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Splitting of Frobenius sandwiches

By

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Abstract

We study Frobenius sandwiches in positive characteristic globally and locally from the viewpoint of Frobenius splitting and F-singularities.

Let X be a smooth variety over an algebraically closed field k of characteristic $p > 0$. A Frobenius sandwich of X is a normal variety Y through which the (relative) Frobenius morphism of X factors as $F\colon X \to Y \to X$. Although X and its Frobenius sandwich Y have the same underlying space in the Zariski topology, it happens that they have very different structures as algebraic varieties. For example, Y may be singular even if X is smooth. More surprisingly, there is a Frobenius sandwich of the projective plane $\mathbb{P}^2$ whose minimal resolution is a uniruled surface of general type called a Zariski surface; see e.g., [6], [15].

Frobenius sandwiches such as Zariski surfaces reflect pathology of purely inseparable morphisms, and so it is difficult to analyze them systematically. In this paper, we will consider Frobenius sandwiches that behave better in the sense of Frobenius splitting. There are a few local and global properties of algebraic varieties in characteristic $p > 0$ defined via splitting of Frobenius maps, among which are local and global F-regularity [16], [17], [26]. These properties have close connection with log terminal singularities and log Fano varieties, respectively [12], [25]. Thus, assuming such Frobenius splitting properties, we can restrict the Frobenius sandwiches under consideration to "well-behaved" ones only, so that we may expect systematic study of them.

Taking the above point of view into account, we propose the following problems. First, we consider a characterization of F-regular Frobenius sandwich singularities. We

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ask if an F-regular Frobenius sandwich singularity is always a toric singularity, and give a partial answer to this problem in Section 2; see also [1], [2], [3]. Second, we consider a global version of the above problem. Namely, we ask if a globally F-regular Frobenius sandwich of a smooth projective toric surface X is always toric. This turns out to be affirmative if X is the projective plane $\mathbb{P}^2$ or a Hirzebruch surface Σ_d. Indeed, we are able to classify globally F-regular Frobenius sandwiches of $\mathbb{P}^2$ and Σ_d in Section 3.

Finally in Section 4, we consider F-blowups of certain surface singularities in characteristic $p > 0$. The notion of F-blowup is introduced by Yasuda [31], and he asks if an F-blowup of a surface singularity coincides with the minimal resolution. We give a counterexample to this question constructed as a non-F-regular Frobenius sandwich surface singularity. On the other hand, we observe that an F-blowup of any F-regular double point is the minimal resolution.

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Notation and Convention. Unless otherwise specified, we work over an algebraically closed field k of characteristic $p > 0$. Let X be an algebraic variety over k. By definition, the absolute Frobenius morphism $F\colon X \to X$ of X is the morphism given by the identity on the underlying space, together with the pth power map on the structure sheaf, which we also denote by $F\colon \mathcal{O}_X \to F_*\mathcal{O}_X = \mathcal{O}_X$. To distinguish the $\mathcal{O}_X$ on the both sides of this Frobenius ring homomorphism, we often identify it with the inclusion map $\mathcal{O}_X \hookrightarrow \mathcal{O}_X^{1/p}$ into the ring of pth roots. On the other hand, the relative (or k-linear) Frobenius $F_{\mathrm{rel}}\colon X \to X^{(-1)}$ is defined by the following Cartesian square:

$$\begin{tikzcd}
X \arrow[dr, "F_{\mathrm{rel}}"] \arrow[drr, bend left, "F"] \arrow[ddr] & & \\
& X^{(-1)} \arrow[r] \arrow[d] & X \arrow[d] \\
& \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

Here $X^{(-1)}$ is the base change of X by the absolute Frobenius of Spec k at the bottom, so that it is isomorphic to X as an abstract scheme but not as a variety over k. We use these variants of Frobenius morphisms interchangeably, since it doesn't matter to identify them in most of our arguments.

§ 1. Local and global splitting of Frobenius

In this section we review some definitions and preliminary results on F-singularities and global F-regularity which motivates us for the present work.

Let X be an algebraic variety over k. We say that X is *Frobenius split*, or *F-split* for short, if the Frobenius ring homomorphism $F\colon \mathcal{O}_X \to F_*\mathcal{O}_X$ splits as an $\mathcal{O}_X$-module homomorphism, i.e., if there exists an $\mathcal{O}_X$-module homomorphism $\phi\colon F_*\mathcal{O}_X \to \mathcal{O}_X$ such that $\phi \circ F$ is the identity on $\mathcal{O}_X$.

The local version of F-splitting is called F-purity, and F-regularity is a local Frobenius splitting property stronger than F-purity:

Definition 1.1 ([18], [17]). Let R be an integral domain of characteristic $p > 0$ which is F-finite (i.e., the inclusion map $R \hookrightarrow R^{1/p}$ is module-finite).

(1) We say that R is *F-pure* if the map $R \hookrightarrow R^{1/p}$ splits as an R-module homomorphism.

(2) We say that R is *strongly F-regular* if for every nonzero element $c \in R$, there exists a power $q = p^e$ such that the inclusion map $c^{1/q}R \hookrightarrow R^{1/q}$ splits as an R-module homomorphism.

Historically, the notions of F-splitting and F-purity appeared in different contexts ([19], [18]), and later, F-regularity was defined in terms of "tight closure" [16]. The above defined strong F-regularity is another version of F-regularity, but they are known to coincide for $\mathbb{Q}$-Gorenstein rings (and in particular in dimension two). Although it is not known whether or not these two variants of F-regularity coincide in general, we sometimes say "F-regular" or "locally F-regular" to mean "strongly F-regular," since we do not treat tight closure in this paper.

Finally, global F-regularity was defined as a global version of strong F-regularity.

Definition 1.2 ([26]). Let X be a projective variety over k and fix any ample line bundle L on X. We say that X is *globally F-regular* if the following equivalent conditions hold.

(1) For any $n \geq 0$ and $0 \neq s \in H^0(X, L^{\otimes n})$, there exists $e \geq 0$ such that the composition map $\mathcal{O}_X \xrightarrow{F^e} F^e_*\mathcal{O}_X \xrightarrow{\cdot s} F^e_*L^{\otimes n}$ splits as an $\mathcal{O}_X$-module homomorphism, where $F^e\colon \mathcal{O}_X \to F^e_*\mathcal{O}_X$ denotes the e-times iterated Frobenius map.

(2) For any effective Cartier divisor D on X, there exists $e \geq 0$ such that the composition map $\mathcal{O}_X \xrightarrow{F^e} F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(D)$ splits as an $\mathcal{O}_X$-module homomorphism.

Remark. (1) Condition (1) of Definition 1.2 does not depend on the choice of an ample line bundle L, and it is equivalent to the strong F-regularity of the section ring $R(X, L) = \bigoplus_{n \geq 0} H^0(X, L^{\otimes n})$; see [14], [26].

(2) The above notions of Frobenius splitting are generalized to those for pairs (X, Δ) consisting of a normal variety X and an effective $\mathbb{R}$-divisor Δ, and even more, those for triples $(X, \Delta, \mathfrak{a}^t)$ where $\mathfrak{a} \subseteq \mathcal{O}_X$ is an ideal sheaf and $0 \leq t \in \mathbb{R}$. For example, global

F-regularity of (X, Δ) is defined by replacing the map $\mathcal{O}_X \xrightarrow{F^e} F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(D)$ in Definition 1.2 (2) by $\mathcal{O}_X \xrightarrow{F^e} F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(\lfloor (p^e-1)\Delta \rfloor + D)$. Similarly, we say that (X, Δ) is F-split if the map $\mathcal{O}_X \xrightarrow{F^e} F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(\lfloor (p^e-1)\Delta \rfloor)$ splits for all $e \geq 0$. See e.g., [13], [23], [24] for more details.

1.3. *Properties of F-singularities.* We collect some fundamental properties of F-purity and F-regularity from [16], [17], [12], [13], [28].

(1) The following implications are known:

$$\text{regular} \Rightarrow \text{strongly } F\text{-regular} \Rightarrow F\text{-pure, normal and Cohen–Macaulay.}$$

(2) F-singularities are related to singularities in MMP as follows:

$$\text{normal, } \mathbb{Q}\text{-Gorenstein and } F\text{-pure} \Rightarrow \text{log canonical singularity;}$$
$$\mathbb{Q}\text{-Gorenstein and strongly } F\text{-regular} \Rightarrow \text{log terminal singularity.}$$

The converse of the implication at the bottom holds in characteristic $p \gg 0$. Namely, if a complex variety X has only log terminal singularities, then reduction modulo p of X is locally F-regular for $p \gg 0$. These results are generalized for pairs.

