RIMS Kôkyûroku Bessatsu
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A note on consistency conditions on dimer models

By

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§ 1. Introduction

Dimer models are introduced by string theorists (see e.g. [6, 7, 8, 11, 12, 13]) to study supersymmetric quiver gauge theories in four dimensions. A dimer model is a bicolored graph on a 2-torus which encode the information of a quiver with relations. If a dimer model is non-degenerate, then the moduli space $\mathcal{M}_\theta$ of stable representations of the quiver with dimension vector $(1, \dots, 1)$ with respect to a generic stability parameter θ in the sense of King [17] is a smooth toric Calabi-Yau 3-fold [15].

Let $\mathcal{V} = \bigoplus_v \mathcal{L}_v$ be the tautological bundle on the moduli space $\mathcal{M}_\theta$ and

$$\phi : \mathbb{C}\Gamma \to \operatorname{End}(\mathcal{V}) \tag{1.1}$$

be the universal morphism from the path algebra $\mathbb{C}\Gamma$ of the quiver with relations associated with a dimer model. This map is not an isomorphism in general, and it is easy to see that the injectivity of this map is equivalent to the *first consistency condition* of Mozgovoy and Reineke [19]. The path algebra $\mathbb{C}\Gamma$ is a Calabi-Yau algebra of dimension three in the sense of Ginzburg [9] if the dimer model satisfies the first consistency condition [19, 4, 3]. This in turn implies [2, 20] that ϕ is an isomorphism, the functor

$$\mathbb{R}\mathrm{Hom}(\mathcal{V}, \bullet) : D^b \operatorname{coh} \mathcal{M}_\theta \to D^b \operatorname{mod} \mathbb{C}\Gamma$$

is an equivalence of triangulated categories, and $\mathbb{C}\Gamma$ is a non-commutative crepant resolution of a Gorenstein affine toric 3-fold.

Received April 21, 2010. Accepted December 24, 2010.
2000 Mathematics Subject Classification(s): 16G20
Key Words: *dimer model, quiver with potential, Calabi-Yau algebra*
Supported by Grant-in-Aid for Scientific Research (No.18540034), Grant-in-Aid for Young Scientists (No.20740037), and Engineering and Physical Sciences Research Council (EP/F055366/1)
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The first consistency condition is an algebraic condition, which is not easy to check in examples. In this paper, we show that a more tractable condition, given in Definition 3.5, is equivalent to the first consistency condition under the non-degeneracy assumption:

Theorem 1.1. *For a non-degenerate dimer model,*

- *the first consistency condition,*
- *the consistency condition in Definition 3.5, and*
- *the* properly-orderedness *in the sense of Gulotta [10]*

are equivalent.

It is known that the consistency condition in Definition 3.5 implies the non-degeneracy [14, Proposition 6.2]. Together with a work of Kenyon and Schlenker [16, Theorem 5.1], Theorem 1.1 implies a result of Broomhead [3] that an *isoradial* dimer model satisfies the first consistency condition. Here we note that isoradiality is a strong condition, and a large number of otherwise well-behaved dimer models fall out of this class.

We recall basic definitions on dimer models in Section 2. The content of Section 3 has bubbled off from [14, Section 5], and the rest of [14] will appear in a separate paper. In Section 4, we show that a dimer model satisfies the consistency condition in Definition 3.5 if and only if it is properly-ordered in the sense of Gulotta [10]. Relations between consistency conditions on dimer models are also discussed by Bocklandt [1, Section 8].

Acknowledgment: We thank Alastair Craw, Nathan Broomhead, Ben Davison, Dominic Joyce, Alastair King, Diane Maclagan, Balazs Szendroi, Yukinobu Toda, Michael Wemyss and Masahito Yamazaki for valuable discussions. We also thank the anonymous referee for carefully reading the manuscript and suggesting a number of improvements.

§ 2. Dimer models and quivers

Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be a real two-torus equipped with an orientation. A *bicolored graph* on T consists of

- a finite set $B \subset T$ of black nodes,
- a finite set $W \subset T$ of white nodes, and
- a finite set E of edges, consisting of embedded closed intervals e on T such that one boundary of e belongs to B and the other boundary belongs to W. We assume that two edges intersect only at the boundaries.

A *face* of a graph is a connected component of $T \setminus \cup_{e \in E} e$. A bicolored graph G on T is called a *dimer model* if every face is simply-connected.

A *quiver* consists of

- a set V of vertices,
- a set A of arrows, and
- two maps $s, t : A \to V$ from A to V.

For an arrow $a \in A$, $s(a)$ and $t(a)$ are said to be the *source* and the *target* of a respectively. A *path* on a quiver is an ordered set of arrows $(a_n, a_{n-1}, \dots, a_1)$ such that $s(a_{i+1}) = t(a_i)$ for $i = 1, \dots, n-1$. We also allow for a path of length zero, starting and ending at the same vertex. The *path algebra* $\mathbb{C}Q$ of a quiver $Q = (V, A, s, t)$ is the algebra spanned by the set of paths as a vector space, and the multiplication is defined by the concatenation of paths;

$$(b_m, \dots, b_1) \cdot (a_n, \dots, a_1) = \begin{cases} (b_m, \dots, b_1, a_n, \dots, a_1) & s(b_1) = t(a_n), \\ 0 & \text{otherwise.} \end{cases}$$

A *quiver with relations* is a pair of a quiver and a two-sided ideal $\mathcal{I}$ of its path algebra. For a quiver $\Gamma = (Q, \mathcal{I})$ with relations, its path algebra $\mathbb{C}\Gamma$ is defined as the quotient algebra $\mathbb{C}Q/\mathcal{I}$. Two paths a and b are said to be *equivalent* if they give the same element in $\mathbb{C}\Gamma$.

A dimer model (B, W, E) encodes the information of a quiver $\Gamma = (V, A, s, t, \mathcal{I})$ with relations in the following way: The set V of vertices is the set of connected components of the complement $T \setminus (\bigcup_{e \in E} e)$, and the set A of arrows is the set E of edges of the graph. The directions of the arrows are determined by the colors of the nodes of the graph, so that the white node $w \in W$ is on the right of the arrow. In other words, the quiver is the dual graph of the dimer model equipped with an orientation given by rotating the white-to-black flow on the edges of the dimer model by minus 90 degrees.

The relations of the quiver are described as follows: For an arrow $a \in A$, there exist two paths $p_+(a)$ and $p_-(a)$ from $t(a)$ to $s(a)$, the former going around the white node connected to $a \in E = A$ clockwise and the latter going around the black node connected to a counterclockwise. Then the ideal $\mathcal{I}$ of the path algebra is generated by $p_+(a) - p_-(a)$ for all $a \in A$.

