Wildness of polynomial automorphisms: Applications of the Shestakov-Umirbaev theory and its generalization

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1 Introduction

For each integral domain R, we denote by $R[\mathbf{x}] = R[x_1, \dots, x_n]$ the polynomial ring in n variables over R, where $n \in \mathbf{N}$, and $\mathbf{x} = \{x_1, \dots, x_n\}$ is a set of variables. For an R-subalgebra A of $R[\mathbf{x}]$, we consider the automorphism group $\mathrm{Aut}(R[\mathbf{x}]/A)$ of the ring $R[\mathbf{x}]$ over A. We say that $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$ is *affine* if $\deg \phi(x_i) = 1$ for $i = 1, \dots, n$, and *elementary* if ϕ belongs to $\mathrm{Aut}(R[\mathbf{x}]/A_i)$ for some i, where $A_i := R[\mathbf{x} \setminus \{x_i\}]$. Here, $\deg f$ denotes the total degree of f for each $f \in R[\mathbf{x}]$. Note that, if ϕ is affine, then we have $(\phi(x_1), \dots, \phi(x_n)) = (x_1, \dots, x_n)A + (b_1, \dots, b_n)$ for some $A \in GL_n(R)$ and $b_1, \dots, b_n \in R$. If ϕ is elementary, then there exist $i \in \{1, \dots, n\}$, $\alpha \in R^{\times}$ and $f \in A_i$ such that $\phi(x_i) = \alpha x_i + f$ and $\phi(x_j) = x_j$ for $j \neq i$. We denote by $\mathrm{Aff}(R, \mathbf{x})$, $\mathrm{E}(R, \mathbf{x})$, and $\mathrm{T}(R, \mathbf{x})$, the subgroups of $\mathrm{Aut}(R[\mathbf{x}]/R)$ generated by all the affine automorphisms, all the elementary automorphisms, and $\mathrm{Aff}(R, \mathbf{x}) \cup \mathrm{E}(R, \mathbf{x})$, respectively. An element of $\mathrm{Aut}(R[\mathbf{x}]/R)$ is sometimes said to be *tame* if it belongs to $\mathrm{T}(R, \mathbf{x})$, and *wild* otherwise.

The following is a fundamental problem in polynomial ring theory.

Tame Generators Problem. When is $\mathrm{T}(R, \mathbf{x})$ equal to $\mathrm{Aut}(R[\mathbf{x}]/R)$?

The equality holds true if $n = 1$, in which case every element of $\mathrm{Aut}(R[\mathbf{x}]/R)$ is affine and elementary.

When $n = 2$, the following result is well-known.

Theorem 1.1. *Assume that $n = 2$, and R is an integral domain. Then,* $\mathrm{T}(R, \mathbf{x})$ *is equal to* $\mathrm{Aut}(R[\mathbf{x}]/R)$ *if and only if R is a field.*

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Here, the "if" part of the above theorem is due to Jung [8] in the case where R is of characteristic zero, and to van der Kulk [9] in the general case. The "only if" part of the above theorem is rather easy (cf. [3, Proposition 5.1.9]).

Throughout this report, we denote by k an arbitrary field of characteristic zero. When $n = 3$, Shestakov-Umirbaev [21] gave a criterion to decide whether a given element of $\mathrm{Aut}(k[\mathbf{x}]/k)$ belongs to $\mathrm{T}(k, \mathbf{x})$. As a consequence, they showed the following theorem ([21, Corollary 10]).

Theorem 1.2 (Shestakov-Umirbaev). $\mathrm{Aut}(k[\mathbf{x}]/k[x_3]) \cap \mathrm{T}(k, \mathbf{x}) = \mathrm{T}(k[x_3], \{x_1, x_2\})$.

Since some automorphisms, including the famous automorphism of Nagata [16], belong to $\mathrm{Aut}(k[\mathbf{x}]/k[x_3])$, but do not belong to $\mathrm{T}(k[x_3], \{x_1, x_2\})$, it was concluded that $\mathrm{T}(k, \mathbf{x})$ is not equal to $\mathrm{Aut}(k[\mathbf{x}]/k)$. At present, the Tame Generators Problem is not solved in the cases where $n \geq 4$, and where $n = 3$ and the field of fractions of R is of positive characteristic.

Recently, the author [10], [11] reconstructed and generalized the theory of Shestakov-Umirbaev. This improvement makes it possible to decide more easily and efficiently whether a given element of $\mathrm{Aut}(k[\mathbf{x}]/k)$ belongs to $\mathrm{T}(k, \mathbf{x})$ when $n = 3$.

The purposes of this report is to announce some recent results obtained as consequences of the Shestakov-Umirbaev theory and its generalization. For details, we refer to our preprints [12], [13] and [14]. This series of papers (with a total of nearly hundred pages) presents various applications of these theories.

In Sections 2, 4, 5 and 6 of this report, we explain the main results of [12]. These results are derived from Theorem 1.2. To illustrate the usefulness of Theorem 1.2, in Section 3, we show the wildness of some concrete automorphisms by means of a criterion derived from this theorem. Sections 7 and 8, and 9 summarize the main results of [13] and [14]. These papers contain strong results obtained as highly technical applications of the generalized Shestakov-Umirbaev theory.

2 Affine reductions and elementary reductions

Let Γ be a finitely generated ordered additive group, and $\mathbf{w} = (w_1, \ldots, w_n)$ an n-tuple of elements of Γ with $\mathbf{w} \neq (0, \ldots, 0)$ and $w_i \geq 0$ for $i = 1, \ldots, n$. For each nonzero polynomial

$$f = \sum_{i_1, \ldots, i_n} \lambda_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n} \in R[\mathbf{x}],$$

we define the $\mathbf{w}$-*degree* $\deg_{\mathbf{w}} f$ of f to be the maximum among $\sum_{l=1}^{n} i_l w_l$ for $i_1, \ldots, i_n$ with $\lambda_{i_1, \ldots, i_n} \neq 0$, where $\lambda_{i_1, \ldots, i_n} \in R$ for each $i_1, \ldots, i_l$. We define $f^{\mathbf{w}}$ to be the sum of

$\lambda_{i_1,\ldots,i_n} x_1^{i_1}\cdots x_n^{i_n}$ for $i_1,\ldots,i_n$ such that $\sum_{l=1}^{n} i_l w_l = \deg_{\mathbf{w}} f$. When $f=0$, we set $f^{\mathbf{w}}=0$ and $\deg_{\mathbf{w}} f = -\infty$, i.e., a symbol which is less than any element of Γ. Then, for each $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$, it holds that

$$\deg_{\mathbf{w}} \phi := \sum_{i=1}^{n} \deg_{\mathbf{w}} \phi(x_i) \geq \sum_{i=1}^{n} w_i := |\mathbf{w}|. \tag{2.1}$$

If $n=2$, then $\deg_{\mathbf{w}} \phi = |\mathbf{w}|$ implies that ϕ belongs to $\mathrm{T}(R,\mathbf{x})$ (see [12, Section 2] for detail).

Now, we consider two kinds of reductions for $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$. We say that ϕ admits an *affine reduction* for the weight $\mathbf{w}$ if there exists $\alpha \in \mathrm{Aff}(R,\mathbf{x})$ such that $\deg_{\mathbf{w}} \phi\circ\alpha < \deg_{\mathbf{w}} \phi$. We say that ϕ admits an *elementary reduction* for the weight $\mathbf{w}$ if there exists $\epsilon \in \mathrm{Aut}(R[\mathbf{x}]/A_i)$ for some i such that $\deg_{\mathbf{w}} \phi\circ\epsilon < \deg_{\mathbf{w}} \phi$.