Remark. It is easy to see that global splitting of Frobenius implies local splitting, i.e., if X is F-split (resp. globally F-regular), then $\mathcal{O}_{X,x}$ is F-pure (resp. strongly F-regular) for all $x \in X$. The converse of this implication does not hold. Actually, global splitting of Frobenius gives strong restriction on the structure of varieties. In particular:

(1) The Kodaira vanishing holds on F-split varieties [19]: If X is an F-split projective variety and L is an ample line bundle on X, then

$$H^i(X, L^{-1}) = 0 \text{ for all } i < \dim X.$$

If X is globally F-regular, then the above vanishing holds for any nef and big line bundle L. (Note that the Kodaira vanishing fails in characteristic p in general.)

(2) If a normal projective variety X is F-split, then $H^0(X, \mathcal{O}_X((1-p)K_X)) \neq 0$. This is because $\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \cong F_*\mathcal{O}_X((1-p)K_X)$ by the adjunction formula (see e.g., [13]), and an F-splitting $\phi\colon F_*\mathcal{O}_X \to \mathcal{O}_X$ is its non-zero global section. Similarly, if X is globally F-regular, then $-K_X$ is big. This is refined as in 1.4 (1) below.

1.4. *Globally F-regular vs. log Fano.* Global F-regularity has a strong connection with log Fano varieties. This connection is a global version of the results in 1.3 (2), and has been taken for granted as a folklore among experts. The following statements are recently established by Schwede and Smith [25]. See also a remarkable application due to Fujino and Gongyo [8].

(1) If X is globally F-regular, then there exists an effective $\mathbb{Q}$-divisor Δ on X such that $-(K_X+\Delta)$ is an ample $\mathbb{Q}$-Cartier divisor and the pair (X,Δ) is globally F-regular (and so, locally F-regular).

(2) If (X,Δ) is a log Fano pair defined over $\mathbb{C}$ in the sense that (X,Δ) is a klt pair and $-(K_X+\Delta)$ is ample, then reduction modulo p of (X,Δ) (and so, reduction modulo p of X) is globally F-regular for $p \gg 0$.

Example 1.5. One of the simplest examples of globally F-regular varieties is the projective n-space $\mathbb{P}^n$. Let $x_0,\dots,x_n$ be the homogeneous coordinates of $\mathbb{P}^n$ and $U_i = D_+(x_i)$ its basic affine open subset for $i=0,\dots,n$. Fix any $U_i \cong \mathbb{A}^n$ and let $y_1 = x_1/x_i,\dots,y_{i-1} = x_{i-1}/x_i, y_{i+1} = x_{i+1}/x_i,\dots,y_n = x_n/x_i$ be its affine coordinates. Since $F^e_*\mathcal{O}_{U_i} \cong \mathcal{O}_{U_i}^{1/q}$ is a free $\mathcal{O}_{U_i}$-module with basis $y_1^{i_1/q}\cdots y_n^{i_n/q}$ $(0 \le i_1,\dots,i_n \le q-1)$, $\mathrm{Hom}_{\mathcal{O}_{U_i}}(F^e_*\mathcal{O}_{U_i},\mathcal{O}_{U_i}) \cong \mathrm{Hom}_{\mathcal{O}_{U_i}}(\mathcal{O}_{U_i}^{1/q},\mathcal{O}_{U_i})$ is generated as an $\mathcal{O}_{U_i}$-module by the dual basis $\phi_{i_1,\dots,i_n}$ $(0 \le i_1,\dots,i_n \le q-1)$. If we identify the anticanonical divisor $-K_{\mathbb{P}^n}$ with the reduced toric divisor Δ consisting of coordinate hyperplanes, then setting $q=p^e$ we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(F^e_*\mathcal{O}_{\mathbb{P}^n},\mathcal{O}_{\mathbb{P}^n}) &\cong H^0(\mathbb{P}^n, F^e_*\mathcal{O}_{\mathbb{P}^n}((1-q)K_{\mathbb{P}^n})) \cong H^0(\mathbb{P}^n, F^e_*\mathcal{O}_{\mathbb{P}^n}((q-1)\Delta))\\ &= \left\langle y_1^{(j_1-q+1)/q}\cdots y_n^{(j_n-q+1)/q} \;\middle|\; \begin{array}{l} j_1,\dots,j_n \ge 0,\\ j_1+\cdots+j_n \le (n+1)(q-1)\end{array}\right\rangle\end{aligned}$$

by the adjunction formula, via which $\phi_{i_1,\dots,i_n}$ corresponds to $y_1^{-i_1/q}\cdots y_n^{-i_n/q}$ in the right-hand side. In particular, $\phi_{0,\dots,0} \in \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(F^e_*\mathcal{O}_{\mathbb{P}^n},\mathcal{O}_{\mathbb{P}^n})$ and it gives a canonical splitting of $F^e\colon \mathcal{O}_{\mathbb{P}^n} \to F^e_*\mathcal{O}_{\mathbb{P}^n}$. Thus $\mathbb{P}^n$ is F-split, and we can see easily that $\mathbb{P}^n$ is globally F-regular by a similar argument. More generally, we have the following.

(1) Any projective toric variety X is globally F-regular. Indeed, the section ring $R = R(X,L)$ of X with respect to an ample line bundle L is a toric ring, so that it is a direct summand of a regular ring. Hence R is strongly F-regular, and so X is globally F-regular.

(2) Any toric variety X is F-split and locally F-regular. Even more, if $\Delta \sim -K_X$ is the reduced toric divisor, then the pair (X,Δ) is F-split; cf. [13, Corollary 2.5].

(3) Let $X \subseteq \mathbb{P}^n$ be a hypersurface defined by a homogeneous polynomial $f \in S = k[x_0,\dots,x_n]$. Then X is F-split if and only if $f^{p-1} \notin (x_0^p,\dots,x_n^p)$ in S. There are similar criteria for global F-regularity, and F-purity and F-regularity of hypersurface singularities. These are special cases of the Fedder-type criteria [7].

We close this section with some conditions for morphisms under which splitting of Frobenius inherits. In the following proposition we assume that the varieties under consideration are projective, whenever we speak of global F-regularity.

Proposition 1.6. *Let $f\colon X \to Y$ be a morphism of varieties over k satisfying either of the following conditions.*

(1) *f is a projective morphism with $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

(2) *The ring homomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ splits as an $\mathcal{O}_Y$-module homomorphism.*

Then, if X is globally F-regular (resp. F-split), so is Y.

Proof. As for case (1), see [14]. Case (2) is proved similarly as a well known fact that strong F-regularity and F-purity inherit to pure subrings. □

§ 2. Frobenius sandwiches

Definition 2.1. Let X be a smooth variety over k and let e be a positive integer. We say that a normal variety Y is an F^e*-sandwich* of X if the eth iterated relative Frobenius morphism of X factors as

$$\begin{tikzcd} X \arrow[rr, "F^e_{\mathrm{rel}}"] \arrow[dr, "\pi"'] & & X^{(-e)} \\ & Y \arrow[ur, "\rho"'] & \end{tikzcd}$$

for some finite k-morphisms $\pi\colon X \to Y$ and $\rho\colon Y \to X^{(-e)}$, which are homeomorphisms in the Zariski topology. An *F-sandwich* will mean an F^1-sandwich.

Remark. (1) By the normality, an F^e-sandwich Y of X is determined by its rational function field $k(Y)$ with $k(X^{(-e)}) \subseteq k(Y) \subseteq k(X)$.

(2) We say that the Frobenius sandwich Y is of *exponent one* if the degree of the morphism $\pi\colon X \to Y$ is p. It is known that F-sandwiches Y of X of exponent one are in one-to-one correspondence with saturated p-closed invertible subsheaves L of the tangent bundle T_X, where L is said to be p-closed if it is closed under p-times iterated composite of differential operators. The correspondence is given by

$$L \mapsto \mathcal{O}_Y = \{f \in \mathcal{O}_X \mid \delta(f) = 0 \text{ for all } \delta \in L\},$$

$$Y \mapsto L = \{\delta \in T_X \mid \delta(f) = 0 \text{ for all } f \in \mathcal{O}_Y\}.$$

See [6] for more details.