A *small cycle* on a quiver coming from a dimer model is the product of arrows surrounding only a single node of the dimer model. A path p is said to be *minimal* if it is not equivalent to a path containing a small cycle. A path p is said to be *minimum* if any path from $s(p)$ to $t(p)$ homotopic to p is equivalent to the product of p and a power of a small cycle. For a pair of vertices of the quiver, a minimum path from one vertex to another may not exist, and will always be minimal when it exists.

Small cycles starting from a fixed vertex are equivalent to each other. Hence the sum ω of small cycles over the set of vertices is a well-defined element of the path algebra. For any arrow a, the small cycles $\omega_{s(a)}$ and $\omega_{t(a)}$ starting from the source $s(a)$ and the target $t(a)$ of a respectively satisfies

$$a\omega_{s(a)} = \omega_{t(a)}a.$$

If follows that ω belongs to the center of the path algebra, and there is the universal map

$$\mathbb{C}\Gamma \to \mathbb{C}\Gamma[\omega^{-1}]$$

into the localization of the path algebra by the multiplicative subset generated by ω. Two paths a and b are said to be *weakly equivalent* if they are mapped to the same element in $\mathbb{C}\Gamma[\omega^{-1}]$, i.e., there is an integer $i \geq 0$ such that $a\omega^i = b\omega^i$ in $\mathbb{C}\Gamma$. Note that the following holds for the paths of the quiver.

Lemma 2.1. *For two paths a and b with the same source and target, the following are equivalent.*

- *a and b are homotopy equivalent.*
- *There are integers $i, j \geq 0$ such that $a\omega^i = b\omega^j$ in $\mathbb{C}\Gamma$.*
- *There is an integer $i \geq 0$ such that either $(a, b\omega^i)$ or $(a\omega^i, b)$ is a weakly equivalent pair.*

For example, the paths p and q shown in Figure 1 are weakly equivalent, but not equivalent. They are homotopic and one has

$$\omega p = \omega q.$$

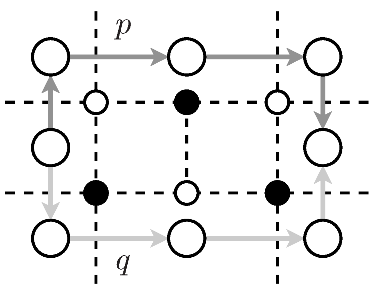

Figure 1. A pair of weakly equivalent paths which are not equivalent

A *perfect matching* (or a *dimer configuration*) on a bicolored graph $G = (B, W, E)$ is a subset D of E such that for any node $v \in B \cup W$, there is a unique edge $e \in D$

connected to v. A dimer model is said to be *non-degenerate* if for any edge $e \in E$, there is a perfect matching D such that $e \in D$.

A *zigzag path* is a path on a dimer model which makes a maximum turn to the right on a white node and to the left on a black node. Note that it is not a path on a quiver. We assume that a zigzag path does not have an endpoint, so that we can regard a zigzag path as a sequence (e_i) of edges e_i parameterized by $i \in \mathbb{Z}$, up to translations of i. Figure 2 shows an example of a part of a dimer model and a zigzag path on it.

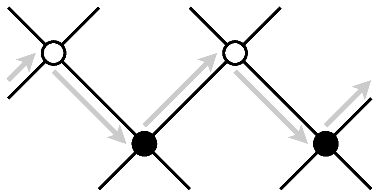

Figure 2. A zigzag path

As an example, consider the dimer model in Figure 3. The corresponding quiver is shown in Figure 4, whose relations are given by

$$\mathcal{I} = (dbc - cbd, dac - cad, adb - bda, acb - bca).$$

This dimer model is non-degenerate, and has four perfect matchings $D_0, \dots, D_3$ shown in Figure 5.

We end this section with the following lemma:

Lemma 2.2. *Assume that a dimer model has a perfect matching D. Then for any path p on the quiver, there are another path q and a non-negative integer k such that p is equivalent to $q\omega^k$ and q is not equivalent to a path containing a small cycle.*

Proof. Consider the number of times the path p crosses D. Then this is a non-negative integer which decreases by one as one removes a small cycle from the path. $\square$

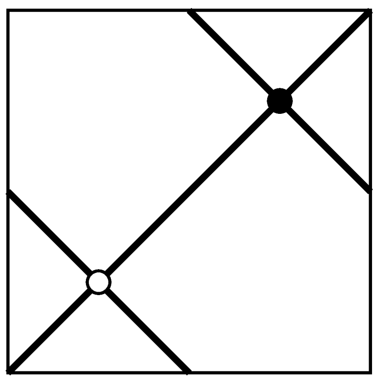

Figure 3. A dimer model

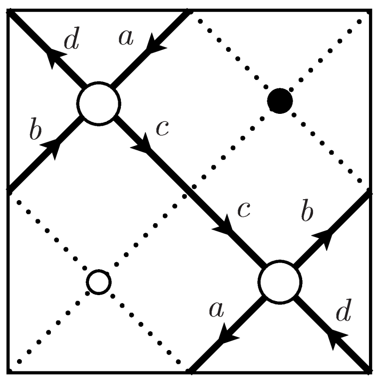

Figure 4. The corresponding quiver

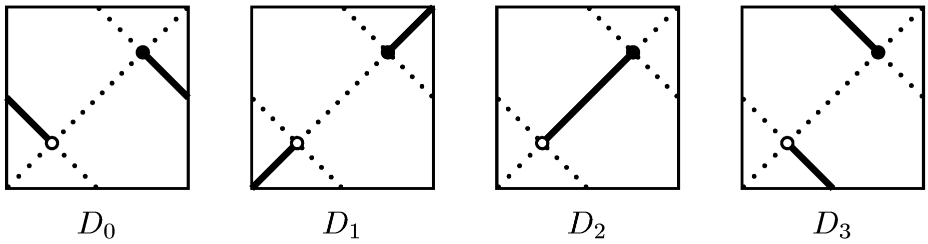

Figure 5. Four perfect matchings

The statement of Lemma 2.2 can be false if there is no perfect matching: Figure 6 shows an example of a dimer model without any perfect matching, which we learned from Broomhead and King. One can see from the relation

$$a = eadcb$$

that

$$cbfead = cbfe^2adc(bd) = cbfe^3adc(bd)c(bd) = \cdots,$$

which shows that the loop $cbfead$ can be divided by any power of the small cycle bd.