Assume that $n=2$ and $w_i \geq 0$ for $i=1,2$. Then, we have $|\mathbf{w}| = w_1+w_2 > 0$ by the assumption that $(w_1,w_2)\neq(0,0)$. Hence, for each $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$, it follows from (2.1) that $\deg_{\mathbf{w}} \phi(x_1) > 0$ or $\deg_{\mathbf{w}} \phi(x_2) > 0$. Let $V(R)$ be the set of $a/b \in K$ for $a,b \in R\setminus\{0\}$ such that $aR+bR=R$, where K is the field of fractions of R. Note that $R\setminus\{0\}$ is contained in $V(R)$, and $V(R)$ is contained in $K^\times$. If R is a PID, then we have $V(R)=K^\times$. By definition, ϕ admits an affine reduction if and only if there exist $a,b,c,d,s,t \in R$ with $ad-bc \in R^\times$ such that

$$\deg_{\mathbf{w}}(a\phi(x_1)+b\phi(x_2)+s)+\deg_{\mathbf{w}}(c\phi(x_1)+d\phi(x_2)+t) < \deg_{\mathbf{w}} \phi(x_1)+\deg_{\mathbf{w}} \phi(x_2).$$

Since $\deg_{\mathbf{w}} \phi(x_1) > 0$ or $\deg_{\mathbf{w}} \phi(x_2) > 0$, this is equivalent to that $a\phi(x_1)^{\mathbf{w}}+b\phi(x_2)^{\mathbf{w}}=0$ or $c\phi(x_1)^{\mathbf{w}}+d\phi(x_2)^{\mathbf{w}}=0$, and is equivalent to that $\phi(x_1)^{\mathbf{w}} = u\phi(x_2)^{\mathbf{w}}$ for some $u \in V(R)$. In particular, we have $\deg_{\mathbf{w}} \phi(x_1) = \deg_{\mathbf{w}} \phi(x_2)$ whenever ϕ admits an affine reduction for the weight $\mathbf{w}$.

Note that ϕ admits an elementary reduction if and only if there exists $f \in R[\phi(x_j)]$ such that $\deg_{\mathbf{w}}(\phi(x_i)-f) < \deg_{\mathbf{w}} \phi(x_i)$ for some $(i,j) \in \{(1,2),(2,1)\}$. Since $\phi(x_i)-f \neq 0$ and $w_l \geq 0$ for $l=1,2$, we have $\deg_{\mathbf{w}}(\phi(x_i)-f) \geq 0$, and hence $\deg_{\mathbf{w}} \phi(x_i) > 0$. It follows that $\deg_{\mathbf{w}} f > 0$, and so $\deg_{\mathbf{w}} \phi(x_j) > 0$. Thus, $f^{\mathbf{w}}$ must be of the form $c(\phi(x_j)^{\mathbf{w}})^l$ for some $c \in R\setminus\{0\}$ and $l \in \mathbf{N}$. Therefore, it holds that $\deg_{\mathbf{w}}(\phi(x_i)-f) < \deg_{\mathbf{w}} \phi(x_i)$ for some $f \in k[\phi(x_j)]$ if and only if $\phi(x_i)^{\mathbf{w}} = c(\phi(x_j)^{\mathbf{w}})^l$ for some $c \in R\setminus\{0\}$ and $l \in \mathbf{N}$.

The following is a basic result on tameness of elements of $\mathrm{Aut}(R[\mathbf{x}]/R)$ for $n=2$. In the case of $\mathbf{w}=(1,1)$, the result is commonly known (cf. [7, Proposition 1]).

Proposition 2.1 ([12, Proposition 3.2])**.** *Assume that $n=2$, and $\mathbf{w} := (w_1,w_2) \in \Gamma^2$ is such that $\mathbf{w} \neq (0,0)$ and $w_i \geq 0$ for $i=1,2$. If $\deg_{\mathbf{w}} \phi > |\mathbf{w}|$ holds for $\phi \in \mathrm{T}(R,\mathbf{x})$, then ϕ admits an affine reduction or elementary reduction for the weight $\mathbf{w}$.*

Next, we recall the notion of coordinate. We call $f \in R[\mathbf{x}]$ a *coordinate* of $R[\mathbf{x}]$ over R if f is equal to $\phi(x_1)$ for some $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$, which is said to be *tame* if ϕ can be taken from $\mathrm{T}(R, \mathbf{x})$, and *wild* otherwise. Let S be an integral domain containing R as a subring. Then, we may regard $\mathrm{Aut}(R[\mathbf{x}]/R)$ as a subgroup of $\mathrm{Aut}(S[\mathbf{x}]/S)$ by identifying $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$ with the automorphism $\mathrm{id}_S \otimes \phi$ of $S \otimes_R R[\mathbf{x}] \simeq S[\mathbf{x}]$ over S. Hence, every coordinate of $R[\mathbf{x}]$ over R is a coordinate of $S[\mathbf{x}]$ over S. On the other hand, not every coordinate of $S[\mathbf{x}]$ over S is a coordinate of $R[\mathbf{x}]$ over R. When $n = 2$, we say that a coordinate f of $S[\mathbf{x}]$ over S is *reduced* over R if

$$\deg_{x_1} \tau(f) + \deg_{x_2} \tau(f) \geq \deg_{x_1} f + \deg_{x_2} f$$

holds for every $\tau \in \mathrm{T}(R, \mathbf{x})$.

For $f \in R[\mathbf{x}]$, we consider the subgroup

$$H(f) := \mathrm{Aut}(R[\mathbf{x}]/R[f]) \cap \mathrm{T}(R, \mathbf{x})$$

of $\mathrm{Aut}(R[\mathbf{x}]/R)$.

The following theorem is a consequence of Proposition 2.1.

Theorem 2.2 ([12, Theorem 4.3]). *Assume that $n = 2$. Let $R \subset S$ be an extension of integral domains, and $f \in R[\mathbf{x}]$ a coordinate of $S[\mathbf{x}]$ over S which is reduced over R.*
(i) *If $\deg_{x_1} f = \deg_{x_2} f$, then $H(f)$ is contained in $\mathrm{Aff}(R, \mathbf{x})$.*
(ii) *If $\deg_{x_i} f < \deg_{x_j} f$ for $(i, j) \in \{(1, 2), (2, 1)\}$, then $H(f)$ is contained in $\mathrm{J}(R; x_j, x_i)$. If $\deg_{x_i} f = 0$, then $H(f) = \mathrm{Aut}(R[\mathbf{x}]/R[x_j])$.*

Here, for a permutation $x_{i_1}, \ldots, x_{i_n}$ of $x_1, \ldots, x_n$, we denote by $\mathrm{J}(R; x_{i_1}, \ldots, x_{i_n})$ the set of $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$ such that $\phi(x_{i_l})$ belongs to $R[x_{i_1}, \ldots, x_{i_l}]$ for $l = 1, \ldots, n$. Note that $\mathrm{J}(R; x_{i_1}, \ldots, x_{i_n})$ forms a subgroup of $\mathrm{T}(R, \mathbf{x})$ consists of the automorphisms ϕ of the form $\phi(x_{i_l}) = a_l x_{i_l} + h_l$ for $l = 1, \ldots, n$, where $a_l \in R^\times$ and $h_l \in R[x_{i_1}, \ldots, x_{i_{l-1}}]$.

Results explained in Sections 4, 5 and 6 are derived from Theorems 1.2 and 2.2.

3 An easy criterion for wildness

The following corollary is an immediate consequence of Theorem 1.2, and Proposition 2.1 applied with $R = k[x_3]$.

Corollary 3.1. *Assume that $n = 3$. Then, $\phi \in \mathrm{Aut}(k[\mathbf{x}]/k[x_3])$ does not belong to $\mathrm{T}(k, \mathbf{x})$ if there exist $w_1, w_2 \in \Gamma$ with $(w_1, w_2) \neq (0, 0)$ and $w_i \geq 0$ for $i = 1, 2$ such that the following conditions hold for $\mathbf{w} := (w_1, w_2, 0)$:*
(i) $\deg_{\mathbf{w}} \phi > |\mathbf{w}|$.
(ii) *There exists $(i, j) \in \{(1, 2), (2, 1)\}$ such that $\deg_{\mathbf{w}} \phi(x_i) < \deg_{\mathbf{w}} \phi(x_j)$, and $\phi(x_j)^{\mathbf{w}}$ is not equal to $c(\phi(x_i)^{\mathbf{w}})^l$ for any $c \in k[x_3]$ and $l \in \mathbf{N}$.*