(3) An F^e-sandwich of a smooth variety may be singular. We call such a singularity an *F^e-sandwich singularity.* Also, an F^e-sandwich of a globally F-regular variety may not be F-split. Indeed, there is an F-sandwich of $\mathbb{P}^2$ whose minimal resolution is a surface of general type (so is not F-split) called a Zariski surface [33]. Zariski surfaces seem to reflect pathological aspects of purely inseparable morphisms. But if we assume global or local F-regularity for Frobenius sandwiches, the situation becomes much simpler.

We rephrase Proposition 1.6 (2) for Frobenius sandwiches as follows.

Proposition 2.2. *Let Y be an F^e-sandwich of X and assume that X is globally F-regular (resp. locally F-regular). Then the following conditions are equivalent:*

(1) *Y is F-split (resp. locally F-pure);*

(2) *Y is globally F-regular (resp. locally F-regular);*

(3) *The ring homomorphism $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ associated to $\pi\colon X \to Y$ splits (resp. splits locally) as an $\mathcal{O}_Y$-module homomorphism.*

Example 2.3. Let $X = \mathbb{A}^n$ with affine coordinate ring $\mathcal{O}_X = k[x_1, \ldots, x_n]$, and let Y be an F-sandwich of X of exponent one. Then there exists a nonzero p-closed vector field $\delta \in T_X$ such that the affine coordinate ring of Y is the *constant ring* of δ:

$$\mathcal{O}_Y = \mathcal{O}_X^\delta := \{f \in \mathcal{O}_X | \delta(f) = 0\} \subset \mathcal{O}_X.$$

Here are examples of F-regular and non-F-regular F-sandwich singularities:

(1) Let $0 \neq \delta = \sum_{i=1}^n a_i x_i \partial/\partial x_i \in T_X$ with $a_1, \ldots, a_n \in \mathbb{F}_p$. This is a p-closed vector field with $\delta^p = \delta$, and $Y = X/\delta$ has a toric singularity of type $\frac{1}{p}(a_1, \ldots, a_n)$, i.e.,

$$\mathcal{O}_X^\delta = k[x_1^{l_1} \cdots x_n^{l_n} | l_1, \ldots, l_n \geq 0; a_1 l_1 + \cdots + a_n l_n \equiv 0 \,(\mathrm{mod}\, p)].$$

(2) Let $n = 2$, $\mathcal{O}_X = k[x, y]$ and let $\delta = x^p \partial/\partial x + y^p \partial/\partial y \in T_X = \mathrm{Der}_k(\mathcal{O}_X)$. This is a nilpotent derivation (i.e., $\delta^p = 0$) and

$$\mathcal{O}_X^\delta = k[x^p, x^p y - x y^p, y^p] \cong k[X, Y, Z]/(Z^p - (X^p Y - X Y^p)).$$

It follows from Fedder's criterion [7] that this is not F-regular; see Example 4.4 for more details.

Since local and global F-regularity impose undoubtedly strong restriction on the structure of singularities and projective varieties in characteristic $p > 0$, respectively, it is natural to ask the following

Problem. Characterize F-regular Frobenius sandwich singularities and globally F-regular Frobenius sandwiches of a given globally F-regular variety. More specifically:

(1) Is an F-regular F^e-sandwich singularity always a toric singularity?

(2) Given a smooth toric surface X, classify globally F-regular F-sandwiches of X.

First we give a partial answer to the problem (1) above. Since the problem is local, the results are stated for a local ring $(R, \mathfrak{m})$. The following proposition is essentially proved by Aramova [1], [2], [3]. A detailed proof is recorded in [21].

Proposition 2.4. *Let R be an F-sandwich singularity of $S = k[[x_1, \dots, x_n]]$ of exponent one, i.e., $R = S^\delta$ for some p-closed vector field $\delta \in \mathrm{Der}_k(S)$. If R is strongly F-regular, then there is a p-closed derivation $\delta' = \sum_{i=1}^n a_i x_i \partial/\partial x_i \in \mathrm{Der}_k(S)$ with $a_1, \dots, a_n \in \mathbb{F}_p$ such that $R \cong S^{\delta'}$.*

Sketch of Proof. For simplicity, we consider the case where $S = k[[x, y]]$, and let $\mathfrak{m} = (x, y)$. If $\delta(S) \not\subseteq \mathfrak{m}$, then $R = S^\delta$ is regular, and the conclusion follows easily. So we may assume $\delta(S) \subseteq \mathfrak{m}$. We have $\delta^p = \alpha\delta$ for some $\alpha \in R$ by the assumption. Since R is strongly F-regular, the inclusion map $R \hookrightarrow S$ splits as an R-module homomorphism. Then we see that $\alpha \in S^\times$ (see the proof of Lemma 3.2), so that $\alpha^{-1/(p-1)} \in S$. Thus $\alpha^{-1/(p-1)} \in R$. Replacing δ by $\alpha^{-1/(p-1)}\delta$, we may assume that $\delta^p = \delta$.

We define $\delta_i \in \mathrm{Der}_k(\mathfrak{m}/\mathfrak{m}^i)$ by $\delta_i(\overline{s}) = \overline{\delta(s)}$. Since $\delta^p - \delta = 0$, the minimal polynomial $\mu_{\delta_i}(t) \in k[t]$ of δ_i divides $t^p - t = t(t-1)(t-2)\cdots(t-(p-1))$. Hence δ_i is diagonalizable with eigenvalues in $\mathbb{F}_p$. Let $z_2, w_2 \in \mathfrak{m}/\mathfrak{m}^2$ be linearly independent eigenvectors of δ_2 and $a, b \in \mathbb{F}_p$ their eigenvalues, respectively. Now we choose $z_i, w_i \in \mathfrak{m}/\mathfrak{m}^i$ for $i \geq 3$ inductively as follows: Given z_{i-1}, w_{i-1}, we can choose z_i, w_i so that (i) they are part of a basis of $\mathfrak{m}/\mathfrak{m}^i$ with eigenvalues a, b; and (ii) their images by a natural surjection $\mathfrak{m}/\mathfrak{m}^i \to \mathfrak{m}/\mathfrak{m}^{i-1}$ are z_{i-1}, w_{i-1}. Let $z = \lim_{i\to\infty} z_i$, $w = \lim_{i\to\infty} w_i \in S$. Then z, w are a regular system of parameters of $S = \varprojlim S/\mathfrak{m}^i$ and $\delta = az\partial/\partial z + bw\partial/\partial w$. $\square$

Proposition 2.5. *Let $(R, \mathfrak{m})$ be a two-dimensional F^e-sandwich double point. If R is strongly F-regular, then R is an A_{q-1}-singularity for some $q \mid p^{2e}$.*

2.6. *Hilbert-Kunz multiplicity.* The above proposition is proved by using the invariant called the Hilbert-Kunz multiplicity introduced by Monsky [20].

Let $(R, \mathfrak{m})$ be an n-dimensional Noetherian local ring over k with $R/\mathfrak{m} = k$ and let I be an $\mathfrak{m}$-primary ideal. Then the Hilbert-Kunz multiplicity $e_{\mathrm{HK}}(I, R)$ is defined by

$$e_{\mathrm{HK}}(I, R) = \lim_{e\to\infty} \frac{\mathrm{length}_R(R/I^{[p^e]})}{p^{ne}} = \lim_{e\to\infty} \frac{\dim_k R/I^{[p^e]}}{p^{ne}}.$$

It is known that this limit exists and is a positive real number.

In general, it is hard to compute $e_{\mathrm{HK}}(I, R)$, but if a regular local ring S is a module-finite extension of R of degree $r = \mathrm{rank}_R S$, it is computed as

$$e_{\mathrm{HK}}(I, R) = \frac{e_{\mathrm{HK}}(IS, S)}{r} = \frac{\mathrm{length}_S(S/IS)}{r}, \tag{2.1}$$

see [30]. In particular, if R is an F^e-sandwich of S, then $e_{\mathrm{HK}}(I, R) \in \frac{1}{p^{ne}}\mathbb{Z}$.

