K-FLIPS AND VARIATION OF MODULI SCHEME OF SHEAVES ON A SURFACE, II

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Introduction

We shall consider some analogy between the wall-crossing problem of moduli schemes of stable sheaves on a surface, and the minimal model program of higher-dimensional varieties. This article is a continuation of [10].

Let X be a non-singular projective surface over $\mathbb{C}$, and H an ample line bundle on X. Denote by $M(H)$ (resp. $M^s(H)$) the coarse moduli scheme of rank-two H-semistable (resp. H-stable) sheaves on X with Chern class $\alpha = (c_1, c_2) \in \mathrm{Pic}(X) \times \mathbb{Z}$.

Let H_- and H_+ be α-generic polarizations such that just one α-wall W separates them For $a \in [0,1]$ one can define the a-semistability of sheaves on X and the coarse moduli scheme $M(a)$ (resp. $M^s(a)$) of rank-two a-semistable (resp. a-stable) sheaves with Chern classes α in such a way that $M(\epsilon) = M(H_+)$ and $M(1-\epsilon) = M(H_-)$ if $\epsilon > 0$ is sufficiently small. $M(a)$ is projective over $\mathbb{C}$. Let $a_- < a_+$ be minichambers separated by only one miniwall a_0, and denote $M_+ = M(a_+)$, $M_- = M(a_-)$ and $M_0 = M(a_0)$. There are natural morphisms $\phi_- : M_- \to M_0$ and $\phi_+ : M_+ \to M_0$ ([1], [2], [8]). One may say they are morphisms of moduli schemes coming from wall-crossing methods. Let $\phi_- : V_- \to V_0$ be a birational projective morphism such that (1) V_- is normal, (2) $-K_{V_-}$ is $\mathbb{Q}$-Cartier and ϕ_--ample, (3) the codimension of the exceptional set $\mathrm{Exc}(\phi_-)$ is more than 1, and (4) the relative Picard number $\rho(V_-/V_0)$ of ϕ_- is 1. After the theory of minimal model program, we say a birational projective morphism $\phi_+ : V_+ \to V_0$ is a *K-flip* of $\phi_- : V_- \to V_0$ if (1) V_+ is normal, (2) K_{V_+} is $\mathbb{Q}$-Cartier and ϕ_+-ample, (3) the codimension of the exceptional set $\mathrm{Exc}(\phi_+)$ is more than 1, and (4) the relative Picard number $\rho(V_+/V_0)$ of ϕ_+ is 1.

Theorem 0.1. *Fix a closed, finite, rational polyhedral cone $\mathcal{S} \subset \overline{\mathrm{Amp}}(X)$ such that $\mathcal{S} \cap \partial\overline{\mathrm{Amp}}(X) \subset \mathbb{R}_{\geq 0} \cdot K_X$. If c_2 is sufficiently large with respect to c_1 and $\mathcal{S}$, then for any α-generic polarizations H_- and H_+ in $\mathcal{S}$ separated by just one α-wall W, and for any adjacent minichambers $a_- < a_+$ separated by a miniwall a_0 we have the following.*
(i) $M_\pm$ are normal and $\mathbb{Q}$-factorial, $K_{M_\pm}$ are Cartier, $M^s_\pm$ are l.c.i., and M_- and M_+ are isomorphic in codimension 1.
(ii) Suppose K_X does not lie in the α-wall, and that K_X and H_+ lie in the same connected components of $\mathrm{NS}(X)_\mathbb{R} \setminus W$. Then $\rho(M_-/M_0) = 1$ and $\phi_+ : M_+ \to M_0$ is a K-flip of $\phi_- : M_- \to M_0$. This morphism ϕ_+ (resp. ϕ_-) is the contraction of an extremal ray of $\overline{\mathrm{NE}}(M_+)$ (resp. $\overline{\mathrm{NE}}(M_-)$), which is described in moduli theory.
(iii) Suppose X is minimal and $\kappa(X) > 0$, which means K_X is not numerically equivalent to 0 and contained in $\overline{\mathrm{Amp}}(X)$. Then there is a polarization, say H_X,

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contained in $\mathcal{S}$ such that no α-wall separates H_X and K_X, and the canonical divisor of $M(H_X)$ is nef.

