr}_m^W K_{\mathbb{C}}[1]$$

for every m, which we denote by $\mathrm{dlog} \wedge$ again. Thus a morphism

$$\mathrm{H}^{2n}(Y, \mathrm{dlog}\, t\wedge) : E_1^{0,2n}(K_{\mathbb{C}}, W) = \mathrm{H}^{2n}(Y, \mathrm{Gr}_0^W K_{\mathbb{C}}) \longrightarrow \mathrm{H}^{2n+1}(Y, \mathrm{Gr}_1^W K_{\mathbb{C}})$$

is obtained.

Lemma 5.3. *We have an isomorphism*

$$\mathrm{H}^{2n+1}(Y, \mathrm{Gr}_1^W K_{\mathbb{C}}) \simeq \bigoplus_{k\geq 0} \mathrm{H}^{2n-2k}(Y_k, \Omega_{Y_k}^{\bullet}).$$

Proof. See [7]. □

Via the isomorphism in the lemma above, we obtain a morphism

$$\bigoplus_{k\geq 0} \epsilon(k+1)(2\pi\sqrt{-1})^{k-n} \int_{Y_k} : \mathrm{H}^{2n+1}(Y, \mathrm{Gr}_1^W K_{\mathbb{C}}) \longrightarrow \mathbb{C},$$

where we set $\epsilon(a) = (-1)^{a(a-1)/2}$ for every integer a.

Definition 5.4. By setting

$$\Theta = (\bigoplus_{k\geq 0} \epsilon(k+1)(2\pi\sqrt{-1})^{k-n} \int_{Y_k}) \cdot \mathrm{H}^{2n}(Y, \mathrm{dlog}\, t\wedge),$$

the morphism $\Theta : E_1^{0,2n}(K_{\mathbb{C}}, W) \longrightarrow \mathbb{C}$ is defined.

The following is a key result in this article:

Theorem 5.5. *There exists a morphism*

$$\mathrm{Tr} : \mathrm{H}^{2n}(Y, K_{\mathbb{C}}) \longrightarrow \mathbb{C}$$

which fits in the commutative diagram

$$\begin{array}{ccc} E_1^{0,2n}(K_{\mathbb{C}}, W) & \xrightarrow{\Theta} & \mathbb{C} \\ \big\downarrow & & \big\| \\ \mathrm{H}^{2n}(Y, K_{\mathbb{C}}) & \xrightarrow[\mathrm{Tr}]{} & \mathbb{C}, \end{array}$$

where the left vertical arrow denotes the surjection in the exact sequence (5.2).

Proof. We can prove the condition $\Theta \cdot d_1 = 0$ by computing the morphism d_1 in terms of Gysin morphisms as in [12]. This implies the conclusion by the exact sequence (5.2). □

§ 6. Polarization

In this final section, we use the identifications

$$\mathrm{H}^q(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \simeq \mathrm{H}^q(Y, K_{\mathbb{C}}) \simeq \mathrm{H}^q(Y, A_{\mathbb{C}})$$

(2.5) and (2.7) freely for every integer q. Because $\mathrm{H}^q(Y, K_{\mathbb{C}}) \simeq \mathrm{H}^q(Y, A_{\mathbb{C}})$ is an isomorphism of mixed Hodge structures by Proposition 3.3, the filtrations W, F on $\mathrm{H}^q(Y, K_{\mathbb{C}})$ and $\mathrm{H}^q(Y, A_{\mathbb{C}})$ induce the same filtrations on $\mathrm{H}^q(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$. Thus we obtain filtrations W and F on $\mathrm{H}^q(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$ such that the triple

$$(\mathrm{H}^q(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y), W[q], F)$$

is (the $\mathbb{C}$-structure of) a mixed Hodge structure. The monodromy logarithm N is induced by the morphism ν via the identification (2.5) and by the morphism d/du via (2.7). We can easily see that the morphism N is a morphism of mixed Hodge structures of type $(-1, -1)$.

As a consequence of Section 4 the cup product (4.1) satisfies the properties

$$\begin{aligned} W_a\mathrm{H}^p(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \cup W_b\mathrm{H}^q(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \\ \subset W_{a+b}\mathrm{H}^{p+q}(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \\ F^a\mathrm{H}^p(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \cup F^b\mathrm{H}^q(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \\ \subset F^{a+b}\mathrm{H}^{p+q}(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \end{aligned}$$

for every a, b, p, q.

On the other hand, the morphism

$$\mathrm{Tr} : \mathrm{H}^{2n}(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \longrightarrow \mathbb{C}$$

is defined in Section 5.

Definition 6.1. The pairing

$$Q : \mathrm{H}^q(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \otimes \mathrm{H}^{2n-q}(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \longrightarrow \mathbb{C}$$

is defined by the formula $Q(x, y) = \epsilon(q)\,\mathrm{Tr}(x \cup y)$ for $x \in \mathrm{H}^q(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$ and for $y \in \mathrm{H}^{2n-q}(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$.