Proof of Proposition 2.5. We have $e_{\mathrm{HK}}(\mathfrak{m}, R) \in \frac{1}{p^{2e}}\mathbb{Z}$ by the assumption. On the other hand, if R is an F-regular double point of $\dim R = 2$, then R has the same ordinary defining equation as in characteristic zero ([4], [11]), and one can take a module-finite extension $R \subseteq S$ with S regular local such that $r = \mathrm{rank}_R S$ is not a power of p unless R is an A_{q-1}-singularity for some power q of p. For example, if R is an E_8-singularity, then $p > 5$ by F-regularity, and $R = S^G$ for a finite group G of order $r = 120$, which is not divisible by p. Then the McKay correspondence holds true as in the formula (4.1) before Proposition 4.9, so that $e_{\mathrm{HK}}(\mathfrak{m}, R) = 2 - 1/r$ by [30]. This leads to a contradiction if R is a rational double point of type D_n, E_6, E_7 or E_8. $\square$

Frobenius sandwich surfaces. We quickly review generalities on Frobenius sandwiches of a smooth surface for the next section. See [6], [15] for further details.

Let X be a smooth projective surface and let $k(X)$ be the function field of X. We define an equivalence relation $\sim$ between rational vector fields $\delta, \delta' \in \mathrm{Der}_k k(X)$ as follows: We write $\delta \sim \delta'$ if there is a nonzero rational function $\alpha \in k(X)$ such that $\delta = \alpha\delta'$.

By a 1-*foliation* of X, we mean a saturated p-closed invertible subsheaf of the tangent bundle T_X. For a 1-foliation L, we have the natural exact sequence

$$0 \longrightarrow L \longrightarrow T_X \longrightarrow I_Z \otimes L' \longrightarrow 0,$$

where I_Z is the defining ideal sheaf of a zero-dimensional subscheme Z and L' is an invertible sheaf. We call Z the singular locus of L and denote it by $\mathrm{Sing}\, L$.

Now we recall the relationship between rational vector fields and 1-foliations. A rational vector field $\delta \in \mathrm{Der}_k k(X)$ is locally expressed as $\alpha(f\partial/\partial s + g\partial/\partial t)$ where s, t are local coordinates, f, g are regular functions without a common factor and $\alpha \in k(X)$. The divisor $(\delta)_0$ associated to δ is defined by glueing the divisors $(\alpha)_0$ on affine open sets. Let Z be the zero-dimensional subscheme of X defined locally by $f = g = 0$. Then we have the natural exact sequence

$$0 \longrightarrow \mathcal{O}_X((\delta)_0) \longrightarrow T_X \longrightarrow I_Z \otimes L' \longrightarrow 0,$$

where I_Z is the defining ideal sheaf of Z and L' is an invertible sheaf. It follows that $\delta \mapsto \mathcal{O}_X((\delta)_0)$ gives a one-to-one correspondence between nonzero p-closed rational vector fields modulo equivalence and 1-foliations.

Theorem 2.7 (Ekedahl [6])**.** *Let X be a smooth projective surface. Then there is a one-to-one correspondence between 1-foliations $L \subset T_X$ and F-sandwich surfaces Y of X of exponent one, given by foliation quotient $\pi \colon X \to Y = X/L$. Furthermore,*

$$\mathrm{Sing}\ Y = \pi(\mathrm{Sing}\ L),$$

where $\mathrm{Sing}\, Y$ *is the singular locus of* Y*, and outside* $\mathrm{Sing}\, L$ *we have the canonical bundle formula*

$$\omega_X \cong \pi^*\omega_Y \otimes L^{\otimes(p-1)}.$$

§ 3. Globally F-regular Frobenius sandwiches of $\mathbb{P}^2$ and Σ_d

In this section we classify globally F-regular F-sandwiches of the projective plane and Hirzebruch surfaces of exponent one. Ganong and Russell showed in [9] that (non-trivial) F-sandwiches of the projective plane are singular and moreover that for each Hirzebruch surface there are at most two smooth F-sandwiches. (The former result was first proved by Bloch.) We study these F-sandwiches from the viewpoint of Frobenius splitting.

Globally F-regular F-sandwiches of $\mathbb{P}^2$. Let X_0, X_1 and X_2 be homogeneous coordinates of $\mathbb{P}^2$, i.e., $\mathbb{P}^2 = \operatorname{Proj} k[X_0, X_1, X_2]$. Let $x = X_1/X_0$, $y = X_2/X_0$ (resp. $z = X_0/X_1$, $w = X_2/X_1$; $u = X_0/X_2$, $v = X_1/X_2$) be the affine coordinates of $U_0 := D_+(X_0)$ (resp. $U_1 := D_+(X_1)$; $U_2 := D_+(X_2)$).

Let $N \cong \mathbb{Z}^2$ be a lattice with standard basis $e_1 = (1,0)$, $e_2 = (0,1)$, $M = \operatorname{Hom}(N, \mathbb{Z})$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. For a fan Δ in N, we denote the associated toric variety over k by $T_N(\Delta)$.

$\mathbb{P}^2$ is a toric surface given by the complete fan whose rays are spanned by $\rho_0 = e_2$, $\rho_1 = e_1$ and $\rho_2 = e_1 - e_2$. For $i = 0, 1, 2$, let D_i be the divisor that corresponds to the ray spanned by ρ_i.

We state here some lemmas which will be used in the proof of Theorem 3.4.

Lemma 3.1. *Let $\delta \in \operatorname{Der}_k k(\mathbb{P}^2)$. If the corresponding 1-foliation $L = \mathcal{O}_{\mathbb{P}^2}((\delta)_0)$ has a nonzero global section, then*

$$\delta \sim (ax + by + cx^2 + dxy + e)\frac{\partial}{\partial x} + (fx + gy + dy^2 + cxy + h)\frac{\partial}{\partial y},$$

where $a, b, c, d, e, f, g, h \in k$.

Proof. Multiplying a suitable rational function on $\mathbb{P}^2$, we may assume that

$$\delta \sim \sum_{0 \le i,j} a_{ij} x^i y^j \frac{\partial}{\partial x} + \sum_{0 \le m,n} b_{mn} x^m y^n \frac{\partial}{\partial y},$$

where $a_{ij}, b_{mn} \in k$, and $\sum_{0 \le i,j} a_{ij} x^i y^j$ and $\sum_{0 \le m,n} b_{mn} x^m y^n$ have no common factor. If we express δ for the local coordinates z, w (resp. u, v), then we easily see that the coefficient of δ in $\partial/\partial z$ (resp. $\partial/\partial u$) equals $-\sum_{0 \le i,j} a_{ij} w^j / z^{i+j-2}$ (resp. $-\sum_{0 \le m,n} b_{mn} v^m / u^{m+n-2}$). Since $\sum_{0 \le i,j} a_{ij} x^i y^j$ and $\sum_{0 \le m,n} b_{mn} x^m y^n$ have no common factor, we have $\deg(\delta)_0 = \operatorname{ord}_{D_2}(\delta)_0$. Now $\deg L \ge 0$, so that $a_{ij} = 0$ for $i + j \ge 3$ (resp. $b_{mn} = 0$ for $m + n \ge 3$). Thus we may assume that $\delta \sim \sum_{0 \le i+j \le 2} a_{ij} x^i y^j \partial/\partial x + \sum_{0 \le m+n \le 2} b_{mn} x^m y^n \partial/\partial y$. Considering δ for the local coordinates z, w again, we see that $b_{20} = a_{02} = 0$, $b_{11} = a_{20}$ and $b_{02} = a_{11}$. This means that $\delta \sim (ax + by + cx^2 + dxy + e)\partial/\partial x + (fx + gy + dy^2 + cxy + h)\partial/\partial y$, where $a, b, c, d, e, f, g, h \in k$. $\square$

Lemma 3.2. *Let $S = k[x, y]$ and let $\delta = f\partial/\partial x + g\partial/\partial x \in \mathrm{Der}_k(S)$, where $f, g \in (x, y)$ and have no common factor. Suppose δ is p-closed. If the inclusion $S^\delta \subset S$ splits as an S^δ-module, then δ is not nilpotent.*

Proof. We have $\delta^p = \alpha\delta$ for some $\alpha \in S$ by the assumption. Now the inclusion $i : S^\delta \hookrightarrow S$ splits as an S^δ-module, so that there is an endomorphism $\varphi \in \mathrm{End}_{S^\delta}(S)$ such that $\varphi \circ i = \mathrm{id}_{S^\delta}$. Since f and g have no common factor, $\mathrm{Der}_k(S)/\langle\delta\rangle$ is torsion-free, where $\langle\delta\rangle$ is the S-submodule of $\mathrm{Der}_k(S)$ spanned by δ^{p^i} $(i \geq 0)$. By the Galois correspondence established in [3], we see that $\langle\delta\rangle \mapsto S^\delta \mapsto \mathrm{Der}_{S^\delta}(S) = \langle\delta\rangle$. Thus by [3, Proposition 2.7] we have $S[\delta] = \mathrm{End}_{S^\delta}(S)$, which means that $\varphi = \sum_{i=0}^{p-1} a_i\delta^i$ for some $a_i \in S$. Since $\varphi \circ i = \mathrm{id}_{S^\delta}$, we have $a_0 = 1$. Considering the coefficient of $\delta \circ \varphi$ $(= 0)$ in δ^1, we see that $\delta(a_1) + 1 + \alpha a_{p-1} = 0$. Since $f, g \in (x, y)$, the linear term of αa_{p-1} is equal to -1. Therefore δ is not nilpotent. □

One can check easily the following lemma.