The greater part of this result has already appeared in [10, Theorem 1.1.]. In Section 1, we shall prove the remaining part of this theorem which has not appeared in [10], that largely is the statement about the $\mathbb{Q}$-factoriality of $M_\pm$ and $\rho(M_\pm/M_0)$. The author was not aware of this part at the time of writing [10]. There is some application; suppose X is minimal and $\kappa(X) > 0$, and fix a polarization L on X. If c_2 is sufficiently large with respect to c_1 and L, then one can observe a moduli-theoretic analogue of the minimal model program of $M(L)$. Here "analogue" means that singularities of $M(H_X)$ are not considered. About this analogy, see Introduction in [10] for detail. We remark that a K-flip differs from a Thaddeus-type flip in [8].

In Section 2, we give some notes about extremal faces of $\overline{\mathrm{NE}}(M(H)) \subset N_1(M(H))$, where H is an α-generic polarization. We shall point out that some extremal faces with dim ≥ 2 can appear in $\overline{\mathrm{NE}}(M(H))$ when H gets closer to more than one α-wall.

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Notation . All schemes are locally of finite type over $\mathbb{C}$ or, more generally, an algebraically closed field of characteristic zero. For a projective scheme V over $\mathbb{C}$, $\mathrm{Num}(V)$ means $\mathrm{Pic}(V)$ modulo numerically equivalence. For any coherent sheaf E on V , $\mathrm{Ext}^i_V(E,E)^0$ means the kernel of trace map $\mathrm{Ext}^i_V(E,E) \to H^i(\mathcal{O}_V)$.

1. Proof of Theorem

There is a union of hyperplanes $W \subset \mathrm{Amp}(X)$ called α-walls in the ample cone $\mathrm{Amp}(X)$ such that $M(H) = M(H,\alpha)$ changes only when H passes through α-walls ([9]). A polarization on X is called α-generic if no α-wall contains it. Now fix a closed, finite, rational polyhedral cone $\mathcal{S} \subset \overline{\mathrm{Amp}}(X)$ as in Theorem 0.1. Refer to [1, Section 3] about the a-stability, minichambers and miniwalls, which appeared in Introduction.

Lemma 1.1. *If c_2 is sufficiently large with respect to c_1 and $\mathcal{S}$, then for any α-generic polarizations H_- and H_+ in $\mathcal{S}$ separated by just one α-wall W, and for any adjacent minichambers a_- and a_+ separated by a miniwall a_0, (i) $M_\pm$ are normal, (ii) $K_{M_\pm}$ are Cartier, (iii) $M^s_\pm$ are l.c.i., (iv) M_- and M_+ are isomorphic in codimension 4, and (v) our natural birational map $M_- \cdots > M_+$ induces* $\mathrm{Pic}(M^s_-) \simeq \mathrm{Pic}(M^s_+)$.

Proof. Fix a polarization $L \in \mathcal{S}$. If c_2 is sufficiently large w.r.t. c_1 and L, then $M(L)$ is normal, $M^s(L)$ is of expected dimension, and the codimension of Sing'$(M(L))$ in $M(L)$ is greater than 4 by [5] and [11], where Sing'$(M(L)) \subset M(L)$ is the closed subset consisting of sheaves E such that E is not L-μ-stable or that $\mathrm{Ext}^2_X(E,E)^0 \neq 0$. One can check (iv) in a similar way to [10, Lemma 2.4.]. Now we compare $M(L)$ with M_+. By (iv) and the deformation theory of simple sheaves, M^s_+ is of expected dimension so it is l.c.i., and