Proof. Suppose to the contrary that ϕ belongs to $\mathrm{T}(k, \mathbf{x})$. Then, ϕ belongs to $\mathrm{T}(k[x_3], \{x_1, x_2\})$ by Theorem 1.2, since ϕ is an element of $\mathrm{Aut}(k[\mathbf{x}]/k[x_3])$ by assumption. Regard $k[\mathbf{x}]$ as the polynomial ring in x_1 and x_2 over $k[x_3]$, where we consider the weight $\mathbf{w}' := (w_1, w_2)$. Then, $\deg_{\mathbf{w}'} m = i_1 w_1 + i_2 w_2 = \deg_{\mathbf{w}} m$ holds for each monomial $m = x_1^{i_1} x_2^{i_2} x_3^{i_3}$. Hence, we get $\deg_{\mathbf{w}'} f = \deg_{\mathbf{w}} f$ and $f^{\mathbf{w}'} = f^{\mathbf{w}}$ for each $f \in k[\mathbf{x}]$. It follows that $\deg_{\mathbf{w}'} \phi = \deg_{\mathbf{w}} \phi$, and is greater than $|\mathbf{w}| = |\mathbf{w}'|$ by (i). By Proposition 2.1, we know that ϕ admits an affine reduction or elementary reduction for the weight $\mathbf{w}'$ as an automorphism of the polynomial ring in x_1 and x_2 over $k[x_3]$. On the other hand, since $\deg_{\mathbf{w}'} \phi(x_l) = \deg_{\mathbf{w}} \phi(x_l)$ and $\phi(x_l)^{\mathbf{w}'} = \phi(x_l)^{\mathbf{w}}$ for $l = 1, 2$, the condition (ii) implies that ϕ does not admit an affine reduction or elementary reduction for the weight $\mathbf{w}'$. This is a contradiction. Therefore, ϕ does not belong to $\mathrm{T}(k, \mathbf{x})$. $\square$

For example, consider Nagata's automorphism [16] given by

$$\phi(x_1) = x_1 - 2(x_1 x_3 + x_2^2) x_2 - (x_1 x_3 + x_2^2)^2 x_3, \quad \phi(x_2) = x_2 + (x_1 x_3 + x_2^2) x_3$$

and $\phi(x_3) = x_3$. Let Γ be the additive group $\mathbf{Z}^2$ equipped with the lexicographic order with $\mathbf{e}_1 > \mathbf{e}_2$, where $\mathbf{e}_1 := (1, 0)$ and $\mathbf{e}_2 := (0, 1)$. Then, for $\mathbf{w} = (\mathbf{e}_1, \mathbf{e}_2, 0)$, we have

$$\deg_{\mathbf{w}} \phi(x_1) = 2\mathbf{e}_1, \quad \deg_{\mathbf{w}} \phi(x_2) = \mathbf{e}_1, \quad \phi(x_1)^{\mathbf{w}} = -x_1^2 x_3^3, \quad \phi(x_2)^{\mathbf{w}} = x_1 x_3^2.$$

One easily checks that (i) and (ii) in Corollary 3.1 are satisfied. Thus, ϕ does not belong to $\mathrm{T}(k, \mathbf{x})$.

Ohta [17, Theorem 3] gave two kinds of automorphisms, one of which is defined by

$$\phi_1(x_1) = \big(g_1(x_3, f_1 + x_1 x_3^3) - x_2\big) x_3^{-3}, \quad \phi_1(x_2) = f_1 + x_1 x_3^3, \quad \phi_1(x_3) = x_3,$$

where f_1 is a *certain* element of $k[x_2, x_3]$, and $g_1(x, y)$ is a polynomial in x and y over k of the form $3x^2 y^5 +$ (terms of lower degree in y). Here, x_1, x_2 and x_3 are denoted by z, y and x, respectively, in the original text. For the same Γ and $\mathbf{w}$ as above, we have

$$\deg_{\mathbf{w}} \phi_1(x_1) = 5\mathbf{e}_1 + 14\mathbf{e}_3, \ \deg_{\mathbf{w}} \phi_1(x_2) = \mathbf{e}_1 + 3\mathbf{e}_3, \ \phi_1(x_1)^{\mathbf{w}} = 3x_1^5 x_3^{14}, \ \phi_1(x_2)^{\mathbf{w}} = x_1 x_3^3.$$

It is easy to check that (i) and (ii) in Corollary 3.1 are satisfied. Hence, we conclude that ϕ_1 does not belong to $\mathrm{T}(k, \mathbf{x})$. As this example shows, we can sometimes decide the wildness of $\phi \in \mathrm{Aut}(k[\mathbf{x}]/k)$ from only partial information on $\phi(x_1)$, $\phi(x_2)$ and $\phi(x_3)$. Tameness of another automorphism of Ohta is determined at the end of the next section.

4 Triangular derivation

Let D be a *locally nilpotent derivation* of $R[\mathbf{x}]$ over R, i.e., an element of $\mathrm{Der}_R R[\mathbf{x}]$ such that $D^l(f) = 0$ holds for some $l \in \mathbf{N}$ for each $f \in R[\mathbf{x}]$. When R contains $\mathbf{Q}$, an element

$\exp D$ of $\mathrm{Aut}(R[\mathbf{x}]/R)$ is defined by

$$(\exp D)(f) = \sum_{i \geq 0} \frac{D^i(f)}{i!}$$

for each $f \in R[\mathbf{x}]$. We say that $D \in \mathrm{Der}_R R[\mathbf{x}]$ is *triangular* if $D(x_i)$ belongs to $R[x_1, \ldots, x_{i-1}]$ for each i. If D is triangular, then D is locally nilpotent, and $(\exp D)(x_i) = x_i + f_i$ for each i, where $f_i \in R[x_1, \ldots, x_{i-1}]$. Hence, $\exp D$ belongs to $\mathrm{J}(R, x_1, \ldots, x_n)$, and so belongs to $\mathrm{T}(R, \mathbf{x})$. For $D \in \mathrm{Der}_R R[\mathbf{x}]$ and $h \in R[\mathbf{x}]$, it is well-known that hD is a locally nilpotent derivation of $R[\mathbf{x}]$ if and only if D is a locally nilpotent derivation of $R[\mathbf{x}]$, and h belongs to $\ker D$ (cf. [3, Corollary 1.3.34]). Even if D is triangular, hD is not always triangular, and so $\exp hD$ may not belong to $\mathrm{T}(k, \mathbf{x})$ for $h \in \ker D \setminus k$. For instance, Nagata's automorphism is wild, and is of the form $\exp hD$, where

$$h = x_1 x_3 + x_2^2, \quad D = -2x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}.$$

This derivation is triangular if x_1 and x_3 are interchanged.

Thus, the following problem arises.

Problem. Assume that $n = 3$. Let D be a triangular derivation of $k[\mathbf{x}]$, and h an element of $\ker D \setminus k$. When does $\exp hD$ belong to $\mathrm{T}(k, \mathbf{x})$?

We completely settle this problem as a consequence of a more general result as follows. Let R be a $\mathbf{Q}$-domain, and D a triangular derivation of $R[x_1, x_2]$ such that

$$D(x_1) = a \qquad D(x_2) = \sum_{i=0}^{l} b_i x_1^i,$$

where $l \geq 0$, and $a, b_0, \ldots, b_l \in R$ with $a \neq 0$ and $b_l \neq 0$. Set

$$I = \{i \geq 0 \mid b_i \notin aR\}, \quad I' = \{1, \ldots, l\} \setminus I,$$

and define $\tau \in \mathrm{Aut}(R[x_1, x_2]/R)$ by

$$\tau(x_1) = x_1, \quad \tau(x_2) = x_2 + \sum_{i \in I'} \frac{b_i}{(i+1)a} x_1^{i+1}.$$

With this notation, we have the following theorem.

Theorem 4.1 ([12, Theorem 6.1]). *Let D be as above, and h an element of $\ker D \setminus R$. Then, $\exp hD$ belongs to $\mathrm{T}(R, \{x_1, x_2\})$ if and only if one of the following conditions holds*:
(i) *$I = \emptyset$.*
(ii) *$I = \{0\}$, and b_0/a belongs to $V(R)$ or $\deg \tau(h) = 1$.*
In particular, when $V(R) = K^\times$, it follows that $\exp hD$ belongs to $\mathrm{T}(R, \{x_1, x_2\})$ if and only if $I = \emptyset$ or $I = \{0\}$, where K is the field of fractions of R.

Applying Theorem 4.1 with $R = k[x_3]$, we get the following theorem with the aid of Theorem 1.2.

Theorem 4.2 ([12, Theorem 6.2])**.** *Assume that* $n = 3$*. Let* D *be a triangular derivation of* $k[\mathbf{x}]$ *with* $D(x_1) = 0$ *and* $D(x_2) \neq 0$*, and* h *an element of* $\ker D \setminus k[x_1]$*. Then,* $\exp hD$ *belongs to* $\mathrm{T}(k, \mathbf{x})$ *if and only if* $\partial D(x_3)/\partial x_2$ *belongs to* $D(x_2)k[x_1, x_2]$*.*

Note that $\partial D(x_3)/\partial x_2$ belongs to $D(x_2)k[x_1, x_2]$ if and only if the coefficient of x_2^i in $D(x_3)$ is divisible by $D(x_2)$ in $k[x_1]$ for each $i \geq 1$, where we regard $D(x_3)$ as a polynomial in x_2 over $k[x_1]$.