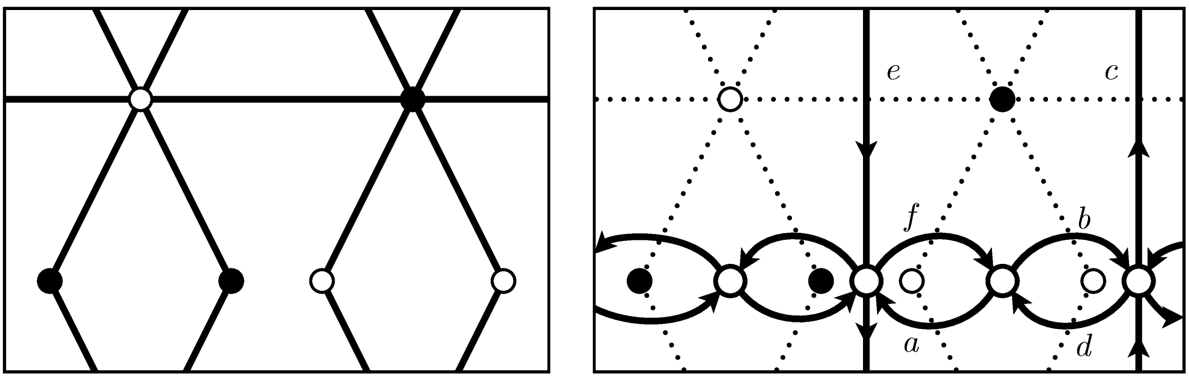

Figure 6. A dimer model without any perfect matching

§ 3. Consistency conditions for dimer models

The following notion is due to Duffin [5] and Mercat [18]:

Definition 3.1. A dimer model is *isoradial* if one can choose an embedding of the graph into the torus so that every face of the graph is a polygon inscribed in a circle of a fixed radius with respect to a flat metric on the torus. Here, the circumcenter of any face must be contained in the face.

A dimer model is isoradial if and only if zigzag paths behave like straight lines:

Theorem 3.2 (Kenyon and Schlenker [16, Theorem 5.1]). *A dimer model is isoradial if and only if the following conditions are satisfied:*

1. *Every zigzag path is a simple closed curve.*
2. *The lift of any pair of zigzag paths to the universal cover of the torus share at most one edge.*

The following condition is introduced by Mozgovoy and Reineke:

Definition 3.3 ([19, Condition 4.12]). A dimer model is said to satisfy the *first consistency condition* if weakly equivalent paths are equivalent.

We regard a zigzag path on the universal cover as a sequence (e_i) of edges e_i parameterized by $i \in \mathbb{Z}$, up to translations of i.

Definition 3.4. Let $z = (e_i)$ and $w = (f_i)$ be two zigzag paths on the universal cover. We say that z and w intersect if there are $i, j \in \mathbb{Z}$ with $e_i = f_j$ such that if u, v are the maximum and the minimum of t with $e_{i+t} = f_{j-t}$ respectively, then $u - v \in 2\mathbb{Z}$. In this case, the sequence $(e_{i+v} = f_{j-v}, e_{i+v+1} = f_{j-v-1}, \dots, e_{i+u} = f_{j-u})$ of intersections is counted as a single intersection. We say that z has a self-intersection if there is a pair $i \neq j$ with $e_i = e_j$ such that the directions of z at e_i and e_j are opposite, and $u - v \in 2\mathbb{Z}$ for u and v defined similarly as above. We say that z is homologically trivial if the map $i \mapsto e_i$ is periodic.

Note that if $u - v > 0$ in the above definition, then the nodes between e_v and e_u are divalent. According to this definition, there are cases where z and w have a common nodes or common edges, but they do not intersect as shown in Figure 7. The assumption $u - v \in 2\mathbb{Z}$ is needed to remove the effect of a divalent node; if there is no divalent node, then a pair of zigzag paths intersect if and only if they have a common edge.

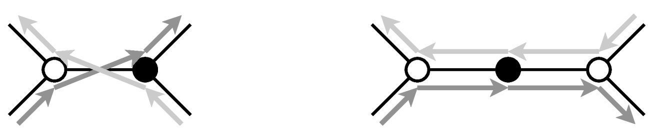

Figure 7. Examples of an intersection (left) and a non-intersection (right)

The following condition is slightly weaker than isoradiality, and easy to check in examples:

Definition 3.5. A dimer model is said to be *consistent* if

- there is no homologically trivial zigzag path,
- no zigzag path has a self-intersection on the universal cover, and
- no pair of zigzag paths intersect each other on the universal cover in the same direction more than once.

Here, the third condition means that if a pair (z, w) of zigzag paths has two intersections a and b and the zigzag path z points from a to b, then the other zigzag path w must point from b to a.

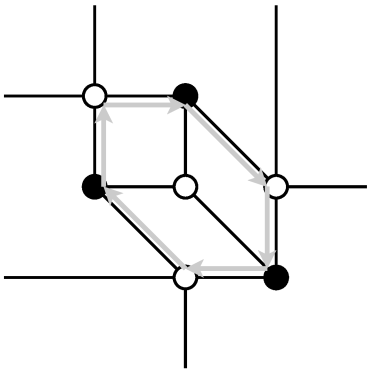

Figure 8. A homologically trivial zigzag path

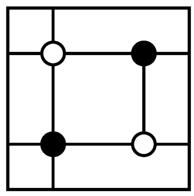

Figure 9. An inconsistent dimer model

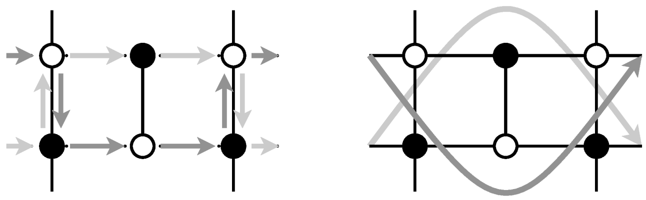

Figure 10. A pair of zigzag paths in the same direction intersecting twice

Figure 8 shows a part of an inconsistent dimer model which contains a homologically trivial zigzag path. Figure 9 shows an inconsistent dimer model, which contains a pair of zigzag paths intersecting in the same direction twice as in Figure 10.

On the other hand, a pair of zigzag paths going in the opposite direction may intersect twice in a consistent dimer model. Figure 12 shows a pair of such zigzag paths on a consistent dimer model in Figure 11.