$$\mathrm{codim}(\mathrm{Sing'}(M_+), M_+) > 4. \tag{1}$$

Thereby M^s_+ is normal. Since $H_\pm$ are α-generic and $a_\pm$ are minichambers, if a rank-two sheaf E with Chern classes α is a_--semistable and not a_+-semistable, then E

is H-semistable for any polarization H, and so our birational map $M_+ \cdots > M_-$ is isomorphic near $M_+ \setminus M_+^s$. Thus M_+ is normal near $M_+ \setminus M_+^s$, and accordingly M_+ itself is normal. Item (v) follows item (iv) and (1) because of Fact 1.3 below.Last, M_+ is the GIT quotient of an open subset R_+ of some Quot-scheme on X. Let $\mathcal{E}$ be a universal family of R_+ on $X \times R_+$. Since a_+ is not a miniwall, one can check that the line bundle $\det R\mathcal{H}om_{p_2}(\mathcal{E}, \mathcal{E})$ on R_+ descends to a line bundle on M_+, that equals K_{M_+}. □

Next we recall a fact concerning $\mathrm{Pic}(M_+^s)$ from [6]. For a moment we assume M_+^s has a universal family $\mathcal{E}$ on $X \times M_+^s$. Let K be the Grothendieck group of $X \times X$ and let $\tilde{K}$ be the kernel of $\xi : K \to \mathbb{Z}$, that is defined by $\xi(C) = \chi(C \boxtimes \pi_1^* \mathcal{E} \boxtimes \pi_2^* \mathcal{E})$. Here $\boxtimes$ denotes the tensor product of complexes. Let $\sigma : X \times X \to X \times X$ be the map exchanging factor and let $\mathrm{Pic}(X \times X)^\sigma$ be the subgroup consisting of line bundles invariant under σ. The map $\psi : \tilde{K} \to \mathrm{Pic}(M_+)$ defined by

$$\psi(C) = \det\left((p_1)_! \left(p_{23}^*(C) \boxtimes p_{12}^* \mathcal{E} \boxtimes p_{13}^* \mathcal{E}\right)\right) \qquad (C \in \tilde{K}) \tag{2}$$

induces a homomorphism

$$\Phi_\pm : \mathrm{Pic}(X \times X)^\sigma \oplus \mathbb{Z} \longrightarrow \mathrm{Pic}(M_\pm^s) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{12}], \tag{3}$$

as explained in [6, p. 132]. One can define Φ also when M_+^s do not necessarily admit a universal family.

Proposition 1.2. *Let $a_\pm$ be a minichamber satisfying assumptions in Lemma 1.1. If c_2 is sufficiently large with respect to c_1 and $\mathcal{S}$, then*

$$\Phi_\pm \otimes \mathbb{Q} : \mathrm{Pic}(X \times X)^\sigma \otimes \mathbb{Q} \oplus \mathbb{Q} \to \mathrm{Pic}(M_\pm^s) \otimes \mathbb{Q} \tag{4}$$

is isomorphic.

Proof. One can verify this from Lemma 1.1 (v) and by reading [6] (especially Lemma 3.10.) carefully. □

Before the proof of Theorem 0.1, recall a useful fact at [SGA2, p.132].

Fact 1.3. *Let W be any quasi-projective and l.c.i. scheme with* $\mathrm{codim}(\mathrm{Sing}(W), W) \geq 4$. *Then for any closed subset $\Lambda \subset W$ of codimension at least two, the restriction map* $\mathrm{Pic}(W) \to \mathrm{Pic}(W \setminus \Lambda)$ *is an isomorphism.*

Now we shall prove two propositions; those and [10] end the proof of Theorem 0.1.