If $D(x_1) = 0$ and h belongs to $k[x_1]$, then hD is triangular, and so $\exp hD$ is tame. If $D(x_1) = D(x_2) = 0$, or if $D(x_1) \neq 0$, then it is easy to check that $\exp hD$ is tame for every $h \in \ker D$ (see [12, Section 6] for detail). Therefore, we have completely answer the problem above.

Now, for $f \in k[x_1] \setminus \{0\}$ and $g \in k[x_1, x_2]$, we define a triangular derivation $T_{f,g}$ of $k[\mathbf{x}]$ by

$$T_{f,g}(x_1) = 0, \quad T_{f,g}(x_2) = f, \quad T_{f,g}(x_3) = -\frac{\partial g}{\partial x_2}.$$

Then, we have $T_{f,g}(fx_3 + g) = 0$, so $hT_{f,g}$ is a locally nilpotent derivation of $k[\mathbf{x}]$ for each $h \in k[x_1, fx_3 + g]$. By Theorem 4.2, it follows that $\Phi_{f,g}^h := \exp hT_{f,g}$ belongs to $\mathrm{T}(k, \mathbf{x})$ if and only if $\partial T_{f,g}(x_3)/\partial x_2 = -\partial^2 g/\partial x_2^2$ belongs to $T_{f,g}(x_2)k[x_1, x_2] = fk[x_1, x_2]$ for each $h \in k[x_1, fx_3 + g] \setminus k[x_1]$. Thus, we get a family

$$\left\{\Phi_{f,g}^h \;\middle|\; (f, g) \in \Lambda,\ h \in k[x_1, fx_3 + g] \setminus k[x_1]\right\}$$

of wild automorphisms of $k[\mathbf{x}]$, where Λ is the set of $(f, g) \in (k[x_1] \setminus \{0\}) \times x_2k[x_1, x_2]$ such that $\partial^2 g/\partial x_2^2$ does not belong to $fk[x_1, x_2]$.

Let us consider the second automorphism of Ohta [17, Theorem 3] defined by

$$\phi_2(x_1) = x_1,\ \phi_2(x_2) = x_2 + x_1 f_2,\ \phi_2(x_3) = x_3 - \sum_{i \geq 0} \sum_{j \geq 1} a_{i,j}\big((x_2 + x_1 f_2)^j - x_2^j\big)x_1^{i-1},$$

where

$$f_2 := x_1x_3 + \sum_{i \geq 0} \sum_{j \geq 1} a_{i,j} x_1^i x_2^j$$

with $a_{i,j} \in k$ for each i, j. Here, x_1, x_2 and x_3 are denoted by x, y and z, respectively, in the original text.

Proposition 4.3. ϕ_2 *belongs to* $\mathrm{T}(k, \mathbf{x})$ *if and only if* $a_{0,j} = 0$ *for every* $j \geq 2$*.*

Proof. Set $D = T_{x_1,g}$ and $\phi = \Phi^{f_2}_{x_1,g} = \exp f_2 D$, where $g := f_2 - x_1x_3$. Then, we claim that $\phi = \phi_2$. In fact, we have $\phi(x_1) = x_1$ and $\phi(x_2) = x_2 + x_1 f_2$, since $D(x_1) = 0$, $f_2 D(x_2) = x_1 f_2$ and $(f_2D)^2(x_2) = 0$. Since $D(f_2) = 0$, we have $\phi(f_2) = f_2$, and hence

$$x_1\phi(x_3) + \sum_{i\geq 0}\sum_{j\geq 1} a_{i,j}x_1^i(x_2 + x_1f_2)^j = \phi(f_2) = f_2 = x_1x_3 + \sum_{i\geq 0}\sum_{j\geq 1} a_{i,j}x_1^i x_2^j.$$

This gives that $\phi(x_3) = \phi_2(x_3)$. Hence, ϕ is equal to ϕ_2. Since f_2 is an element of $k[x_1, f_2] \setminus k[x_1]$, it holds that $\phi = \Phi^{f_2}_{x_1,g}$ belongs to $\mathrm{T}(k, \mathbf{x})$ if and only if $\partial^2 g/\partial x_2^2 = \sum_{i\geq 0}\sum_{j\geq 1} j(j-1)a_{i,j}x_1^i x_2^{j-2}$ belongs to $x_1 k[x_1, x_2]$ as mentioned. This condition is equivalent to the condition that $a_{0,j} = 0$ for every $j \geq 2$. □

5 Tameness and triangularizability

We say that $D \in \mathrm{Der}_k\, k[\mathbf{x}]$ is *triangularizable* if $\tau^{-1} \circ D \circ \tau$ is triangular for some $\tau \in \mathrm{Aut}(k[\mathbf{x}]/k)$. When this is the case, D is locally nilpotent. Moreover, $\exp D$ belongs to $\mathrm{T}(k, \mathbf{x})$ if so does τ, since $\exp(\tau^{-1} \circ D \circ \tau) = \tau^{-1} \circ (\exp D) \circ \tau$. If τ does not belong to $\mathrm{T}(k, \mathbf{x})$, however, $\exp D$ does not belong to $\mathrm{T}(k, \mathbf{x})$ in general. Actually, as will be remarked after Theorem 8.1, $\exp D$ can be wild even if $\tau^{-1} \circ D \circ \tau = \partial/\partial x_1$ for some τ. On the other hand, it is also not clear whether D is always triangularizable if $\exp D$ belongs to $\mathrm{T}(k, \mathbf{x})$. When $n = 2$, every locally nilpotent derivation of $k[\mathbf{x}]$ is triangularizable due to Rentschler [19]. When $n \geq 4$, there exists a locally nilpotent derivation D of $k[\mathbf{x}]$ which is not triangularizable, but $\exp D$ belongs to $\mathrm{T}(k, \mathbf{x})$, by the results of Bass [1], Popov [18] and Smith [20] (see also [6, Sections 3.9]). So Freudenburg [6, Section 5.3] raised the following question.

Question (Freudenburg)**.** Assume that $n = 3$. Is a locally nilpotent derivation D of $k[\mathbf{x}]$ always triangularizable if D is *tame*, i.e., $\exp D$ belongs to $\mathrm{T}(k, \mathbf{x})$?

We give a partial affirmative answer to this question as follows.

Theorem 5.1 ([12, Theorem 1.2])**.** *Assume that $n = 3$, and D is a locally nilpotent derivation of $k[\mathbf{x}]$ such that $\ker D$ contains a tame coordinate of $k[\mathbf{x}]$ over k. Then, $\exp D$ belongs to $\mathrm{T}(k, \mathbf{x})$ if and only if $\tau^{-1} \circ D \circ \tau$ is triangular for some $\tau \in \mathrm{T}(k, \mathbf{x})$.*

We note that there exists a locally nilpotent derivation of $k[\mathbf{x}]$ for $n = 3$ such that $\ker D$ contains no coordinate of $k[\mathbf{x}]$ over k (cf. [4] and [5]). Such a locally nilpotent derivation is never conjugate to a triangular derivation multiplied by an element of $\ker D$. In fact, every triangular derivation of $k[\mathbf{x}]$ kills a tame coordinate of $k[\mathbf{x}]$ over k if $n \geq 2$. In Section 9, we will discuss tameness of $\exp D$ for such special D.

Theorem 5.1 is obtained by Theorem 1.2 and the following theorem.

Theorem 5.2 ([12, Theorem 5.1]). *Let R be a $\mathbf{Q}$-domain, and D a locally nilpotent derivation of $R[x_1, x_2]$ over R such that $\exp D$ belongs to $\mathrm{T}(R, \{x_1, x_2\})$. Then, there exists $\tau \in \mathrm{T}(R, \mathbf{x})$ such that $D' := \tau^{-1} \circ D \circ \tau$ is triangular, or $\deg D'(x_i) \leq 1$ for $i = 1, 2$. If $V(R) = K^\times$, then there exists $\tau \in \mathrm{T}(R, \mathbf{x})$ such that D' is triangular, where K is the field of fractions of R.*

It is interesting to ask the following question.