To obtain a criterion for the minimality of a path, we discuss the intersection of a path of the quiver and a zigzag path. Note that paths of the quiver and zigzag paths

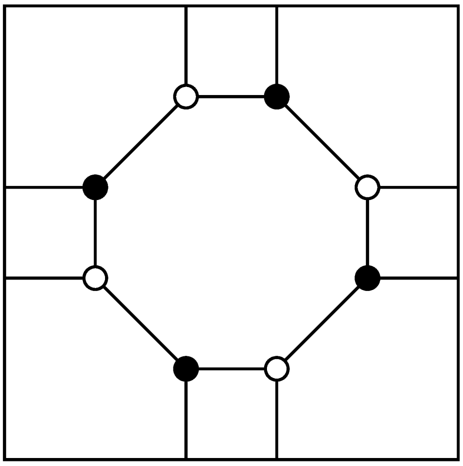

Figure 11. A consistent non-isoradial dimer model

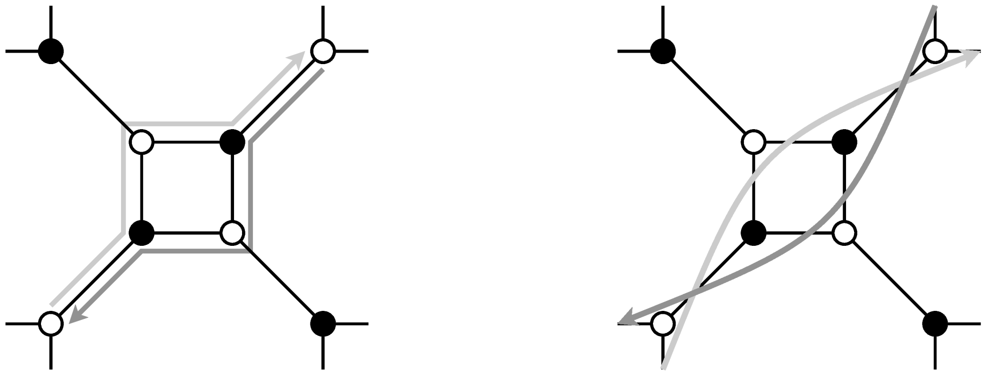

Figure 12. A pair of zigzag paths in the opposite direction intersecting twice

are both regarded as sequences of arrows of the quiver, where the former are finite and the latter are infinite.

Definition 3.6. Let $p = a_1 a_2 \dots$ be a path of the quiver ($a_i \in A$) and $z = (b_i)_{i \in \mathbb{Z}}$ be a zigzag path. We say p intersects z at an arrow a if there are i, j with $a = a_i = b_j \in A = E$, satisfying the following condition: If u, v denote the maximum and the minimum of t with $a_{i+t} = b_{j-t}$ respectively, then $u - v$ is even. In this case, the sequence $(a_{i+v} = b_{j-v}, \dots, a_{i+u} = b_{j-u})$ is counted as a single intersection.

Figure 13 shows an example of a non-intersection; the path shown in dark gray does not intersect the zigzag path shown in light gray. Note that the dark gray path is equivalent to the dashed path, which does not have a common edge (or an arrow) with the light gray path.

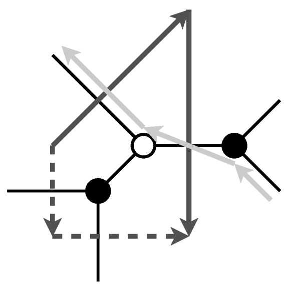

Figure 13. An example of a non-intersection

The following lemma is obvious from the definition of the equivalence relations of paths:

Lemma 3.7. *Let z be a zigzag path on the universal cover. Suppose that a path p' is obtained from another path p by replacing $p_+(a) \subset p$ with $p_-(a)$ or the other way around for a single arrow a, as in the definition of the equivalence relations of paths. If neither $ap_+(a)$ nor $p_+(a)a$ is a part of p, then there is a natural bijection between the intersections of z and p and those of z and p'. If a is not a part of p, then this bijection preserves the order of intersections along z.*

Because $ap_+(a)$ and $p_+(a)a$ are small cycles, the first half of Lemma 3.7 immediately gives the following:

Corollary 3.8. *A minimal path which does not intersect a zigzag path z cannot be equivalent to a path intersecting z.*

Lemma 3.7 also gives the following:

Corollary 3.9. *Let p be a path of the quiver. If there is no zigzag path that intersects p more than once in the same direction on the universal cover, then p is minimal.*

Proof. Assume that there is no zigzag path that intersects p more than once in the same direction on the universal cover. If p contains an arrow a and either $p_+(a)$ or $p_-(a)$, then one of two zig-zag paths containing the edge corresponding to a intersects p more than once in the same direction on the universal cover. It follows that if p contains $p_+(a)$ or $p_-(a)$ for an arrow a, then p does not contain a. Let p' be a path related to p as in Lemma 3.7. Since p does not contain small cycles $ap_+(a)$ or $p_+(a)a$, Lemma 3.7 implies that p' also satisfies the assumption and hence does not contain a small cycle. By repeating this argument, we can see that if a path is equivalent to p, then it does not contain a small cycle. □

The following lemma shows that the consistency condition implies the first consistency condition of Mozgovoy and Reineke:

Lemma 3.10. *If weak equivalence does not imply equivalence, then the dimer model is not consistent.*

Proof. Assume for contradiction that a consistent dimer model has a pair of weakly equivalent paths which are not equivalent. Then there is a pair (a, b) of paths on the universaly cover such that

- There is an integer $i \geq 0$ such that either $(a, b\omega^i)$ or $(a\omega^i, b)$ is weakly equivalent but not equivalent.
- If one of a and b contains loops, then it is a loop and the other one is a trivial path.
- a and b meet only at the endpoints.

Choose one of such pairs, without fixing the endpoints, so that the area bounded by a and b is minimal with respect to the inclusion relation.

Figure 14 shows a pair (a, b) of such paths. We may assume that a is a non-trivial path. Let v_1 and v_2 be the source and the target of a respectively. To show the inconsistency of the dimer model, consider the zigzag path z which starts from the white node just on the right of the first arrow in the path a as shown in light gray in Figure 14.

We show that if z crosses a, then it contradicts the minimality of the area. Assume that z crosses a, and consider the path c which goes along z as in Figure 14. Since z crosses a, the path c also crosses a. Let v_3 be the vertex where a and c intersects, and