Proposition 1.4. *Let a_+ be a minichamber satisfying assumptions in Lemma 1.1. Suppose c_2 is so large with respect to c_1 and $\mathcal{S}$ that $M_\pm$ are normal, $M_\pm^s$ are l.c.i.,* $\mathrm{codim}(\mathrm{Sing}(M_\pm^s), M_\pm^s) \geq 4$, $\mathrm{codim}(M_\pm \setminus M_\pm^s, M_\pm) \geq 2$, *and the homomorphisms at* (4) *are isomorphic. Then $M_\pm$ are $\mathbb{Q}$-factorial.*

Proof. First remark that assumptions in this proposition holds for $c_2 \gg 0$ from Lemma 1.1, Proposition 1.2, [11], and [3, Theorem 9.1.2.]. We shall verify this only for M_+. Let U be the open set $M_+ \setminus \mathrm{Sing}(M_+)$ in M_+. If $\mathrm{Cl}(M_+)$ means its divisor class group generated by Weil divisors, then we have

$$\mathrm{Cl}(M_+) \longrightarrow \mathrm{Cl}(U) \simeq \mathrm{Pic}(U) \longrightarrow \mathrm{Pic}(M_+^s),$$

where the first map is restriction, the second map is isomorphism since U is smooth, and the third map is an extension map, which is assured by Fact 1.3. Next, we have the following diagram.

$$\begin{array}{ccc} \bar{\Phi}_+ \otimes \mathbb{Q} : \mathrm{Pic}(X \times X)^\sigma \otimes \mathbb{Q} \oplus \mathbb{Q} & \longrightarrow & \mathrm{Pic}(M_+) \otimes \mathbb{Q} \\ \| & & \downarrow \\ \Phi_+ \otimes \mathbb{Q} : \mathrm{Pic}(X \times X)^\sigma \otimes \mathbb{Q} \oplus \mathbb{Q} & \longrightarrow & \mathrm{Pic}(M_+^s) \otimes \mathbb{Q}, \end{array}$$

where $\bar{\Phi}_+$ is defined at the equation (1.13) in [6] since $H_\pm$ are α-generic and a_+ is not a miniwall, and the second column is a restriction map. Proposition 1.2 implies that the second column is surjective. On the other hand, the assumptions in this proposition implies that the second column is injective. As a result we get a homomorphism $\mathrm{Cl}(M_+) \to \mathrm{Pic}(M_+) \otimes \mathbb{Q}$. Thus we end the proof. □

For a projective morphism f, we define $N_1(f)$ and $\overline{\mathrm{NE}}(f)$ according to [4, Example 2.16], an *extremal ray* or *extremal face* of $\overline{\mathrm{NE}}(f)$ according to [4, Definition 1.15], and the *contraction of an extremal ray or face* according to [4, Definition 1.25].

Proposition 1.5. *Let $a_\pm$ be minichambers as in Theorem 0.1. Suppose c_2 is sufficiently so large with respect to c_1 and $\mathcal{S}$ that conclusions in Lemma 1.1 and Proposition 1.2 hold good. Then we have the following. Let t be any point in $\phi_+(\mathrm{Exec}(\phi_+)) \subset M_0$, and let $l \simeq \mathbf{P}^1$ be any line in $\phi_+^{-1}(t) \simeq \mathbf{P}^{N_t}$. Then $\mathbb{R}_{\geq 0} \cdot l$ is an extremal ray of $\overline{\mathrm{NE}}(M_+)$, and ϕ_+ is the contraction of this extremal ray. In particular $\rho(M_+/M_0) = 1$. The similar statement holds also for $\phi_- : M_- \to M_0$.*

Proof. We check it for a_+; the proof is the same for a_-. For simplicity suppose that M_+^s has a universal family $\mathcal{E}$ on $X \times M_+$, but the proof goes in a similar way for general case. The set

$$M_+ \supset P_+ = \big\{ [E] \bigm| E \text{ is not } a_-\text{-semistable} \big\} \tag{5}$$

is contained in M_+^s since we consider rank-two case. Take a point $t \in \phi_+(P_+)$. By Proposition 2.1. in [10], it holds that $\phi_+^{-1}(t) \simeq \mathbf{P}^N$, and there is a nontrivial exact sequence on $X \times \mathbf{P}^N$