Question. Let R be a $\mathbf{Q}$-domain, and D a locally nilpotent derivation of $R[\mathbf{x}]$ over R. Does $\exp D$ belong to $\mathrm{T}(R, \mathbf{x})$ whenever $\exp hD$ belongs to $\mathrm{T}(R, \mathbf{x})$ for some $h \in \ker D \setminus \{0\}$?

As a corollary to Theorem 5.2, we get the following result.

Corollary 5.3 ([12, Corollary 5.2]). *Let R be a $\mathbf{Q}$-domain, and D a locally nilpotent derivation of $R[\mathbf{x}]$ over R for $n = 2$. If $\exp hD$ belongs to $\mathrm{T}(R, \mathbf{x})$ for some $h \in \ker D \setminus \{0\}$, then $\exp D$ belongs to $\mathrm{T}(R, \mathbf{x})$.*

6 Invariant coordinates

In Theorem 2.2, we described a rough structure of the subgroup $H(f)$ of $\mathrm{Aut}(R[\mathbf{x}]/R)$ for a coordinate $f \in R[\mathbf{x}]$ of $S[\mathbf{x}]$ over S which is reduced over R. In this section, we assume that the field K of fractions of R is of characteristic zero, and determine the precise structure of $H(f)$ and classify f such that $H(f)$ has at least two elements.

By the following lemma, we may assume that $S = K$.

Lemma 6.1 ([12, Lemma 7.1]). *Assume that K is of characteristic zero. If $f \in R[x_1, x_2]$ is a coordinate of $S[x_1, x_2]$ over S, then f is a coordinate of $K[x_1, x_2]$ over K.*

Now, we define four types of elements of $R[\mathbf{x}]$ which are coordinates of $K[\mathbf{x}]$ over K as follows: For $a \in R \setminus \{0\}$, $g \in R[x_1]$ with $\deg g \geq 2$, and $u(z) \in K[z^l] \setminus K$ and $1 \neq \zeta \in R^\times$ with $\zeta^l = 1$ for some $l \geq 2$, we define

$$f_1 = ax_2 + g, \qquad f_2 = ax_1 + u\big((\zeta - 1)x_2 + g\big),$$

where we assume that g, ζ and $u(z)$ are such that $u\big((\zeta-1)x_2 + g\big)$ belongs to $R[\mathbf{x}]$. For $\tau \in \mathrm{Aff}(K, \mathbf{x})$ such that $\tau(x_1) = \alpha x_1 + \beta x_2 + \gamma$ for $\alpha, \beta, \gamma \in K$ with $\alpha, \beta \neq 0$, and for $v \in K[x_1]$ with $\deg v = 2$, or $v \in K[x_1^l] \setminus K$ for some $l \geq 2$, we define

$$f_3 = \tau(x_1), \qquad f_4 = \tau(x_2 + v),$$

where we assume that τ and v are such that $\tau(x_2 + v)$ belongs to $R[\mathbf{x}]$.

For $f_1, \ldots, f_4$ as above, we define subsets $H_1, \ldots, H_4$ of $\mathrm{Aut}(R[\mathbf{x}]/R)$ as follows:

- $H_1 = \mathrm{J}(R; x_1, x_2) \cap \mathrm{Aut}(R[\mathbf{x}]/R[f_1])$.
- H_2 is the set of $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R[x_1])$ such that $\phi(x_2) = \xi x_2 + (\xi - 1)(\zeta - 1)^{-1} g$. Here, $\xi \in R$ is such that $(\xi - 1)(\zeta - 1)^{-1} g$ belongs to $R[x_1]$, and $\xi^m = 1$, where m is the maximal integer for which $u(z)$ belongs to $R[z^m]$.
- $H_3 = \mathrm{Aff}(R, \mathbf{x}) \cap \tau \circ \mathrm{Aut}(K[\mathbf{x}]/K[x_1]) \circ \tau^{-1}$.
- $H_4 = \mathrm{Aff}(R, \mathbf{x}) \cap \tau \circ \mathrm{Aut}(K[\mathbf{x}]/K[x_2 + v]) \circ \tau^{-1}$.

In the notation above, we have the following result.

Theorem 6.2 ([12, Theorem 7.2])**.** *Assume that $n = 2$ and K is of characteristic zero.*
(i) *Let $f \in R[\mathbf{x}]$ be a coordinate of $K[\mathbf{x}]$ over K which is reduced over R. If $\deg_{x_1} f \geq \deg_{x_2} f \geq 1$ and $H(f) \neq \{\mathrm{id}_{R[\mathbf{x}]}\}$, then f has the form of f_i for some $i \in \{1, 2, 3, 4\}$.*
(ii) *If f_i is reduced over R for $i \in \{1, 2, 3, 4\}$, then we have $H(f_i) = H_i$.*

In the case where $R = k[x_3]$, the above theorem and Theorem 1.2 imply the following corollary.

Corollary 6.3 ([12, Corollary 7.5])**.** *Assume that $n = 3$. Let $f \in k[\mathbf{x}]$ be a coordinate of $k(x_3)[x_1, x_2]$ over $k(x_3)$. If $H := \mathrm{Aut}(k[\mathbf{x}]/k[x_3, f]) \cap \mathrm{T}(k, \mathbf{x})$ has at least two elements, then one of the following holds for some $\tau \in \mathrm{T}(k[x_3], \{x_1, x_2\})$:*
(i) *$\tau(f) = ax_1 + b$ for some $a, b \in k[x_3]$ with $a \neq 0$, and $H = \mathrm{Aut}(k[\mathbf{x}]/k[x_1, x_3])$.*
(ii) *$\tau(f) = ax_2 + g$ for some $a \in k[x_3]$ and $g \in k[x_1, x_3]$ with $\deg_{x_1} g \geq 2$ for which the leading coefficient of g, as a polynomial in x_1 over $k[x_3]$, does not belong to $ak[x_3]$. Moreover, we have $H = \mathrm{J}(k[x_3]; x_1, x_2) \cap \mathrm{Aut}(k[\mathbf{x}]/k[ax_2 + g, x_3])$.*

If K is of positive characteristic, the statements of Lemma 6.1 and Theorem 6.2 do not hold in general (cf. [12, Section 7]).

7 Generalized Shestakov-Umirbaev theory

In the following sections, we explain the main results of [13] and [14]. These papers are devoted to applications of the generalized Shestakov-Umirbaev theory [10], [11]. In this section, we mention some consequences of this theory used in [13] and [14]. In what follows, we assume that $n = 3$ unless otherwise stated, and a *wild automorphism* always means an element of $\mathrm{Aut}(k[\mathbf{x}]/k)$ not belonging to $\mathrm{T}(k, \mathbf{x})$.

For $\mathbf{w} = (w_1, w_2, w_3) \in \Gamma^3$, we define $\mathrm{rank}\, \mathbf{w}$ to be the rank of the $\mathbf{Z}$-submodule of Γ generated by w_1, w_2 and w_3. For $\phi \in \mathrm{Aut}(k[\mathbf{x}]/k)$, consider the following conditions:

(1) $\phi(x_1)^{\mathbf{w}}$, $\phi(x_2)^{\mathbf{w}}$ and $\phi(x_3)^{\mathbf{w}}$ are algebraically dependent over k, and are pairwise algebraically independent over k;

(2) $\phi(x_i)^{\mathbf{w}}$ does not belong to $k[\{\phi(x_j)^{\mathbf{w}} \mid j \neq i\}]$ for $i = 1, 2, 3$.

The generalized Shestakov-Umirbaev theory implies the following sufficient condition for wildness, where $\Gamma_{>0} := \{\alpha \in \Gamma \mid \alpha > 0\}$.

Proposition 7.1 ([13, Section 1]). *If $\phi \in \mathrm{Aut}(k[\mathbf{x}]/k)$ is such that* (1) *and* (2) *hold for some $\mathbf{w} \in (\Gamma_{>0})^3$ with $\operatorname{rank} \mathbf{w} = 3$, then ϕ is wild.*

We call $P \in k[\mathbf{x}]$ a *W-test polynomial* if, for each $\phi \in \mathrm{Aut}(k[\mathbf{x}]/k)$, it holds that ϕ is wild whenever there exist a totally ordered additive group Γ and $\mathbf{w} \in (\Gamma_{>0})^3$ with $\operatorname{rank} \mathbf{w} = 3$ as follows:

(a) $\deg_{\mathbf{w}} \phi(P) < \deg_{\mathbf{w}} \phi(x_{i_1})$ for some $i_1 \in \{1, 2, 3\}$;

(b) $\deg_{\mathbf{w}} \phi(x_{i_2})$ and $\deg_{\mathbf{w}} \phi(x_{i_3})$ are linearly independent over $\mathbf{Z}$ for some $i_2, i_3 \in \{1, 2, 3\}$.