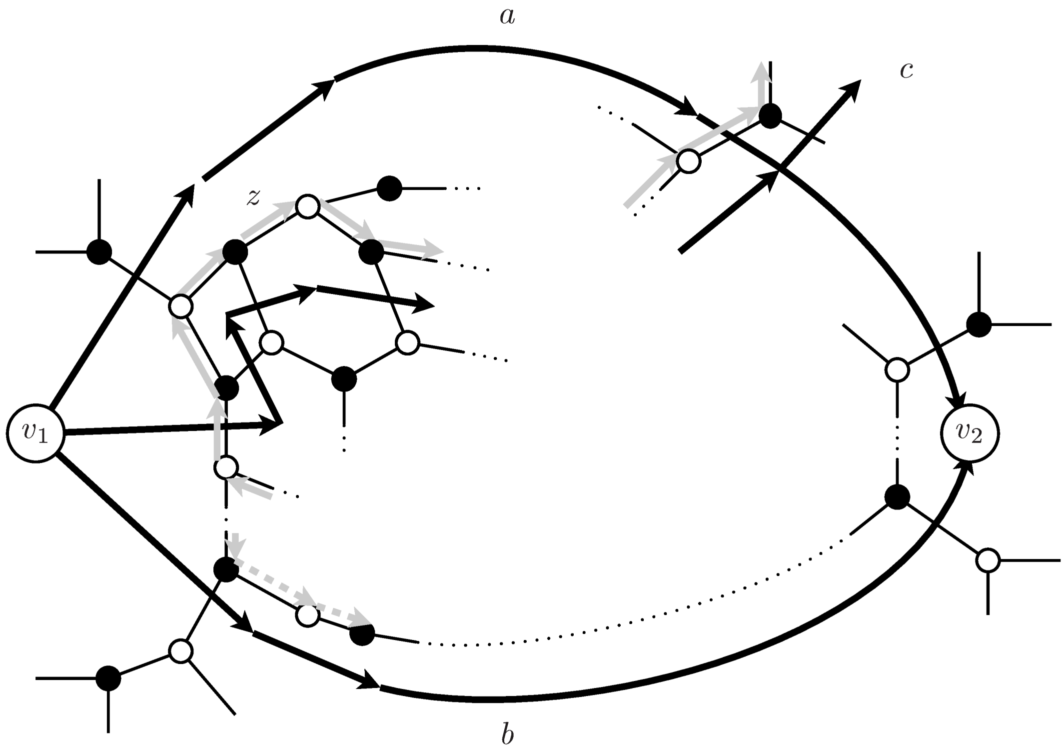

Figure 14. A pair of inequivalent paths which are weakly equivalent

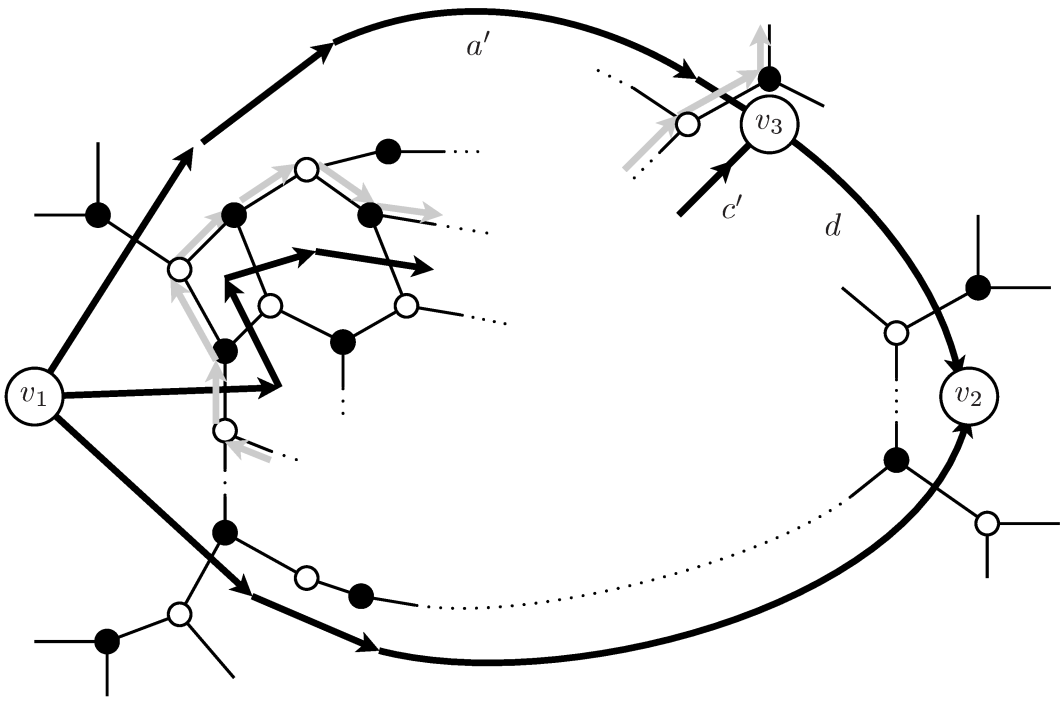

Figure 15. The paths a' and c'

a' and c' be the parts of a and c from v_1 to v_3 respectively. The part of a from v_3 to v_2 will be denoted by d as in Figure 15.

If there is a zigzag path w which intersects c' more than once in the same direction, then w also intersects z more than once in the same direction, which contradicts the assumption that the dimer model is consistent. Hence no zigzag path intersects c' more than once in the same direction, so that c' is minimal by Corollary 3.9.

Suppose dc' is different from b. Then by the minimality of the area and the minimality of c', there are non-negative integers i and j such that a' is equivalent to $c'\omega^i$ and either $(dc'\omega^j, b)$ or $(dc', b\omega^j)$ are equivalent pairs. Then one of $(a, b\omega^{i-j})$, $(a\omega^{j-i}, b)$ and $(a, b\omega^{i+j})$ is an equivalent pair, which contradicts the assumption. If dc' coincides with b, then b is equivalent to a path that goes along the opposite side of z as in Figure 16, which contradicts the minimality of the area.

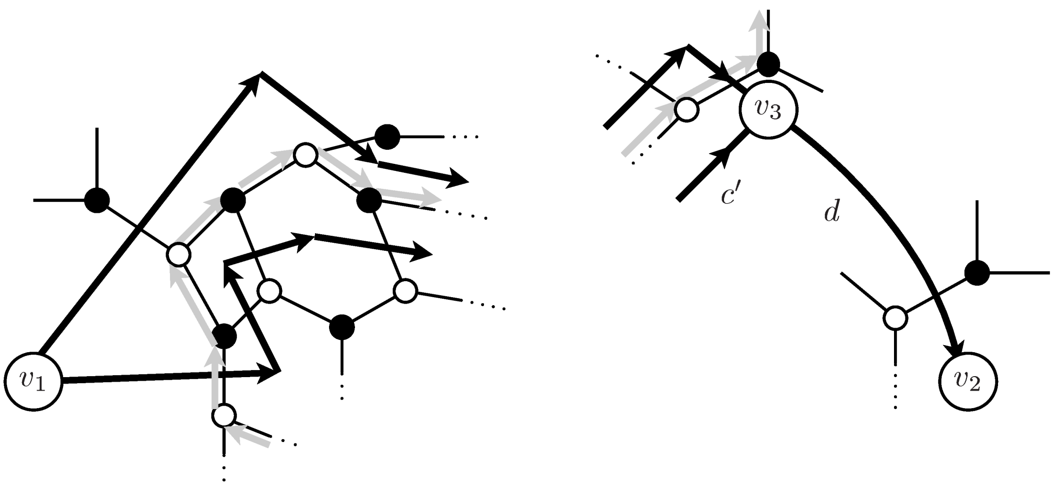

Figure 16. A path equivalent to dc'

Hence the zigzag path z cannot cross the path a. In the same way, the dashed gray zigzag path in Figure 14 cannot cross the path b. It follows that if we extend these two zigzag paths in both directions, then they will intersect in the same direction more than once or have a self-intersection. This contradicts the consistency of the dimer model, and Lemma 3.10 is proved.