$$0 \longrightarrow \pi_1^* F \otimes \mathcal{O}_{\mathbf{P}^N}(1) \longrightarrow \mathcal{E}|_{\phi_+^{-1}(t)} \otimes \pi_2^* L \longrightarrow \pi_1^* G \longrightarrow 0, \tag{6}$$

where F and G are coherent sheaves on X, which depends on the choice of t, and L is a line bundle on $\phi_+^{-1}(t)$. Let $l \simeq \mathbf{P}^1$ be a line in $\phi_+^{-1}(t)$. Then (6) implies that $ch(\mathcal{E}|_l) = ch(E) + \mathcal{O}_l(1) \cdot ch(F)$ in $A(X \times l)$, where E is a rank-two sheaf with Chern classes α. Let C be a class in $\tilde{K}$. Because of the definition of $\tilde{K}$ and the G.R.R. theorem, we have

$$\begin{aligned} \deg(\psi(C) \cdot l) &= [p_{1*} (ch (p_{23}^* C \boxtimes p_{12}^* \mathcal{E}|_l \boxtimes p_{13}^* \mathcal{E}|_l) \cdot p_{23}^* td(X \times X))]_{1, l \times X \times X} \cdot \mathcal{O}_l(1) \\ &= [p_{1*} (p_{23}^* ch(C) \cdot \{p_2^* ch(E) + p_1^* \mathcal{O}_l(1) \cdot p_2^* ch(F)\} \cdot \\ &\qquad \{p_3^* ch(E) + p_1^* \mathcal{O}_l(1) \cdot p_3^* ch(F)\} \cdot p_{23}^* td(X \times X))]_{1, l \times X \times X} \cdot \mathcal{O}_l(1) \\ &= [ch(C) \cdot td(X \times X) \cdot \{\pi_1^* ch(F) \pi_2^* ch(E) + \pi_2^* ch(F) \pi_1^* ch(E)\}]_{0, X \times X} \\ &= \chi \left(X \times X, C \boxtimes (\pi_1^* F \boxtimes \pi_2^* E + \pi_2^* F \boxtimes \pi_1^* E) \right). \end{aligned}$$

By the projection formula and again by the definition of $\tilde{K}$, the last term equals

$$\chi\left(X\times X, C\boxtimes\{\pi_1^*(F+G+F-G)\boxtimes\pi_2^*(E)+\pi_2^*(F+G+F-G)\boxtimes\pi_1^*(E)\}\right)/2$$
$$=\chi\left(X\times X, C\boxtimes\{\pi_1^*(F-G)\boxtimes\pi_2^*(E)+\pi_2^*(F-G)\boxtimes\pi_1^*(E)\}\right)/2=$$
$$\left[\pi_1^*td(X)\cdot\pi_2^*td(X)\cdot ch(C)\cdot\{\pi_1^*ch(F-G)\cdot\pi_2^*ch(E)+\pi_2^*ch(F-G)\cdot\pi_1^*ch(E)\}\right]_0/2=$$
$$\left[\{\pi_{1*}\left(ch(C)\cdot\pi_2^*(td(X)ch(E))\right)+\pi_{2*}\left(ch(C)\cdot\pi_1^*(td(X)ch(E))\right)\}\cdot td(X)ch(F-G)\right]_0/2.$$

From [1, Section 3], if we denote $\xi=c_1(F)-c_1(G)\in\mathrm{NS}(X)$, $n=c_2(F)$ and $m=c_2(G)$, then $W^{\xi}=\{H\in\mathrm{Amp}(X)|H\cdot\xi=0\}$ equals W and one can check that $td(X)\cdot ch(F-G)=(0,\xi,(a_0-1)(H_+-H_-)\cdot\xi)$. Thereby one can verify that

$$\begin{aligned}(7)\quad \deg(\psi(C)\cdot l)=\big[&\{\pi_{1*}(C\cdot\pi_2^*td(X))+\pi_{2*}(C\cdot\pi_1^*td(X))\}^1+\\ &(a_0-1)\{\pi_{1*}(C\cdot\pi_2^*td(X))+\pi_{2*}(C\cdot\pi_1^*td(X))\}^0\cdot(H_+-H_-)\big]\cdot\xi/2.\end{aligned}$$