It is sometimes useful to use a W-test polynomial for showing that an automorphism is wild. The following proposition follows from Proposition 7.1.

Proposition 7.2 ([14, Proposition 6.1]). *Let P be an element of $k[\mathbf{x}]$ not belonging to $k[\mathbf{x} \setminus \{x_i\}]$ for $i = 1, 2, 3$. Then, P is a W-test polynomial if the following conditions hold for every totally ordered additive group Γ and $\mathbf{w} \in (\Gamma_{>0})^3$ such that $P^{\mathbf{w}}$ is not a monomial*:

(i) *$P^{\mathbf{w}}$ is not divisible by $x_i - g$ for any $g \in k[\mathbf{x} \setminus \{x_i\}] \setminus k$ for $i = 1, 2, 3$;*

(ii) *$P^{\mathbf{w}}$ is not divisible by $x_i^{s_i} - c x_j^{s_j}$ for any $c \in k^{\times}$, $s_i, s_j \in \mathbf{N}$ and $i, j \in \{1, 2, 3\}$ with $i \neq j$.*

By this proposition, we can check that $P = x_1 x_3 - \sum_{i=1}^{t} \alpha_i x_2^{i-1}$ and $x_2 - P x_3$ are W-test polynomials if $t \geq 2$, where $\alpha_1, \ldots, \alpha_{t-1} \in k$ and $\alpha_t \in k^{\times}$. This result is used to prove Theorems 9.1 and 9.3.

8 Absolutely wild and totally wild coordinates

We say that a coordinate f of $k[\mathbf{x}]$ over k is *absolutely wild* if $D(f) = 0$ implies that $\exp D$ is wild for every nonzero locally nilpotent D of $k[\mathbf{x}]$, and *totally wild* if $\phi(f) = f$ implies that ϕ is wild for every $\mathrm{id}_{k[\mathbf{x}]} \neq \phi \in \mathrm{Aut}(k[\mathbf{x}]/k)$. Since $D(f) = 0$ implies $(\exp D)(f) = f$, "totally wild" implies "absolutely wild". We claim that "absolutely wild" implies "wild". In fact, if f is a tame coordinate, then there exists $\sigma \in \mathrm{T}(k, \mathbf{x})$ such that $\sigma(x_1) = f$, for which we have $D(f) = 0$, and $\exp D$ belongs to $\mathrm{T}(k, \mathbf{x})$, where $D := \sigma \circ (\partial/\partial x_2) \circ \sigma^{-1}$. In [13], we construct totally wild coordinates, and absolutely wild coordinates which are not totally wild as follows.

For $\theta(z) \in k[z] \setminus k$, we define a locally nilpotent derivation D_θ of $k[\mathbf{x}]$ by

$$D_\theta(x_1) = -\theta'(x_2), \quad D_\theta(x_2) = x_3, \quad D_\theta(x_3) = 0,$$

where $\theta'(z)$ is the derivative of $\theta(z)$. Then, $f_\theta := x_1x_3 + \theta(x_2)$ belongs to $\ker D_\theta$. Hence, $f_\theta D_\theta$ is a locally nilpotent derivation of $k[\mathbf{x}]$. Set $\sigma_\theta = \exp f_\theta D_\theta$, and $y_1 = \sigma_\theta(x_1)$. We consider the subgroup

$$G_\theta := \mathrm{Aut}(k[\mathbf{x}]/k[y_1]) \cap \mathrm{T}(k, \mathbf{x})$$

of $\mathrm{Aut}(k[\mathbf{x}]/k)$. Note that $G_\theta = \{\mathrm{id}_{k[\mathbf{x}]}\}$ if and only if y_1 is a totally wild coordinate. If G_θ is a finite group, then y_1 is an absolutely wild coordinate. Actually, $\exp D$ has an infinite order for every locally nilpotent derivation $D \neq 0$, since $(\exp D)^l = \exp lD \neq \mathrm{id}_{k[\mathbf{x}]}$.

Let a and b be the coefficients of z^d and z^{d-1} in $\theta(z)$, respectively, where $d := \deg \theta(z)$. We set $c = -b/(ad)$ and write $\theta(z) = \sum_{i=0}^{d} u_i(z-c)^i$, where $u_i \in k$ for each i. Then, we have $u_d = a$, $u_{d-1} = 0$ and $u_0 = \theta(c)$. Let $e \in \mathbf{N}$ be the the positive generator of the ideal of $\mathbf{Z}$ generated by $2i - 1$ for $1 \leq i \leq d$ with $u_i \neq 0$, and define

$$T_\theta = \{\zeta \in k^\times \mid \zeta^e = 1\}.$$

For each $\zeta \in T_\theta$, we define an element ϕ_ζ of $\mathrm{J}(k; x_3, x_2, x_1)$ by $\phi_\zeta(x_3) = \zeta x_3$, and

$$\phi_\zeta(x_2 - c) = \zeta^2(x_2 - c) + \zeta(\zeta - 1)\theta(c)x_3, \quad \phi_\zeta(x_1) = x_1 + g_\zeta,$$

where

$$g_\zeta := \big(\zeta\theta(x_2) - \theta(\phi_\zeta(x_2)) + (1-\zeta)\theta(c)\big)(\zeta x_3)^{-1}.$$

Here, we note that g_ζ always belongs to $k[\mathbf{x}]$ for $\zeta \in T_\theta$.

In the notation above, we have the following theorem.

Theorem 8.1 ([13, Theorem 6.1]). *For each $\zeta \in T_\theta$, the automorphism ϕ_ζ belongs to G_θ. The map $\iota : T_\theta \ni \zeta \mapsto \phi_\zeta \in G_\theta$ is an injective homomorphism of groups. If $d \geq 9$ and $d \neq 10, 12$, then ι is an isomorphism.*

By this theorem, we know that there exist a number of totally wild coordinates, and absolutely wild coordinates which are not totally wild as follows. If $d \geq 9$ and $d \neq 10, 12$, then G_θ is isomorphic to T_θ. Since T_θ is a finite group, it follows that G_θ is a finite group. Hence, y_1 is an absolutely wild coordinate as mentioned. Furthermore, y_1 is a totally wild coordinate if and only if $T_\theta = \{1\}$. Since some $\theta(z)$'s satisfy $T_\theta = \{1\}$ and others do not, it follows that there exist various totally wild coordinates, and absolutely wild coordinates which are not totally wild.

Note that

$$\mathrm{Aut}(k[\mathbf{x}]/k[y_1]) = \sigma_\theta \circ \mathrm{Aut}(k[\mathbf{x}]/k[x_1]) \circ \sigma_\theta^{-1},$$

so every element of $\sigma_\theta \circ \mathrm{Aut}(k[\mathbf{x}]/k[x_1]) \circ \sigma_\theta^{-1}$ not belonging to G_θ is wild. Hence, if $d \geq 9$ and $d \neq 10, 12$, then $\exp D$ is wild even for the locally nilpotent derivation

$$D := \sigma_\theta \circ \left(\frac{\partial}{\partial x_3}\right) \circ \sigma_\theta^{-1},$$

since $\exp D = \sigma_\theta \circ (\exp \partial/\partial x_3) \circ \sigma_\theta^{-1}$, and $\exp D$ has an infinite order. If y_1 is a totally wild coordinate, then $\sigma_\theta \circ \tau \circ \sigma_\theta^{-1}$ is wild even for $\tau \in \operatorname{Aut}(k[\mathbf{x}]/k)$ defined by

$$\tau(x_1) = x_1, \quad \tau(x_2) = x_2, \quad \tau(x_3) = -x_3.$$

As these examples show, the existence of absolutely wild or totally wild coordinates means the existence of a very large class of wild automorphisms of $k[\mathbf{x}]$.