□

Lemma 3.11. *For a path p in a consistent dimer model, the following are equivalent:*

1. *p is minimal.*

2. *p is minimum.*

3. *There is no zigzag path that intersects* p *more than once in the same direction on the universal cover.*

Proof. It is clear that 2 implies 1. To show the converse, take a minimal path p and a path q from $s(p)$ to $t(p)$ homotopic to p. Then $(p, q\omega^i)$ or $(p\omega^i, q)$ is weakly equivalent, hence equivalent. By the minimality of p, $p\omega^i$ is equivalent to q, which means p is minimum.

Corollary 3.9 states that 3 implies 1. To show the converse, suppose there is a zigzag path z as above. Let a_1 and a_2 be arrows on the intersection of z and p such that the directions are from a_1 to a_2 on both z and p, and their parts between a_1 and a_2 do not meet each other. Let p' be the part of p from $s(a_1)$ to $t(a_2)$. There is a path q from $s(a_1)$ to $t(a_2)$ which is parallel to z. Since q is minimal by Corollary 3.9, it is minimum and there is an integer $i \geq 0$ such that p' is equivalent to $q\omega^i$. If p' is also minimal, i must be zero and therefore p' is equivalent to q. This contradicts Lemma 3.7 and thus p is not minimal. $\square$

The following lemmas show that the first consistency condition of Mozgovoy and Reineke together with the existence of a perfect matching implies the consistency condition:

Lemma 3.12. *Assume that a dimer model has a perfect matching and a pair of zigzag paths intersecting in the same direction twice on the universal cover, none of which has a self-intersection. Then there is a pair of inequivalent paths which are weakly equivalent.*

Proof. For a pair (z, w) of zigzag paths intersecting in the same direction twice on the universal cover, consider the pair (a, b) of paths as shown in dark gray in Figure 17. Our assumption that w does not have a self-intersection implies that a does not intersect w. We claim that there is a minimal path a' which does not intersect w such that $a = a'\omega^k$ for some $k \in \mathbb{N}$. The existence of such a' and k follows from Corollary 3.8 and the existence of a perfect matching: A perfect matching intersects a in a finite number of points, and the number of intersection decreases by one as one factors out a small cycle. Hence the process of

- deforming the path without letting it intersect w (Lemma 3.7), and
- factoring out a small cycle if any

must terminate in finite steps. Moreover, the resulting path a' cannot be equivalent to a path intersecting w by Corollary 3.8. Similarly, there is a minimal path b' from v_1 to v_2 which does not intersect z. On the other hand, a' and b' intersect z and w respectively

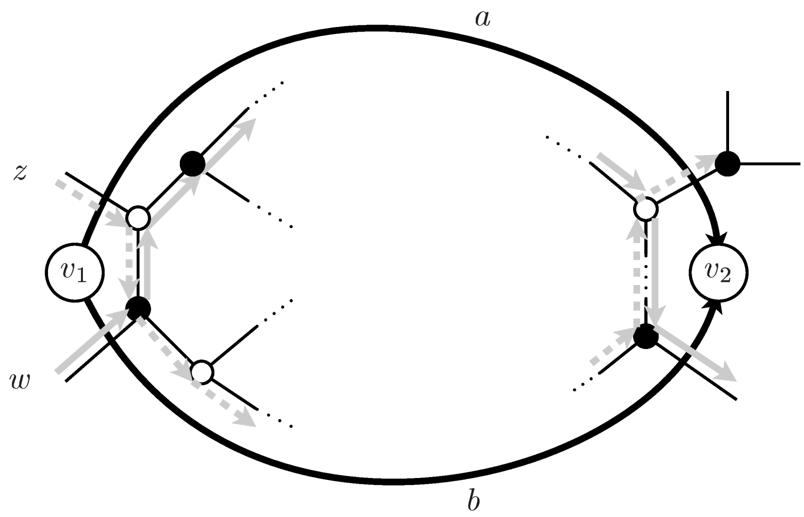

Figure 17. A pair of inequivalent paths which are weakly equivalent

for topological reason. It follows that $(a', b'\omega^i)$ or $(a'\omega^i, b')$ for some non-negative integer i gives a pair of weakly equivalent paths which are not equivalent. □

Lemma 3.13. *Assume that a dimer model has a perfect matching and a zigzag path with a self-intersection on the universal cover. Then there is a pair of inequivalent paths which are weakly equivalent.*

Proof. Let z be a zigzag path on the universal cover with a self-intersection and $e_0 e_1 e_2 \dots e_n e_0$ be a loop in z, where z has a self-intersection at e_0 and does not have any self-intersection in $(e_1, \dots, e_n)$. The union of the edges $e_1, \dots, e_n$ will be denoted by C.

Regarding e_0 as an arrow, we put $v_1 = s(e_0)$ and $v_2 = t(e_0)$. There is a path b from v_1 to v_2 which goes along z. The edge e_0 as an arrow of the quiver also forms a path from v_1 to v_2. We show that the path e_0 is minimal, and

- there is a minimal path b' from v_1 to v_2 which is not equivalent to e_0, or
- there is a non-trivial cyclic path which is not equivalent to any positive power of a small cycle.

In the latter case, since we are working on the universal cover, this cyclic path is homologically trivial, and the pair consisting of this cyclic path and a suitable power

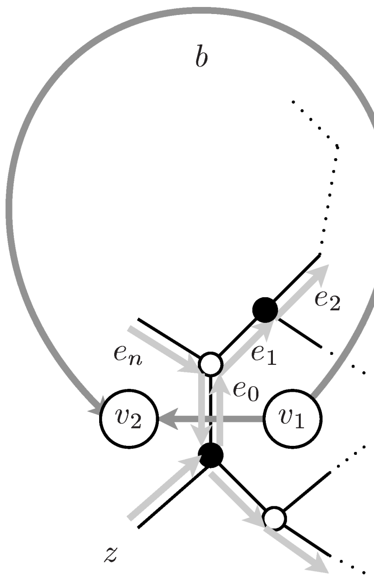

Figure 18. A pair of inequivalent paths which are weakly equivalent

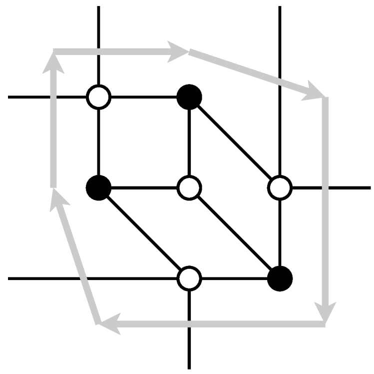

Figure 19. Homologically trivial zigzag path and a cyclic path on the quiver

of a small cycle gives a pair of inequivalent paths which are weakly equivalent. In the former case, there is a non-negative integer i such that either $(e_0, b'\omega^i)$ or $(e_0\omega^i, b')$ is a pair of weakly equivalent paths, since both e_0 and b' are paths from v_1 to v_2 on the universal cover, and hence homotopic. This pair of paths cannot be equivalent since e_0 and b' are minimal.