Lemma 3.3. *Let $\delta = (ax + by + c)\partial/\partial x + (dy + e)\partial/\partial y \in \mathrm{Der}_k(k[x, y])$, where $a, b, c, d, e \in k$, and $ax+by+c$ and $dy+e$ have no common factor. Suppose δ is p-closed. If δ is not nilpotent, then $a, d \neq 0$.*

For $i = 1, \ldots, p-1$, let Δ^i be the complete fan whose rays are spanned by e_2, $pe_1 - ie_2$ and $-pe_1 + (i-1)e_2$.

Theorem 3.4. *A globally F-regular F-sandwich of $\mathbb{P}^2$ of exponent one is isomorphic to either one of the singular toric surfaces $T_N(\Delta^i)$ $(1 \leq i \leq p-1)$. In particular, there are just $p-1$ isomorphism classes of globally F-regular F-sandwiches.*

Proof. Let $\pi : \mathbb{P}^2 \to Y$ be a globally F-regular F-sandwich surface and let $L \subset T_{\mathbb{P}^2}$ (resp. $\delta \in \mathrm{Der}_k k(\mathbb{P}^2)$) be the corresponding 1-foliation (resp. the rational vector field). Since the associated ring homomorphism $\mathcal{O}_Y \to \pi_*\mathcal{O}_{\mathbb{P}^2}$ splits by Proposition 2.2, there is a nonzero $\mathcal{O}_Y$-module homomorphism $\pi_*\mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_Y$. Outside $\mathrm{Sing}\, Y$ we have

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_Y}(\pi_*\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_Y) &\cong \mathcal{H}om_{\mathcal{O}_Y}(\pi_*\mathcal{O}_{\mathbb{P}^2}, \omega_Y) \otimes \omega_Y^{-1} \\ &\cong \pi_*(\omega_{\mathbb{P}^2}) \otimes \omega_Y^{-1} \cong \pi_*(\omega_{\mathbb{P}^2} \otimes \pi^*(\omega_Y^{-1})) \cong \pi_*(L^{\otimes(p-1)}),\end{aligned}$$

which gives a (global) isomorphism $\mathrm{Hom}_{\mathcal{O}_Y}(\pi_*\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_Y) \cong H^0(\mathbb{P}^2, L^{\otimes(p-1)})$ since Y is normal. Thus L has a nonzero global section. In particular, $\deg L \geq 0$. On the other hand, since the tangent bundle $T_{\mathbb{P}^2}$ is stable, we have $\deg L < 3/2$. Therefore we conclude that $L \cong \mathcal{O}_{\mathbb{P}^2}(1)$ or $\mathcal{O}_{\mathbb{P}^2}$. Now we have the induced exact sequence

$$0 \longrightarrow L \longrightarrow T_{\mathbb{P}^2} \longrightarrow I_{\mathrm{Sing}\, L} \otimes L' \longrightarrow 0,$$

where $I_{\operatorname{Sing} L}$ is the ideal sheaf of $\operatorname{Sing} L$ and L' is an invertible sheaf. From this sequence we see that the second Chern class $c_2(I_{\operatorname{Sing} L}) = 1$ or 3, which means that there is a singular point on Y. After a suitable change of coordinates, we may assume that Y is singular at the point corresponding to the origin of U_0. Then by Lemma 3.1 we may assume that

$$\delta = (ax + by + cx^2 + dxy)\frac{\partial}{\partial x} + (ex + fy + dy^2 + cxy)\frac{\partial}{\partial y},$$

where $a, b, c, d, e, f \in k$.

First suppose that the 1-foliation $L \subset T_{\mathbb{P}^2}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(1)$. If we express δ for the local coordinates z, w, we have $\delta = -(bzw+az+dw+c)\partial/\partial z - (bw^2+(a-f)w-e)\partial/\partial w$. Suppose that $ax + by + cx^2 + dxy$ and $ex + fy + dy^2 + cxy$ have no common factor. Then we have $\deg(\delta)_0 = \operatorname{ord}_{D_2}(\delta)_0$. Now $L \cong \mathcal{O}_{\mathbb{P}^2}(1)$, so that $bzw + az + dw + c$ and $bw^2 + (a-f)w - e$ must have the common factor z, which implies $b = c = d = e = 0$ and $a = f$. Thus $\delta \sim x\partial/\partial x + y\partial/\partial y$. Similarly we have $\delta \sim x\partial/\partial x + y\partial/\partial y$ in the case where $ax + by + cx^2 + dxy$ and $ex + fy + dy^2 + cxy$ have a common factor of degree 1. If they have a common factor of degree 2, then Y is smooth on the image of U_0, which is a contradiction. Now we easily see that the corresponding globally F-regular F-sandwich Y is isomorphic to the singular toric surface $T_N(\Delta^1)$. In this case, Y has only one singular point.

Next suppose that the 1-foliation $L \subset T_{\mathbb{P}^2}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}$. We will show that after suitable changes of coordinates, we have

$$\delta \sim \alpha\frac{\partial}{\partial \alpha} + i\beta\frac{\partial}{\partial \beta},$$

where α (resp. β) is the local coordinate z or u (resp. w or v) and $i \in (\mathbb{Z}/p\mathbb{Z})^\times$, $i \not\equiv 1 \pmod p$.

Suppose that $ax + by + cx^2 + dxy$ and $ex + fy + dy^2 + cxy$ have no common factor. Then δ is not nilpotent by Lemma 3.2. Suppose $b = 0$. For the local coordinate z, w, we have $\delta = -(az + dw + c)\partial/\partial z + ((f - a)w + e)\partial/\partial w$. By Lemma 3.3 we have $a, f - a \neq 0$. After a change of coordinates as $z - (ac - cf + de)/a(f - a) \mapsto z$ and $w - e/(a - f) \mapsto w$, we have $\delta = -(az + dw)\partial/\partial z + (f - a)w\partial/\partial w$. Since $a \neq 0$ and δ is p-closed, we have $\delta \sim (z + sw)\partial/\partial z + iw\partial/\partial w$, where $s \in k$, $i \in (\mathbb{Z}/p\mathbb{Z})^\times$. For the local coordinates u, v, we have $(z + sw)\partial/\partial z + iw\partial/\partial w = ((1 - i)u + s)\partial/\partial u - iv\partial/\partial v$. After a suitable change of coordinates, we eventually have $\delta \sim u\partial/\partial u + iv\partial/\partial v$, where $i \in (\mathbb{Z}/p\mathbb{Z})^\times$, $i \not\equiv 1 \pmod p$. (Since Y is singular on the image of U_0, we see that $i \not\equiv 1 \pmod p$.)

Next suppose that $b \neq 0$. We may assume $b = 1$. For the local coordinates z, w, we have $\delta = -(zw + az + dw + c)\partial/\partial z - (w^2 + (a - f)w - e)\partial/\partial w$. Let $A_+, A_- \in k$ be the roots of the quadratic equation $w^2 + (a - f)w - e = 0$.

Now we show that $a + A_+ \neq 0$. If $a + A_+ = 0$, then we have $\delta = -(zw + az + dw + c)\partial/\partial z - (w + a)(w - f)\partial/\partial w$. If $a + f = 0$, then after a change of a coordinate as $w - f \mapsto w$, we have $\delta = -(zw + dw + fd + c)\partial/\partial z - w^2\partial/\partial w$. For the local coordinates u, v, we have $\delta = -((fd + c)v + d)\partial/\partial u + \partial/\partial v$. This implies that δ is nilpotent or not p-closed, which is a contradiction. Next suppose that $a + f \neq 0$. After a change of coordinates as $z + (df + c)/(a + f) \mapsto z$ and $w - f \mapsto w$, we have

$$\delta = \frac{-ad + c}{a + f}\frac{\partial}{\partial u} + ((a + f)v + 1)\frac{\partial}{\partial v}$$

for the local coordinates u, v. If $-ad + c \neq 0$, then δ is not p-closed, which is a contradiction. If $-ad + c = 0$, then δ defines a nonzero divisor on U_0, which is a contradiction. Therefore we conclude that $a + A_+ \neq 0$.