9 Local slice constructions

The *rank* $\operatorname{rank} D$ of $D \in \operatorname{Der}_k k[\mathbf{x}]$ is by definition the minimal number $r \geq 0$ for which $D(\sigma(x_i)) \neq 0$ holds for $i = 1, \ldots, r$ for some $\sigma \in \operatorname{Aut}(k[\mathbf{x}]/k)$ (cf. [5]). As mentioned after Theorem 5.1, every triangular derivation of $k[\mathbf{x}]$ is of rank less than n when $n \geq 2$. If $n = 2$, every locally nilpotent derivation of $k[\mathbf{x}]$ is of rank at most one by Rentschler [19]. Freudenburg [4], [5] first gave locally nilpotent derivations of $k[\mathbf{x}]$ of rank n for $n \geq 3$ using his method of *local slice constructions*. It is not easy to construct such a locally nilpotent derivation D, for which it is previously not known whether $\exp D$ is tame.

In this section, we summarize the main results of [14], where we give a large family of locally nilpotent derivations of $k[\mathbf{x}]$ by means of local slice construction, and determine tameness of $\exp hD$ for each D and $h \in \ker D \setminus \{0\}$. The family includes the locally nilpotent derivations of Freudenburg, and many other locally nilpotent derivations of rank three. The result is that $\exp hD$ is always wild unless hD is triangularizable by a tame automorphism, i.e., $\tau^{-1} \circ (hD) \circ \tau$ is triangular for some $\tau \in \mathrm{T}(k, \mathbf{x})$. This gives a partial affirmative answer to the question of Freudenburg (Section 5).

Now, for $i = 0, 1$, let t_i be a positive integer, and α_j^i an element of k for $j = 1, \ldots, t_i$ with $\alpha_{t_i}^i = 1$. We define a sequence $(b_i)_{i=0}^\infty$ of integers by

$$b_0 = b_1 = 0 \quad \text{and} \quad b_{i+1} = t_i b_i - b_{i-1} + \xi_i \quad \text{for} \quad i \geq 1,$$

where $t_i := t_0$ if i is an even number, and $t_i := t_1$ otherwise, and where $\xi_i := 1$ if $i \equiv 0, 1 \pmod 4$, and $\xi_i := -1$ otherwise. For each $i \geq 1$, we define $\eta_i(y, z) \in k[y, z]$ by

$$\eta_i(y,z) = z^{t_i b_i + 1} + \sum_{j=1}^{t_i} \alpha_j^i y^j z^{(t_i - j) b_i} \qquad \text{if} \quad i \equiv 0, 1 \pmod 4$$

$$\eta_i(y,z) = y^{t_i} + \sum_{j=1}^{t_i} \alpha_j^i z^{j b_i - 1} y^{t_i - j} \qquad \text{otherwise,}$$

where $\alpha_j^i := \alpha_j^0$ if i is an even number, and $\alpha_j^i := \alpha_j^1$ otherwise for each j. Set

$$r = x_1 x_2 x_3 - \sum_{i=1}^{t_0} \alpha_i^0 x_2^i - \sum_{j=1}^{t_1} \alpha_j^1 x_1^j,$$

and define a sequence $(f_i)_{i=0}^{\infty}$ of rational functions by $f_0 = x_2$, $f_1 = x_1$ and $f_{i+1} = f_{i-1}^{-1} q_i$ for each $i \geq 1$ by induction on i, where $q_i = \eta_i(f_i, r)$. Note that

$$q_1 = r + \sum_{j=1}^{t_1} \alpha_j^1 x_1^j = x_1 x_2 x_3 - \sum_{i=1}^{t_0} \alpha_i^0 x_2^i, \quad f_2 = x_1 x_3 - \sum_{i=1}^{t_0} \alpha_i^0 x_2^{i-1}, \tag{9.1}$$

and $r = x_2 f_2 - \sum_{j=1}^{t_1} \alpha_j^1 x_1^j$. If $t_0 = 2$, then we have $f_2 = x_1 x_3 - x_2 - \alpha_1^0$. In this case, we can define $\tau_2 \in \mathrm{T}(k, \mathbf{x})$ by

$$\tau_2(x_1) = f_2, \quad \tau_2(x_2) = x_1, \quad \tau_2(x_3) = x_3.$$

We can also construct the sequence $(f_i)_{i=0}^{\infty}$ from the same data t_0, t_1, $(\alpha_j^0)_{j=1}^{t_0-1}$ and $(\alpha_j^1)_{j=1}^{t_1-1}$ by interchanging the role of t_0 and t_1, and $(\alpha_j^0)_{j=1}^{t_0-1}$ and $(\alpha_j^1)_{j=1}^{t_1-1}$. To distinguish it from the original one, we denote it by $(f_i')_{i=0}^{\infty}$. If $t_1 = 2$, then we can define $\tau_2' \in \mathrm{T}(k, \mathbf{x})$ by $\tau_2'(x_1) = f_2'$, $\tau_2'(x_2) = x_1$ and $\tau_2'(x_3) = x_3$ as above.

When f_i and f_{i+1} belong to $k[\mathbf{x}]$, we consider the derivation $D_i := \Delta_{(f_i, f_{i+1})}$ of $k[\mathbf{x}]$. Here, for $g_1, g_2 \in k[\mathbf{x}]$, we define a derivation $\Delta_{(g_1, g_2)}$ of $k[\mathbf{x}]$ by $\Delta_{(g_1, g_2)}(g_3) = \det(\partial g_i / \partial x_j)_{i,j}$ for each $g_3 \in k[\mathbf{x}]$. For example, we have

$$D_1(x_1) = 0, \quad D_1(x_2) = -x_1 \quad \text{and} \quad D_1(x_3) = -\sum_{i=2}^{t_0} (i-1) \alpha_i^0 x_2^{i-2} \tag{9.2}$$

by (9.1). Hence, D_1 is triangular. When f_i' and f_{i+1}' belong to $k[\mathbf{x}]$, we define $D_i' = \Delta_{(f_i', f_{i+1}')}$ similarly.

Set $a_i = t_i b_i + \xi_i$ for each $i \geq 0$, and let I be the set of $i \in \mathbf{N}$ such that $a_j > 0$ for $j = 1, \ldots, i$. Then, we have $a_0 = 0$, $a_1 = 1$ and $a_{i+1} = t_{i+1} a_i - a_{i-1}$ for $i \geq 1$. From this, we get

$$I = \begin{cases} \{1\} & \text{if } t_0 = 1 \\ \{1, 2\} & \text{if } (t_0, t_1) = (2, 1) \\ \{1, 2, 3, 4\} & \text{if } (t_0, t_1) = (3, 1) \\ \mathbf{N} & \text{otherwise.} \end{cases} \tag{9.3}$$

In the notation above, we have the following result.

Theorem 9.1 ([13, Theorem 2.1]). *The following assertions hold for each $i \in I$:*
(i) *f_i and f_{i+1} belong to $k[\mathbf{x}]$, and D_i is a locally nilpotent derivation of $k[\mathbf{x}]$ such that $D_i(r) = -f_i f_{i+1}$. Furthermore, we have the following*:
(1) *If i is the maximum of I, then D_i is not irreducible and* $\ker D_i \neq k[f_i, f_{i+1}]$.
(2) *If i is not the maximum of I, then D_i is irreducible and* $\ker D_i = k[f_i, f_{i+1}]$.
(ii) *Assume that $t_0 = 2$. Then, we have $\tau_2^{-1} \circ D_i \circ \tau_2 = D_{i-1}'$. Hence, D_2 is triangularizable by a tame automorphism. Moreover, the following assertions hold*:

(a) *If* $t_1 = 2$, *then we have* $\tau^{-1} \circ D_i \circ \tau = D_0$, *where* $\tau := (\tau_2 \circ \tau_2')^{i/2}$ *if* i *is an even number, and* $\tau := (\tau_2 \circ \tau_2')^{(i-1)/2} \circ \tau_2$ *otherwise.*
(b) *If* $t_1 \geq 3$ *and* $i \geq 3$, *then* $\exp hD_i$ *is wild for every* $h \in \ker D_i \setminus \{0\}$.
(iii) *If* $t_0 \geq 3$ *and* $i \geq 2$, *then* $\exp hD_i$ *is wild for every* $h \in \ker D_i \setminus \{0\}$.

Here, $D \in \mathrm{Der}_k\, k[\mathbf{x}]$ is said to be *irreducible* if $D(k[\mathbf{x}])$ is contained in no proper principal ideal of $k[\mathbf{x}]$.