To obtain a minimal path from b, we first remove as many small cycles from b as possible without making it intersect C. This process terminates in finite steps just as in the proof of Lemma 3.12. The resulting path b_1 may not be minimal yet since it might allow a deformation first to a path intersecting C and then to a path containing small cycles. Assume that another path b_1' from v_1 to v_2 intersecting C is obtained from b_1 by replacing $p_-(a) \subset b_1$ with $p_+(a)$ (or the other way around, depending on the color of the node at $e_0 \cap e_1$) for a single arrow a. Since C is a part of a zigzag path, it follows, from the definitions of a zigzag path and the equivalence of paths just as in Lemma 3.7, that the arrow a must be e_0. (Lemma 3.7 roughly states that one needs a small cycle to deform a path across a zigzag path. Since b_1 does not contain a small cycle, the only way to deform it across C is to deform it by the equivalence relation at e_0. Unfortunately, one cannot apply Lemma 3.7 directly in the present situation since C may intersect $z \setminus C$.)

Thus b_1 contains $p_-(e_0)$ (or $p_+(e_0)$) and is written as $b_1 = cp_-(e_0)d$ (or $b_1 = cp_+(e_0)d$), where c and d are paths from v_1 to v_2. At least one of them (say, c) is not homotopic to the arrow e_0 in $\mathbb{R}^2 \setminus C$. Take a perfect matching D and count the number $|c \cap D|$ of edges of D which meet the path c.

If the number $|c \cap D|$ is equal to $|b_1 \cap D|$, then $p_-(e_0)d$ is a non-trivial cyclic path on the quiver which does not meet D at all. Note that for any given perfect matching, equivalent paths have the same numbers of arrows meeting that perfect matching. Since a small cycle meet any perfect matching at exactly one edge, the cyclic path $p_-(e_0)d$ cannot be equivalent to any positive power of a small cycle.

If the number $|c \cap D|$ is smaller than $|b_1 \cap D|$, then we set $b_2 = c$ and repeat this process. After finitely many steps, we obtain a path $b' = b_n$ such that

- b' is not homotopic to e_0 in $\mathbb{R}^2 \setminus C$, and
- b' is not equivalent to a path containing a small cycle, so that b' is minimal,

or a cyclic path which is not equivalent to any positive power of a small cycle.

To show that the path e_0 is minimal, note that the arrow e_0 can be equivalent to another path only if the edge e_0 is the first of several consecutive edges connected by divalent nodes. Since z has a self-intersection at e_0, the number of consecutive edges connected by divalent nodes must be odd and e_0 can be equivalent only to arrows. This shows that the path e_0 is minimal.

It is clear that b' is a path of length greater than one. This shows that e_0 is not equivalent to b', and Lemma 3.13 is proved. □

The following lemma can be shown in an analogous way:

Lemma 3.14. *Assume that a dimer model has a perfect matching and a zigzag path with the trivial homology class, then there is a cyclic path on the quiver which is weakly equivalent to some power of a small cycle but not equivalent.*

Indeed, consider the path which goes around the zigzag path, and factor out all the possible small cycles. Then one ends up with a path weakly equivalent to a power of a small cycle but not equivalent to it.

For example, the path on the quiver shown in Figure 19 is weakly equivalent to a small cycle as shown in Figure 20, although it is not equivalent; if we call the idempotent element in the path algebra corresponding to the top-left vertex and the path shown in Figure 19 starting from the top-left vertex as e and p respectively, then one has $p \neq e\omega$ and $p\omega = e\omega^2$.

§ 4. Properly-ordered dimer models

For a node in a dimer model, the set of zigzag paths going through the edges adjacent to it has a natural cyclic ordering given by the directions of the outgoing paths

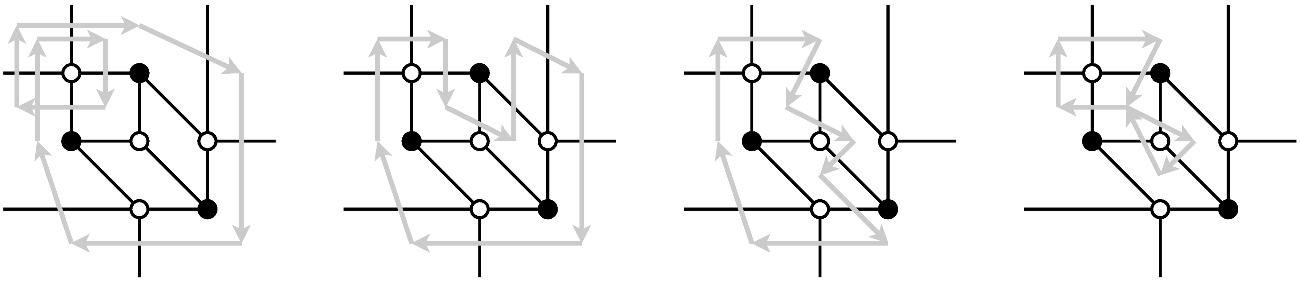

Figure 20. Deforming a path on the quiver

from the node. On the other hand, the homology classes of these zigzag paths determine another cyclic ordering if these classes are distinct.

Definition 4.1 (Gulotta [10, section 3.1]). A dimer model is *properly ordered* if

1. there is no homologically trivial zigzag path,
2. no zigzag path has a self-intersection on the universal cover,
3. no pair of zigzag paths in the same homology class have a common node, and
4. for any node of the dimer model, the cyclic order on the set of zigzag paths going through that node coincides with the cyclic order determined by their homology classes.

Here, the homology group of the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ is identified with $\mathbb{Z}^2$ in a natural way. The *slope* of a zigzag path is

$$\frac{(u, v)}{\sqrt{u^2 + v^2}} \in S^1,$$

where $(u, v) \in \mathbb{Z}^2$ is the homology class of the zigzag path. The lack of a self-intersection implies that (u, v) is a primitive element, so that a set of zigzag paths with distinct homology classes has a well-defined counter-clockwise cyclic order.

A consistent dimer model is properly ordered:

Lemma 4.2. *In a consistent dimer model, the cyclic order of the zigzag paths around any node of the dimer model is compatible with the cyclic order determined by their slopes.*

Proof. Let z_1, z_2 and z_3 be a triple of zigzag paths passing through a node of the dimer model along neighboring edges at the node whose cyclic order around the node does not respect the cyclic order determined by their slopes. Then two of them must intersect more than once in the same direction on the universal cover. $\square$

The converse is also true:

Lemma 4.3. *A properly-ordered dimer model is consistent.*

Proof. Assume for contradiction that a properly-ordered dimer model has a pair $z_1 = (e_k)_{k\in\mathbb{Z}}$ and $z_2 = (f_\ell)_{\ell\in\mathbb{Z}}$ of zigzag paths intersecting in the same direction more than once on the universal cover. We show that there is an infinite sequence $(z_3, z_4, \dots)$ of zigzag paths on the universal cover with distinct slopes, which contradicts the finiteness of the set of slopes.