Now we shall show that $\mathrm{rk}N_1(M_+/M_0)=1$. If we pick two points t_1 and t_2 in $\phi_+(P_+)$, then $\phi_+^{-1}(t_i)\simeq\mathbf{P}^{N_i}$ for $i=1,2$. Fix lines $l_i\subset\phi_+^{-1}(t_i)$. Then there are exact sequences on $X\times l_i$

$$0\longrightarrow\pi_1^*F_i\otimes\mathcal{O}_{\mathbf{P}^N}(1)\longrightarrow\mathcal{E}|_{l_i}\otimes\pi_2^*L_i\longrightarrow\pi_1^*G_i\longrightarrow 0,$$

where F_i and G_i are coherent sheaves on X, and L_i is a line bundle on l_i, for $i=1,2$. Since the wall defined by $\xi_i=c_1(F_i)-c_1(G_i)$ equals W for $i=1,2$, there is a rational number r such that $\xi_1=r\xi_2$ in $\mathrm{Num}(X)$. Then (4) and (7) imply that $l_1\equiv r\cdot l_2$ in $N_1(M_+/M_0)$. As a result, we have $\overline{\mathrm{NE}}(\phi_+)=\mathbb{R}_{\geq 0}\cdot l$.

Now $\mathbb{R}_{\geq 0}\cdot l$ is an extremal ray of $\overline{\mathrm{NE}}(M_+)$. Indeed, let $u_i\in\overline{\mathrm{NE}}(M_+)$ $(i=1,2)$ satisfy that $u_1+u_2\in\mathbb{R}_{\geq 0}\cdot l$. Then, for any $H\in\mathrm{Amp}(M_0)$, $0=(u_1+u_2)\cdot\phi_+^*(H)=u_1\cdot\phi_+^*(H)+u_2\cdot\phi_+^*(H)$. Since $u_i\in\overline{\mathrm{NE}}(M_+)$, we have $u_i\cdot\phi_+^*(H)\geq 0$, and hence $u_i\cdot\phi_+^*(H)=0$ for $i=1,2$. Recall that, by Example-Exercise 3-5-1 in [7], a natural inclusion $N_1(\phi_+)\subset N_1(M_+)$ identifies $\overline{\mathrm{NE}}(\phi_+)$ with

$$\left\{z\in\overline{\mathrm{NE}}(M_+)\;\middle|\;z\cdot\phi_+^*(H)=0\text{ for any }H\in\mathrm{Amp}(M_0)\right\}.$$

Thereby $u_i\in\overline{\mathrm{NE}}(\phi_+)=\mathbb{R}_{\geq 0}\cdot l$.

Last, ϕ_+ is the contraction of $\mathbb{R}_+\cdot l$. Indeed, for any irreducible curve $C\subset M_+$, one can verify that $\phi_+(C)$ is a point if and only if $C\in\mathbb{R}_+\cdot l$ by using arguments above. Also it holds that $\phi_{+*}(\mathcal{O}_{M_+})\simeq\mathcal{O}_{M_0}$, since one can show that M_0 is normal from conclusions in Lemma 1.1 and Serre's criterion of normality, and so we conclude the proof of this proposition. □

2. Some extremal faces of $M(H)$

Now we suppose that a polarization H_+ is α-generic and contained in an α-chamber $\mathcal{C}$, with which two different α-walls W_1 and W_2 contact, that a polarization H_0 is contained in $W_1\cap W_2\cap\overline{\mathcal{C}}$, and that no α-wall except W_1 and W_2 contains H_0. Similarly to [1, Section 3], for $a\in[0,1]$ one can define the a-stability of a coherent sheaf on X and the moduli scheme $M(a)$ of a-semistable rank-two sheaves on X with fixed Chern classes in such a way that $M(1)=M(H_0)$ and $M(\epsilon)=M(H_+)$ if $\epsilon\geq 0$ is sufficiently small. Let $a_{\pm}$ be minichambers separated by just one miniwall a_0. Then Proposition 2.1 below says that $\rho(M_+/M_0)$ can be greater than 1, $\overline{\mathrm{NE}}(M_+)$ can have an extremal face with dim ≥ 2, and so $\overline{\mathrm{NE}}(M_+)$ can admit a "polyhedral-like part".