After a change of coordinates as $z + (dA_+ + c)/(a + A_+) \mapsto z$ and $w - A_- \mapsto w$, we have

$$\delta = -\left((a + A_-)u + \frac{ad - c}{a + f}\right)\frac{\partial}{\partial u} + ((a + A_+)v + 1)\frac{\partial}{\partial v}$$

for the local coordinates u, v. After a suitable change of coordinates, we have $\delta \sim u\partial/\partial u + iv\partial/\partial v$, where $i \in (\mathbb{Z}/p\mathbb{Z})^\times$, $i \not\equiv 1 \pmod p$.

Therefore we conclude that

$$\delta \sim x\frac{\partial}{\partial x} + iy\frac{\partial}{\partial y},$$

where $i \in (\mathbb{Z}/p\mathbb{Z})^\times$, $i \not\equiv 1 \pmod p$. The same holds for the case where $ax+by+cx^2+dxy$ and $ex + fy + dy^2 + cxy$ have a common factor of degree 1. If they have a common factor of degree 2, then Y is smooth on the image of U_0, which is a contradiction. Now we easily see that the corresponding globally F-regular F-sandwich Y is isomorphic to the singular toric surface $T_N(\Delta^i)$. In this case, Y has 3 singular points. $\square$

Globally F-regular F-sandwiches of Hirzebruch surfaces. The Hirzebruch surface Σ_d $(d \geq 0)$ is the $\mathbb{P}^1$-bundle associated to the vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d)$ on $\mathbb{P}^1$. It is well-known that Σ_d is the union of affine planes whose affine coordinate rings are $k[x, y]$, $k[x^d y, 1/x]$, $k[1/x, 1/x^d y]$ and $k[1/y, x]$, respectively.

As with Lemma 3.1, we can prove the following lemma.

Lemma 3.5. *Let $\delta \in \operatorname{Der}_k k(\Sigma_d)$. Suppose that the corresponding 1-foliation $\mathcal{O}_{\Sigma_d}((\delta)_0)$ has a nonzero global section. Then we have the following.*

(1) *If $d = 0$, then $\delta \sim (ax^2 + bx + c)\partial/\partial x + (ey^2 + fy + g)\partial/\partial y$, where $a, b, c, e, f, g \in k$.*

(2) *If $d \geq 1$, then $\delta \sim (ax^2 + bx + c)\partial/\partial x + (F(x)y - dax + e)y\partial/\partial y$, where $a, b, c, e \in k$ and $F(x) \in k[x]$ with $\deg F(x) \leq d$.*

Let Δ_{di} $(1 \leq i \leq p-1)$ (resp. Δ_{d0} ; Δ_{dp}) be the complete fan whose rays are spanned by e_2, $pe_1 - ie_2$, $-e_2$ and $-pe_1 + (i+d)e_2$ (resp. e_2, e_1, $-e_2$ and $-pe_1 + de_2$; e_2, e_1, $-e_2$ and $-e_1 + dpe_2$).

By Lemma 3.5, we easily see that a globally F-regular F-sandwich of Σ_0 of exponent one is isomorphic to either one of the toric surfaces $T_N(\Delta_{0i})$ $(0 \leq i \leq p-1)$. (In particular, there are just p isomorphism classes of globally F-regular F-sandwiches.) We can prove similar results in the case where $d \geq 1$. See [21], [22].

Theorem 3.6. *A globally F-regular F-sandwich of Σ_d $(d \geq 1)$ of exponent one is isomorphic to either one of the toric surfaces $T_N(\Delta_{di})$ $(0 \leq i \leq p)$. In particular, there are just $p+1$ isomorphism classes of globally F-regular F-sandwiches.*

§ 4. F-blowups vs. minimal resolution for surface singularities

In this section, we consider the F-blowups of surface singularities, and give examples of an F-sandwich surface singularity whose F-blowup is not the minimal resolution.

In what follows, we denote by $q = p^e$ a power of char $k = p > 0$. For a variety X defined over k, we identify the eth iterate of Frobenius map $F^e \colon \mathcal{O}_X \to F^e_* \mathcal{O}_X$ with the inclusion map $\mathcal{O}_X \hookrightarrow \mathcal{O}_X^{1/q}$. In this manner the induced morphism $X^{1/q} := \mathrm{Spec}_X \mathcal{O}_X^{1/q} \to X$ is identified with the eth absolute Frobenius morphism $F^e \colon X \to X$. (We also abuse the absolute and relative Frobenius, since it is harmless under our assumption that k is algebraically closed.)

Definition 4.1 (Yasuda [31]). Let X be a variety of $\dim X = n$ over k. The eth *F-blowup* $\mathrm{FB}_e(X)$ of X is defined to be the irreducible component $\mathrm{Hilb}_{p^{ne}}(X^{1/p^e}/X)^\circ$ of the relative Hilbert scheme $\mathrm{Hilb}_{p^{ne}}(X^{1/p^e}/X)$ that dominates X, together with the projective birational morphism $\varphi \colon \mathrm{FB}_e(X) \to X$.

By definition, $\varphi \colon Z = \mathrm{FB}_e(X) \to X$ satisfies the property that the torsion-free pullback $\varphi^\star \mathcal{O}_X^{1/q} := \varphi^* \mathcal{O}_X^{1/q}/\text{torsion}$ is a flat (equivalently, locally free) $\mathcal{O}_Z$-algebra of rank $q^n = p^{ne}$, and $\mathrm{FB}_e(X)$ is universal with respect to this property.

Through the remainder of this section, we consider the surface case $n = 2$ and work under the following notation: Let (X, x) be a normal surface singularity defined over k. Since we are interested in birational modifications of an isolated singularity (X, x), we will presumably put $X = \mathrm{Spec}\, \mathcal{O}_{X,x}$. Let $f \colon \widetilde{X} \to X$ be the minimal resolution.

We will consider the following question raised by Yasuda:

Question. Is $\mathrm{FB}_e(X)$ equal to the minimal resolution $\widetilde{X}$?

It is proved that $\mathrm{FB}_e(X) = \widetilde{X}$ for $e \gg 0$ if X is either a toric singularity [31] or a tame quotient singularity [29]. When $X = S/G$ is a quotient of smooth S by a finite group G of order not divisible by p, the essential part is to prove the isomorphism $\mathrm{FB}_e(X) \cong \mathrm{Hilb}^G(S)$ of the F-blowup with the G-Hilbert scheme. In general, it is easy to see that F-blowups of a quotient singularity are dominated by the G-Hilbert scheme, and the following is a slight generalization thereof; see [31].

Proposition 4.2. *Suppose that $\pi\colon S \to X$ is a finite morphism of degree d with S smooth such that the associated ring homomorphism $\mathcal{O}_X \to \pi_*\mathcal{O}_S$ splits as an $\mathcal{O}_X$-module homomorphism. Then the F-blowups of X are dominated by $\mathrm{Hilb}_d(S/X)^\circ$, i.e., for all $e \geq 0$ there exists a morphism over X,*

$$\mathrm{Hilb}_d(S/X)^\circ \to \mathrm{FB}_e(X).$$

Proof. We will show that there exist morphisms over X,

$$\mathrm{Hilb}_d(S/X) \xrightarrow{\sim} \mathrm{Hilb}_{dq^2}(S^{1/q}/X) \to \mathrm{Hilb}_{q^2}(X^{1/q}/X).$$

The isomorphism on the left exists because $S^{1/q} \to S$ is faithfully flat of degree q^2 by the smoothness of S. The morphism on the right is constructed similarly as in [32]: The map $\mathcal{O}_X^{1/q} \to \pi_*\mathcal{O}_S^{1/q}$ splits as an $\mathcal{O}_X$-module homomorphism by the assumption, so that its torsion-free pullback to $W = \mathrm{Hilb}_{dq^2}(S^{1/q}/X)$ by $\psi\colon W \to X$, $\psi^\star\mathcal{O}_X^{1/q} \to \psi^\star\pi_*\mathcal{O}_S^{1/q}$ splits as an $\mathcal{O}_W$-module homomorphism. Since $\psi^\star\pi_*\mathcal{O}_S^{1/q}$ is a flat $\mathcal{O}_W$-module, its direct summand $\psi^\star\mathcal{O}_X^{1/q}$ is also flat (of rank q^2) over W. Thus we have $\mathrm{Hilb}_{dq^2}(S^{1/q}/X) \to \mathrm{Hilb}_{q^2}(X^{1/q}/X)$ by the universal property of $\mathrm{Hilb}_{q^2}(X^{1/q}/X)$.