An intersection $(e_i = f_{j+u}, e_{i+1} = f_{j+u-1}, \dots, e_{i+u} = f_j)$ of z_1 and z_2 where $i, j \in \mathbb{Z}$ and $u \in 2\mathbb{N}$ is called a *last* intersection if $(e_k)_{k>i+u}$ does not intersect $(f_\ell)_{\ell>j+u}$. Another intersection $(e_{i'} = f_{j'+u'}, e_{i'+1} = f_{j'+u'-1}, \dots, e_{i'+u'} = f_{j'})$ for $i' + u' < i$ is called the *second last* intersection along z_1 if $(e_k)_{i'+u'<k<i}$ does not intersect $(f_\ell)_{\ell<j}$. Although a last intersection may not be unique, and not all last intersections may have the second last intersection, the assumption that z_1 and z_2 intersect in the same direction more than once implies the existence of at least one last intersection having the second last intersection.

Figure 21 shows a part of a pair of zigzag paths near a last and the second last intersections. We have suppresed the rest of the paths, which may also intersect this part. We choose the names z_1 and z_2 for these zigzag paths, so that the node $e_{i+u} \cap e_{i+u+1}$ at the last intersection is white as in Figure 22. Although the second last intersection in this figure may be the one along z_2 instead of the one along z_1, this does not affect the discussion below.

Now choose the third zigzag path $z_3 = (g_m)_{m\in\mathbb{Z}}$ as the one going in the direction opposite to z_2 from the second last intersection as shown in dotted arrow in Figure 22, so that $g_0 = f_{j'-1}$. Note that z_2 and z_3 may not intersect at $g_0 = f_{j'-1}$ if the node at $g_0 \cap g_1$ is divalent. The cyclic order on the set of zigzag paths, passing through the node $g_{-1} \cap g_0$ where z_1, z_2 and z_3 meet, is given by $(z_1, z_2, z_3, \cdots)$. Since the dimer model is properly-ordered, the slopes of z_1, z_2 and z_3 have this cyclic order. The slope of a zigzag path determines the asymptotic behavior of the zigzag path on the universal cover, so that the zigzag paths z_1, z_2 and z_3 must have this cyclic order outside of a compact set. Combined with the assumption that the intersection $(e_i = f_{j+u}, e_{i+1} = f_{j+u-1}, \dots, e_{i+u} = f_j)$ is a last intersection of z_1 and z_2, this implies that

- $(g_m)_{m>0}$ intersects $(e_k)_{k>i'+u'}$, or
- $(g_m)_{m>0}$ intersects $(f_\ell)_{\ell>j'+u'}$.

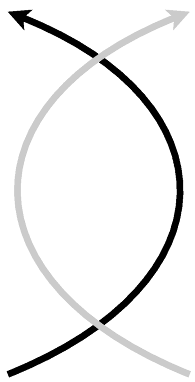

Figure 21. A pair of intersections of zigzag paths

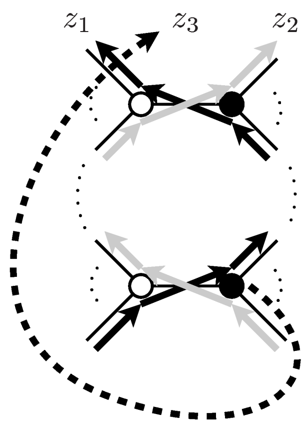

Figure 22. z_3 bending over to the left

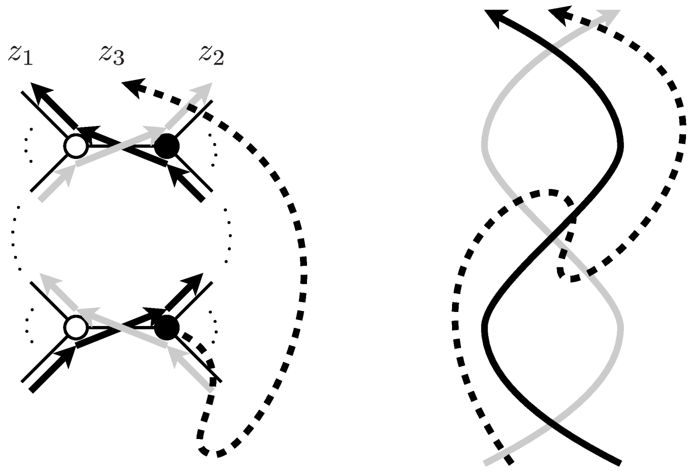

Figure 23. z_3 bending over to the right

Schematic pictures of examples of the former case and the latter case are shown in Figure 22 and Figure 23. It may also happen that $(g_m)_{m>0}$ intersect both $(e_k)_{k>i'+u'}$ and $(f_\ell)_{\ell>j'+u'}$.

In the former case, the part $(g_m)_{m>0}$ of the zigzag path z_3 intersects the zigzag path z_1 in the same direction more than once, and one can find a pair of a last and the second last intersection as in Figure 21, where the solid arrow represents z_1 and the gray arrow represents z_3 this time. Now we can repeat the same argument to obtain another zigzag path z_4 such that

- the cyclic order of the slopes is (z_2, z_3, z_4, z_1), and
- z_4 intersects z_1 or z_3 in the same direction more than once.

In the latter case, the lack of self-intersection of zigzag paths in a properly-ordered dimer model implies that the part $(g_m)_{m<0}$ of the zigzag path z_3 intersects the part $(f_\ell)_{\ell>j'+u'}$ of the zigzag path z_2, and one can find a pair of a last and the second last intersections as in Figure 21, where the solid arrow represents z_3 and the gray arrow represents z_2 this time. Now we can repeat the same argument to obtain another zigzag path z_4 such that

- the cyclic order of the slopes is (z_2, z_4, z_3, z_1), and
- z_4 intersects z_2 or z_3 in the same direction more than once.

In both cases, we obtain a zigzag path z_4 whose slope is different from the slope of any of z_1, z_2 and z_3. By continuing this process, we obtain an infinite sequence $(z_5, z_6, \dots)$ of zigzag paths with distinct slopes, and Lemma 4.3 is proved.

□

By combining Lemma 4.2 with Lemma 4.3, one obtains the equivalence between consistency condition in Definition 3.5 and Gulotta's condition:

Proposition 4.4. *A dimer model is consistent if and only if it is properly-ordered.*

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