Let $P_+ \subset M_+^s$ be the set defined at (5). Every member $E \in P_+$ has a Harder-Narasimhan filtration with respect to a_-, that is given by a nontrivial exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0,$$

and then one can check that the wall defined by $\xi(E) := c_1(F) - c_1(G) \in \mathrm{NS}(X)$ equals W_1 or W_2 because of the way to derive $a_\pm$ from $H_\pm$. For $j = 1, 2$, we define a set

$$P_+ \supset P_+^{(j)} = \left\{ [E] \in P_+ \;\middle|\; \text{the wall defined by } \xi(E) \text{ equals } W_j \right\}.$$

Then, from the uniqueness of a_--HNF, $P_+^{(j)}$ is a union of some connected components of P_+, and it holds that $P_+^{(1)} \cap P_+^{(2)} = \emptyset$.

Proposition 2.1. *Suppose that both $P_+^{(1)}$ and $P_+^{(2)}$ are non-empty. Then $\overline{\mathrm{NE}}(M_+)$ has a two-dimensional extremal face spanned by $\mathbb{R}_{\geq 0} \cdot l_1$ and $\mathbb{R}_{\geq 0} \cdot l_2$, where $l_j \simeq \mathbf{P}^1$ is a line contained in $\phi_+^{-1}(t_j) \simeq \mathbf{P}^{N_j}$ with some $t_j \in \phi_+(P_+^{(j)})$, for $j = 1, 2$. The morphism ϕ_+ is the contraction of this extremal face.*

Proof. If a sheaf $E_j \in M_+^s$ is a member of $l_j \subset P_+$, then one can check that $\mathbb{R} \cdot \xi(E_1)$ does not contain $\xi(E_2)$ in $\mathrm{Num}(X)$ since $W_1 \neq W_2$. Thus it follows from (7) that the ray $\mathbb{R}_{\geq 0} \cdot l_1$ does not contain l_2 in $N_1(M_+)$. In a similar way to the proof of Proposition 1.5, we can check that (i) $\overline{\mathrm{NE}}(\phi_+) = \mathbb{R}_{\geq 0} \cdot l_1 + \mathbb{R}_{\geq 0} \cdot l_2$, (ii) this is a two-dimensional extremal face of $\overline{\mathrm{NE}}(M_+)$, and (iii) ϕ_+ is the contraction of this extremal face. □

Similarly, suppose that different α-walls W_j $(1 \leq j \leq N)$ contact with an α-chamber $\mathcal{C}$ containing H_+ and satisfy that $\cap_{j=1}^N W_j \cap \overline{\mathcal{C}}$ is non-empty. Then $\rho(M_+/M_0)$ can be N or more, and $\overline{\mathrm{NE}}(M_+)$ can have an extremal face with $\dim \geq N$.

Remark 2.2. There does exist an example of a surface X, a class α with $4c_2 - c_1^2 \gg 0$, an α-chamber $\mathcal{C}$, two α-walls W_1 and W_2, an α-generic polarization H_+, a polarization H_0, a minichamber a_+ and a miniwall a_0 such that both $P_+^{(1)}$ and $P_+^{(2)}$ are non-empty. We leave it to the reader to find such examples. In rank-two case, the definition of α-walls is rather numerical. Hence if one grasps the structure of $\mathrm{Amp}(X)$, then it may be just a calculating exercise to find such an example. Remark that, when X is an Abelian surface, $\mathrm{Amp}(X)$ is just a connected component of the big cone of X.

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