Composing the above, we get a morphism $\mathrm{Hilb}_d(S/X) \to \mathrm{Hilb}_{q^2}(X^{1/q}/X)$ over X, and taking the irreducible components dominating X gives the desired morphism. $\square$

For Frobenius sandwiches we have even more:

Proposition 4.3. *Suppose X is an F^e-sandwich of a smooth surface S, i.e., the e-th iterate of the Frobenius of S factors as $F^e\colon S \xrightarrow{\pi} X \to S$. If $d = \deg\pi$, then*

$$\mathrm{Hilb}_d(S/X)^\circ \cong \mathrm{FB}_e(X).$$

Proof. By the assumption the eth iterate of Frobenius of X factors as $F^e\colon \mathcal{O}_X \to \mathcal{O}_S \to \mathcal{O}_X^{1/q}$, via which $\mathcal{O}_X^{1/q}$ is a reflexive $\mathcal{O}_S$-module. Since S is a smooth surface, this implies that $\mathcal{O}_X^{1/q}$ is a locally free $\mathcal{O}_S$-module. On the other hand, the torsion-free pullback $\psi^\star\mathcal{O}_S$ by $\psi\colon W = \mathrm{Hilb}_d(S/X)^\circ \to X$ is a flat (hence locally free) $\mathcal{O}_W$-module, so that $\psi^\star\mathcal{O}_X^{1/q}$ is also a locally free $\mathcal{O}_W$-module. Hence ψ factors as $\psi\colon \mathrm{Hilb}_d(S/X)^\circ \to \mathrm{FB}_e(X) \to X$.

To give the inverse morphism $\mathrm{FB}_e(X) \to \mathrm{Hilb}_d(S/X)^\circ$, we note that the map $\mathcal{O}_S \to \mathcal{O}_X^{1/q}$ splits as an $\mathcal{O}_S$-module homomorphism since $\mathcal{O}_S \to \mathcal{O}_S^{1/q}$ splits by the smoothness of S. Then by a similar argument as in the proof of Proposition 4.2, we obtain $\mathrm{FB}_e(X) \to \mathrm{Hilb}_d(S/X)^\circ$. $\square$

Now we give a counterexample to the Question above.

Example 4.4. Let $X = S/\delta$ be the quotient of $S = \mathbb{A}^2 = \operatorname{Spec} k[x,y]$ by a p-closed vector field $\delta = x^p\partial/\partial x + y^p\partial/\partial y$. Then $\mathcal{O}_X = \mathcal{O}_S^\delta = k[x^p, x^py - xy^p, y^p] \subset \mathcal{O}_S = k[x,y]$. Let $S' \to S$ be the blowup at the origin and let $g\colon X' \to X$ be the induced morphism. Then we have a commutative diagram

$$\begin{tikzcd} S' \arrow[r] \arrow[d] & S \arrow[d, "\pi"] \\ X' \arrow[r, "g"] & X, \end{tikzcd}$$

where the vertical arrows are homeomorphic in the Zariski topology, and the exceptional set of g is a single $\mathbb{P}^1$, on which X' has $p+1$ A_{p-1}-singularities. Resolving these A_{p-1}-singularities, we obtain the minimal resolution $f\colon \widetilde{X} \to X$.

We will show that $\mathrm{FB}_e(X)$ is dominated by X' for all $e \geq 0$, so that the F-blowups of X do not coincide with any resolution of X. To see this it suffices to show that $g^\star\pi_*\mathcal{O}_S$ is locally free, which implies that $\mathrm{Hilb}_p(S/X)^\circ$ is dominated by X' and so does $\mathrm{FB}_e(X)$ by virtue of Proposition 4.3.

Let S'_1 and S'_2 be the affine open subsets of S' with affine coordinates $x, y/x$ and $y, x/y$, respectively, and let X'_1 and X'_2 be the corresponding affine open subsets covering X'. We verify that $g^\star\pi_*(\mathcal{O}_S)|_{X'_i}$ is a free $\mathcal{O}_{X'_i}$-module of rank 2 for $i = 1, 2$. By symmetry it is enough to consider the case $i = 1$. Denote the affine coordinates of S'_1 by $x, z = y/x$. Then $\mathcal{O}_S = k[x, xz] \subset \mathcal{O}_{S'_1} = k[x,z]$ and

$$\mathcal{O}_X = k[x^p, x^{p+1}z(1 - z^{p-1}), x^pz^p] \subset \mathcal{O}_{X'_1} = k[x^p, xz(1-z^{p-1}), z^p].$$

Hence

$$g^\star\pi_*(\mathcal{O}_S)|_{X'_1} = \mathrm{Im}(\mathcal{O}_{X'_1} \otimes_{\mathcal{O}_X} \mathcal{O}_S \to \mathcal{O}_{S'_1}) = k[x, xz, z^p],$$

and this is a free $\mathcal{O}_{X'_1}$-module with basis $1, x, \ldots, x^{p-1}$.

Remark. In the above example, the singularity of X is a rational singularty if and only if $p = 2$, and in this case it is a rational double point of type D_4, which is not F-regular. In Artin's list [4] of rational double points in characteristic $p \leq 5$, we can find similar examples of F-sandwiches whose F-blowups are singular. We can obtain more detailed information about these rational double points. First of all, we shall recall

Lemma 4.5 (See e.g., [5]). *Let (X,x) be a rational surface singularity and let $f\colon \widetilde{X} \to X$ be any resolution of the singularity. If M is a reflexive $\mathcal{O}_X$-module, then the torsion-free pullback $f^\star M = f^*M/$*torsion *is a locally free $\mathcal{O}_{\widetilde{X}}$-module.*

Since $\mathcal{O}_X^{1/q}$ is a reflexive $\mathcal{O}_X$-module of rank q^2, $f^\star\mathcal{O}_X^{1/q}$ is locally free of rank q^2 by Lemma 4.5. In particular, we have

Corollary 4.6. *The F-blowup $\mathrm{FB}_e(X)$ of a rational surface singularity X is dominated by the minimal resolution $\widetilde{X}$ for all $e \geq 0$.*

Lemma 4.7 (Artin–Verdier [5])**.** *Let (X, x) be a two-dimensional rational double point and let $f \colon \widetilde{X} \to X$ be the minimal resolution. Let $E_1, \dots, E_s$ be the irreducible exceptional curves of f and write the fundamental cycle as $Z_0 = \sum_{i=1}^{s} r_i E_i$.*

(1) *There is a one-to-one corrspondence between the exceptional curves E_i of f and the isomorphism classes of non-trivial indecomposable reflexive $\mathcal{O}_X$-modules M_i.*

(2) *The torsion-free pullback $\widetilde{M_i} = f^\star M_i$ of each M_i is a locally free $\mathcal{O}_{\widetilde{X}}$-module of rank r_i, and $c_1(\widetilde{M_i})E_j = \delta_{ij}$.*

Example 4.8. Recall the case $p = 2$ of Example 4.4. In this case, the reflexive $\mathcal{O}_X$-module $\pi_*\mathcal{O}_S$ is of rank $p = 2$, and it is an indecomposable reflexive module corresponding to the central curve of the exceptional set of f (the solid circle in the figure below).

$$\begin{array}{ccccc} & & \circ & & \\ & & | & & \\ \circ & - & \bullet & - & \circ \end{array}$$

Similarly we can observe that for non-F-regular F-sandwich rational double points listed below, $\pi_*\mathcal{O}_S$ is an indecomposable reflexive $\mathcal{O}_X$-module corresponding to the solid circle; see [4] for the defining equations of E_6^0, E_7^0 and E_8^0-singularities.