Lemma 6.2. *We have*

$$Q(W_a\mathrm{H}^q(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y), W_b\mathrm{H}^{2n-q}(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)) = 0$$

if $a + b \leq -1$.

Proof. Because we have

$$\begin{aligned} W_a\mathrm{H}^q(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \cup W_b\mathrm{H}^{2n-q}(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) & \\ \subset W_{a+b}\mathrm{H}^{2n}(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) & \\ \subset W_{-1}\mathrm{H}^{2n}(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) = 0 & \end{aligned}$$

if $a + b \leq -1$, we obtain the conclusion. □

Definition 6.3. By the above lemma, the pairing Q induces a pairing

$$\mathrm{Gr}^W_m\,\mathrm{H}^q(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \otimes \mathrm{Gr}^W_{-m}\,\mathrm{H}^{2n-q}(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \longrightarrow \mathbb{C}$$

for every integer m, which is denoted by the same letter Q.

Definition 6.4. The morphism $\mathrm{dlog} : \mathcal{O}^*_X \longrightarrow \Omega^1_X$ is given by sending $g \in \mathcal{O}^*_X$ to the 1-form $\mathrm{dlog}\, g = dg/g$. This defines a morphism of complexes $\mathcal{O}^*_X \longrightarrow \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y[1]$ denoted by dlog again. Thus the morphism

$$\mathrm{H}^1(X, \mathrm{dlog}) : \mathrm{H}^1(X, \mathcal{O}^*_X) \longrightarrow \mathrm{H}^2(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$$

is obtained.

Since the morphism f is projective, there exists a relatively ample line bundle $\mathcal{L}$ on X. The isomorphism class of $\mathcal{L}$ is denoted by $[\mathcal{L}]$, which is an element of $\mathrm{H}^1(X, \mathcal{O}^*_X)$. Then the image of $[\mathcal{L}]$ by the morphism $\mathrm{H}^1(X, \mathrm{dlog})$ is denoted by $[\omega]$, which is an element of $\mathrm{H}^2(Y, \Omega^{\bullet}_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$.

Definition 6.5. The morphism

$$l : \mathrm{H}^p(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y) \longrightarrow \mathrm{H}^{p+2}(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$$

is defined by taking cup product with $-[\omega]$, that is, $l(x) = -[\omega] \cup x$ for an element x of $\mathrm{H}^p(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$. It is easy to see that the morphism l is a morphism of mixed Hodge structures of type $(1,1)$.

Remark. We remark that the usual Chern class, which is defined by using the exact sequence involving the exponential map, is different from the above by the sign (see Deligne [3]). This is the reason why we take $-[\omega]$ instead of $[\omega]$.

Definition 6.6. We set

$$L^{i,j} = \mathrm{Gr}^W_{-i} \mathrm{H}^{n+j}(Y, \Omega^\bullet_{X/\Delta}(\log Y) \otimes \mathcal{O}_Y)$$

for every integer i, j, which is (the $\mathbb{C}$-structure of) a Hodge structure of weight $n+j-i$. The monodromy logarithm N induces a morphism $L^{i,j} \longrightarrow L^{i+2,j}$, which is denoted by l_1, and the morphism l in Definition 6.5 induces a morphism $l_2 : L^{i,j} \longrightarrow L^{i,j+2}$ for every i, j. Moreover we set $L = \bigoplus_{i,j} L^{i,j}$. By setting

$$\psi(x,y) = \begin{cases} (-1)^{n+j-i} Q(x,y) & \text{if } x \in L^{-i,-j}, y \in L^{i,j}, \\ 0 \qquad \text{otherwise,} \end{cases}$$

a bilinear form

$$\psi : L \otimes L \longrightarrow \mathbb{C}$$

is defined.

Now we state the main result of this article:

Theorem 6.7. *The quadruple (L, l_1, l_2, ψ) is a polarized bigraded Hodge-Lefschetz module in the sense of Guillén-Navarro Aznar* [8].

Proof. By using the comparison morphism $\varphi : A_{\mathbb{C}} \longrightarrow K_{\mathbb{C}}$ and by the careful computation on the sign in question, we can prove that these data (L, l_1, l_2, ψ) coincide with the data induced from the E_1-terms of the weight spectral sequence associated to $(A_{\mathbb{C}}, W)$, which are treated in [15] and in [8]. Thus Theorem 4.5 in [8] implies the conclusion. □

Remark. Once the above theorem is established, we can obtain a polarization by the standard procedure. We remark that the primitive part of the morphism l commutes with taking Gr^W because l is a morphism of mixed Hodge structures of type $(1,1)$.

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