Recall that $\mathrm{pl}\, D := D(k[\mathbf{x}]) \cap \ker D$ forms an ideal of $\ker D$ for each $D \in \mathrm{Der}_k\, k[\mathbf{x}]$, and is called the *plinth ideal* of D. Assume that D is locally nilpotent. Then, we have $\mathrm{pl}\, D \neq \{0\}$ unless $D = 0$. Owing to Miyanishi [15], it holds that $\mathrm{pl}\, D = \ker D$ if and only if D is irreducible and of rank one when $n = 3$. By Daigle-Kaliman [2, Theorem 1], $\mathrm{pl}\, D$ is always a principal ideal of $\ker D$ when $n = 3$.

We use the following lemma to determine the rank of a locally nilpotent derivation.

Lemma 9.2 ([14, Lemma 2.5]). *Let* $D \neq 0$ *be an irreducible locally nilpotent derivation of* $k[\mathbf{x}]$. *If* $\ker D$ *contains a coordinate* p *of* $k[\mathbf{x}]$ *over* k, *then there exists* $s \in k[\mathbf{x}]$ *such that* $D(s)$ *belongs to* $k[p] \setminus \{0\}$.

Since $\mathrm{pl}\, D$ is a principal ideal of $\ker D$, Lemma 9.2 implies that $\mathrm{pl}\, D$ is generated by an element of $k[p] \setminus \{0\}$ if D is irreducible, and $\ker D$ contains a coordinate p of $k[\mathbf{x}]$ over k. On the other hand, if $t_0 = 2$, $t_1 \geq 3$ and $i \geq 3$, or if $t_0 \geq 3$, $(t_0, t_1) \neq (3, 1)$ and $i \geq 2$, then we have $\mathrm{pl}\, D_i = f_i f_{i+1} \ker D_i$ (cf. [14, Proposition 1.2]). Since f_i and f_{i+1} are algebraically independent over k, we see that $f_i f_{i+1}$ does not belong to $k[p]$ for any coordinate p of $k[\mathbf{x}]$ over k. Thus, we conclude that D_i is of rank three. In [14], we also determined the rank of D_i for the other cases.

Next, take $i \in \mathbf{N}$ with $i \geq 2$, and assume that $t_0 \geq 3$ if $i = 2$, and $t_0 \geq 3$ and $(t_0, t_1) \neq (3, 1)$ if $i \geq 3$. Let $\lambda(y) \in k[y] \setminus \{0\}$ and $\mu(y, z) = \sum_{j \geq 1} \mu_j(y) z^j \in zk[y, z] \setminus \{0\}$ be such that $\gcd(\lambda(y), \mu_j(y)) = 1$ for some $j \geq 1$, where $\mu_j(y) \in k[y]$ for each $j \geq 1$. We set

$$r_i = \lambda(f_i)\tilde{r} - \mu(f_i, f_{i-1}), \quad \text{where} \quad \tilde{r} := \begin{cases} x_2 & \text{if} \quad i = 2 \\ r & \text{if} \quad i \geq 3. \end{cases}$$

Then, we define

$$\tilde{f}_{i+1} = \tilde{\eta}_i\big(f_i, r_i \lambda(f_i)^{-1}\big) \lambda(f_i)^{a_i} f_{i-1}^{-1},$$

where

$$\tilde{\eta}_2(y, z) := y + \sum_{j=1}^{t_0} \alpha_j^0 z^{j-1}$$

and $\tilde{\eta}_i(y, z) := \eta_i(y, z)$ for $i \geq 3$.

With this notation and assumptions, we have the following result.

Theorem 9.3 ([13, Theorem 3.1]). (i) *$\tilde{f}_{i+1}$ belongs to $k[\mathbf{x}]$, and $\tilde{D}_i := \Delta_{(f_i,\tilde{f}_{i+1})}$ is an irreducible locally nilpotent derivation of $k[\mathbf{x}]$ such that $\ker \tilde{D}_i = k[f_i, \tilde{f}_{i+1}]$. Moreover, we have $\tilde{D}_i(r_i) = -\lambda(f_i)\tilde{f}_{i+1}$ if $i = 2$, and $\tilde{D}_i(r_i) = -\lambda(f_i)f_i\tilde{f}_{i+1}$ if $i \geq 3$.*
(ii) *Assume that $\lambda(y)$ belongs to $k^\times$, $\mu(y,z)$ belongs to $zk[z]$, and $i = 2$. Then, $\exp h\tilde{D}_2$ is tame if and only if h belongs to $k[\tilde{f}_3]$ for $h \in \ker \tilde{D}_2$. In the other case, $\exp h\tilde{D}_i$ is wild for each $h \in \ker \tilde{D}_i \setminus \{0\}$.*

In the same situation, the following proposition holds.

Proposition 9.4 ([14, Proposition 1.5]). (i) *If $i \geq 3$, then we have $\operatorname{rank} \tilde{D}_i = 3$.*
(ii) *If $\lambda(y)$ belongs to $k^\times$, then $\operatorname{rank} \tilde{D}_2 = 2$, and $\tilde{f}_3$ is a coordinate of $k[\mathbf{x}]$ over k.*
(iii) *Assume that $\lambda(y)$ does not belong to k. If $t_0 \geq 4$, or $\mu_j(y)$ does not belong to $\sqrt{\lambda(y)k[y]}$ for some $j \geq 2$, then we have $\operatorname{rank} \tilde{D}_2 = 3$. If $t_0 = 3$, and $\mu_j(y)$ belongs to $\sqrt{\lambda(y)k[y]}$ for every $j \geq 2$, then we have $\operatorname{pl} \tilde{D}_2 = \tilde{f}_3 \ker \tilde{D}_2$.*

The locally nilpotent derivations of Freudenburg [4] are obtained as follows. Assume that $t_j = 3$ and $\alpha_1^j = \alpha_2^j = 0$ for $j = 0, 1$. Then, we have $I = \mathbf{N}$ by (9.3). By Theorem 9.1 (i), it follows that f_i and f_{i+1} belong to $k[\mathbf{x}]$, and D_i is an irreducible locally nilpotent derivation of $k[\mathbf{x}]$ with $\ker D_i = k[f_i, f_{i+1}]$ for each $i \geq 1$. It is easy to check that $a_1 = 1$, $a_2 = 2$, $f_2 = x_1x_3 - x_2^2$ and $a_{i+1} = 3a_i - a_{i-1}$ for every $i \geq 2$. Moreover, we have $r = x_2f_2 - x_1^3$, $f_0 = x_2$, $f_1 = x_1$ and $f_{i+1} = f_{i-1}^{-1}(r^{a_i} + f_i^3)$ for every $i \geq 2$. From this, we see that $(\iota^{-1} \circ D_i \circ \iota)_{i=1}^\infty$ is the same as the sequence of locally nilpotent derivations of "Fibonacci type" given by Freudenburg [4], where $\iota \in \operatorname{Aut}(k[\mathbf{x}]/k)$ is such that $\iota(x_2) = -x_2$ and $\iota(x_i) = x_i$ for $i = 1, 3$. According to Theorem 9.1 (ii), $\exp hD_i$ is wild for each $h \in \ker D_i \setminus \{0\}$ for every $i \geq 2$. Next, for $l, m \in \mathbf{N}$, set $\lambda(y) = y^l$ and $\mu(y,z) = -z^m$. Then, it follows from Theorem 9.3 that $\tilde{f}_3$ belongs to $k[\mathbf{x}]$, $\tilde{D}_2$ is an irreducible locally nilpotent derivation of $k[\mathbf{x}]$ such that $\ker \tilde{D}_2 = k[f_2, \tilde{f}_3]$, and $\exp h\tilde{D}_2$ is wild for each $h \in \ker \tilde{D}_2 \setminus \{0\}$. In this case, we have $r_2 = f_2^l x_2 + x_1^m$. Since $\tilde{\eta}_2(y,z) = y + z^2$, and $a_2 = t_0 - 1 = 2$, we get

$$\tilde{\eta}_2\left(f_2, \frac{r_2}{\lambda(f_2)}\right)(f_2^l)^2 = \left(f_2 + \frac{r_2^2}{f_2^{2l}}\right) f_2^{2l} = f_2^{2l+1} + r_2^2 = x_1(f_2^{2l}x_3 - 2f_2^l x_1^{m-1}x_2 + x_1^{2m-1}),$$

and so $\tilde{f}_3 = f_2^{2l}x_3 - 2f_2^l x_1^{m-1}x_2 + x_1^{2m-1}$. We note that, if $m = 2l+1$, then $\tilde{D}_2$ is the same as the homogeneous locally nilpotent derivation of "type $(2, 4l+1)$" given by Freudenburg [4].

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