(1) E_7^0-singularity in $p = 2$:

$$\begin{array}{ccccccccccc} & & & & \bullet & & & & & & \\ & & & & | & & & & & & \\ \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \end{array}$$

(2) E_8^0-singularity in $p = 2$:

$$\begin{array}{ccccccccccccc} & & & & \circ & & & & & & & & \\ & & & & | & & & & & & & & \\ \bullet & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \end{array}$$

(3) E_6^0-singularity in $p = 3$:

$$\begin{array}{ccccccccc} & & & & \circ & & & & \\ & & & & | & & & & \\ \circ & - & \circ & - & \bullet & - & \circ & - & \circ \end{array}$$

(4) E_8^0-singularity in $p = 3$:

$$\begin{array}{ccccccccccccc} & & & & \bullet & & & & & & & & \\ & & & & | & & & & & & & & \\ \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \end{array}$$

(5) E_8^0-singularity in $p=5$:

$$\begin{array}{ccccccccccccc} & & & & \circ & & & & & & & & \\ & & & & | & & & & & & & & \\ \circ & - & \circ & - & \circ & - & \bullet & - & \circ & - & \circ & - & \circ \end{array}$$

Remark. The above examples reflect pathology of non-F-regular F-sandwich rational double points in characteristic $p \le 5$. Contrary to this, if a rational double point (X,x) is a tame quotient singularity, that is, a quotient of a smooth surface S by a finite group of order not divisible by p, then by the McKay correspondence [10] one has

$$\pi_* \mathcal{O}_S \cong \mathcal{O}_X \oplus \bigoplus_{i=1}^{s} M_i^{\oplus r_i}, \tag{4.1}$$

in which every indecomposable reflexive module M_i appears as a direct summand with multiplicity $r_i = \operatorname{rank} M_i > 0$. This is also verified to be true whenever the rational double point under consideration is F-regular. Explicitly, this is the case for A_n-singularities in arbitrary characteristic p, D_n-singularities in $p \ne 2$, E_6 and E_7-singularities in $p > 3$ and E_8-singularities in $p > 5$; see [11] for a classification of F-regular surface singularities. The formula (4.1) is also stated in [30], in which there seems to be an ambiguity about the difference of F-regular singularity and tame quotient singularity.

The following proposition follows from [31] for A_n-singularities (which are toric) and from [29] for E_6, E_7 and E_8-singularities (which are tame quotients), but only the case $2 \ne p | n-2$ for D_n-singularities is not covered by [31], [29].

Proposition 4.9. *If (X,x) is a two-dimensional F-regular double point, then*

$$\mathrm{FB}_e(X) \cong \widetilde{X} \text{ for } e \gg 0.$$

To prove this, we need a result which enables us to compare $F_*^e \mathcal{O}_X$ with $\pi_* \mathcal{O}_S$ for a finite covering $\pi \colon S \to X$ by a smooth surface S. The following lemma is a slight improvement of the formula (2.1) for Hilbert-Kunz multiplicity. (Note that it is straightforward in the case where X is a Frobenius sandwich of S.)

Lemma 4.10. *Let (X,x) be a two-dimensional F-regular double point, and let $\pi \colon S \to X$ be a finite covering by a smooth surface S of degree r. Decompose $\pi_* \mathcal{O}_S$ and $F_*^e \mathcal{O}_X = \mathcal{O}_X^{1/q}$ into direct sums of indecomposable reflexive $\mathcal{O}_X$-modules as* (4.1) *and*

$$F_*^e \mathcal{O}_X = \mathcal{O}_X^{\oplus a_0^{(e)}} \oplus \bigoplus_{i=1}^{s} M_i^{\oplus a_i^{(e)}}.$$

Then for all $i = 0, 1, \dots, s$, one has

$$\lim_{e \to \infty} \frac{a_i^{(e)}}{p^{2e}} = \frac{r_i}{r} > 0.$$

Proof. Since $\mathcal{O}_X$ is F-regular, the limit $\lim_{e\to\infty}\frac{a_i^{(e)}}{p^{2e}}$ exists by [27, Proposition 3.3.1]. Let $f\colon \widetilde{X}\to X$ be the minimal resolution, $Z=\sum_{i=1}^s z_iE_i$ any anti-f-nef cycle on $\widetilde{X}$, and let $I=f_*\mathcal{O}_X(-Z)\subset\mathcal{O}_X$. Then by Kato's Riemann-Roch (see e.g., [30]),

$$\mathrm{length}_{\mathcal{O}_X}\mathcal{O}_X/I^{[q]}=-\frac{Z(K_{\widetilde{X}}+Z)}{2}q^2+c_1(f^\star\mathcal{O}_X^{1/q})Z.$$

Since $K_{\widetilde{X}}=0$ and $c_1(\widetilde{M_i})Z=z_i$, one has

$$e_{\mathrm{HK}}(I,\mathcal{O}_X)=-\frac{Z^2}{2}+\sum_{i=1}^s\lim_{e\to\infty}\frac{a_i^{(e)}}{p^{2e}}z_i.$$

On the other hand, one has

$$e_{\mathrm{HK}}(I,\mathcal{O}_X)=\frac{1}{r}\mathrm{length}_{\mathcal{O}_X}\mathcal{O}_S/I\mathcal{O}_S=-\frac{Z^2}{2}+\sum_{i=1}^s\frac{r_i}{r}z_i$$

again by Kato's Riemann-Roch applied to (4.1). Thus $\displaystyle\sum_{i=1}^s\lim_{e\to\infty}\frac{a_i^{(e)}}{p^{2e}}z_i=\sum_{i=1}^s\frac{r_i}{r}z_i$. Since the cone of f-nef divisors has dimension equal to s, one can choose s linearly independent vectors in $\mathbb{Z}^s$ as $(z_1,\dots,z_s)$, from which follows that $\displaystyle\lim_{e\to\infty}\frac{a_i^{(e)}}{p^{2e}}=\frac{r_i}{r}$. The right-hand side of this equality is positive, since $r_i=\operatorname{rank}M_i>0$ from the McKay correspondence, which holds true for any F-regular double point. $\square$

Proof of Proposition 4.9. Let n be the minimal number of generators of $\mathcal{O}_X^{1/q}$ as an $\mathcal{O}_X$-module and pick a surjection $\mathcal{O}_X^{\oplus n}\to\mathcal{O}_X^{1/q}$. Since $f^\star\mathcal{O}_X^{1/q}$ is a locally free $\mathcal{O}_{\widetilde{X}}$-module of rank q^2 by Lemma 4.5, the induced surjection $\mathcal{O}_{\widetilde{X}}^{\oplus n}\to f^\star\mathcal{O}_X^{1/q}$ gives rise to a morphism $\Phi_e\colon\widetilde{X}\to\mathbb{G}$ over X to the Grassmannian $\mathbb{G}=\mathrm{Grass}(q^2,\mathcal{O}_X^{\oplus n})$ such that $f^\star\mathcal{O}_X^{1/q}$ is isomorphic to the pull back of the universal quotient bundle of $\mathbb{G}$.

Similarly, since the torsion-free pullback $\varphi^\star\mathcal{O}_X^{1/q}$ to $Z=\mathrm{FB}_e(X)$ is locally free, the surjection $\mathcal{O}_Z^{\oplus n}\to\varphi^\star\mathcal{O}_X^{1/q}$ gives rise to a morphism $Z=\mathrm{FB}_e(X)\to\mathbb{G}$ over X, through which Φ_e factors as

$$\Phi_e\colon\widetilde{X}\to\mathrm{FB}_e(X)\to\mathbb{G}$$

by Corollary 4.6. Composing with the Plücker embedding $\mathbb{G}\hookrightarrow\mathbb{P}$ over X, we have

$$\Phi_{|L|}\colon\widetilde{X}\to\mathrm{FB}_e(X)\to\mathbb{P},$$

the morphism over X given by the f-generated line bundle $L=c_1(f^\star\mathcal{O}_X^{1/q})$. Now by Lemma 4.7, the intersection number of L with each exceptional curves E_i is $L\cdot E_i=a_i^{(e)}$, so that L is f-very ample for $e\gg0$ by Lemma 4.10. It follows that Φ_e is a closed immersion for $e\gg0$, so that $\widetilde{X}\cong\mathrm{FB}_e(X)$. $\square$

Remark. The above proof also shows that $\mathrm{Hilb}_n(S/X)^\circ \cong \mathrm{FB}_e(X)$ holds for $e \gg 0$ in the notation of Proposition 4.2.

Conjecture. If X is an F-regular surface singularity, then $\mathrm{FB}_e(X)$ is the minimal resolution of X for $e \gg 0$.

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